On Some Operator Inequalities

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ABSTRACT

For Hilbert-space operators $S, T$ with $S$ invertible and self-adjoint, Corach, Porta, and Recht recently proved that $\|STS^{-1} + S^{-1}TS\| \geq 2\|T\|$. A generalization of this inequality to larger classes of operators and norms is obtained as an immediate consequence of the operator form of the arithmetic-geometric-mean inequality. Some related inequalities are also discussed.

1. INTRODUCTION

Let $B(H)$ be the space of bounded operators on a complex Hilbert space $H$, and let $\| \|$ denote the usual operator norm on $B(H)$. In their work on the geometry of the space of self-adjoint invertible elements of a $C^*$-algebra, Corach, Porta, and Recht proved in [6] that if $S$ and $T$ are operators in $B(H)$ with $S$ invertible and self-adjoint, then

$$\|STS^{-1} + S^{-1}TS\| \geq 2\|T\|. \quad (1)$$

Besides the operator $\Phi_S$ defined on $B(H)$ as

$$\Phi_S(T) =STS^{-1} + S^{-1}TS, \quad (2)$$

where $S$ is invertible and self-adjoint, they also considered the companion...
The objective of this paper is to give a proof of a generalized version of (1) which is much simpler and shorter than the one given in [6]. In Section 2 we will extend (1) to larger classes of norms and operators. In fact our extended form of (1) will be realized as a special case of the arithmetic-geometric-mean inequality for operators. In Section 3, we will use some well-known operator inequalities to compare \( \Phi_s(T) \) and \( \psi_s(T) \) in various norms. The techniques in Section 3 will be employed in Section 4 to discuss a result of Berberian [3] concerning operators unitarily equivalent to their adjoints. It has been shown in [3] that if \( A \) and \( X \) are elements of a \( C^* \)-algebra with \( A \) positive and \( X \) self-adjoint such that \( AXA + X = 0 \), then \( X = 0 \). A quantitative version of this result will be presented in Section 4.

In addition to the usual operator norm on \( B(H) \), there are other interesting unitarily invariant or symmetric norms defined on ideals contained in the ideal of compact operators. When \( H \) is infinite-dimensional, a unitarily invariant norm \( \| \| \) is defined only on a norm ideal associated with it. For the sake of brevity, we will make no explicit mention of this ideal. It is understood that when we consider \( \| X \| \) we are assuming that \( X \) belongs to the norm ideal associated with \( \| \| \).

Especially well known among the unitarily invariant norms are the Schatten \( p \)-norms defined on the Schatten \( p \)-classes as

\[
\| X \|_p = \left( \sum_j s_j^p(X) \right)^{1/p} \quad \text{for} \quad 1 \leq p \leq \infty,
\]  

where \( s_j(X) \) are the singular values of the compact operator \( X \) arranged in decreasing order \( s_1(X) \geq s_2(X) \geq \cdots \), with multiplicities counted. When \( p = \infty \), the norm \( \| X \|_\infty \) coincides with the usual operator norm \( \| X \| = s_1(X) \). Note that \( \| X \|_2 \) is the Hilbert-Schmidt norm and \( \| X \|_1 \) is the trace norm.
For a complete account of the theory of unitarily invariant norms the reader is referred to [8], [17], or [18].

2. ON THE CORACH-PORTA-RECHT INEQUALITY

We begin this section by presenting the following operator form of the arithmetic–geometric mean inequality. For different proofs and several applications of this inequality the reader is referred to [4], [5], [9], [12], and [15].

**Theorem 1.** If $A$, $B$, and $X$ are operators in $B(H)$, then for every unitarily invariant norm we have

$$\|AA^*X + XBB^*\| \geq 2\|A^*XB\|.$$  

As a corollary to Theorem 1 we obtain the following considerable generalization of the inequality (1).

**Corollary 1.** If $R$, $S$, and $T$ are operators in $B(H)$ such that $R$ and $S$ are invertible, then for every unitarily invariant norm we have

$$\|R^*TS^{-1} + R^{-1}TS^*\| \geq 2\|T\|.$$  

**Proof.** The inequality (7) follows from (6) upon letting $A = R^*$, $B = S$, and $X = R^{-1}TS^{-1}$.

Specializing (7) to the particularly important Schatten $p$-norms, we have

$$\|R^*TS^{-1} + R^{-1}TS^*\|_p \geq 2\|T\|_p \quad \text{for} \quad 1 \leq p < \infty.$$  

The following result (see [12]) enables us to investigate the case when the inequality (8) for the Schatten $p$-norms, with $1 < p < \infty$, degenerates to equality.

**Theorem 2.** If $A$, $B$, and $X$ are operators in $B(H)$ and if $1 < p < \infty$, then

$$\|AA^*X + XBB^*\|_p = 2\|A^*XB\|_p$$  

if and only if $AA^*X = XBB^*$.  

For the Schatten $p$-norms with $1 < p < \infty$ we utilize Theorem 2 to give a necessary and sufficient condition for equality in (8) to hold.
COROLLARY 2. If R, S, and T are operators in \( B(H) \) such that R and S are invertible and if \( 1 < p < \infty \), then

\[
\| R^*TS^{-1} + R^{-1}TS^* \|_p = 2\| T \|_p
\]

if and only if \( TS^*S = RR^*T \).

Proof. As in the proof of Corollary 1, the assertion of Corollary 2 follows from Theorem 2 by setting \( A = R^* \), \( B = S \), and \( X = R^{-1}TS^{-1} \).

It should be pointed out here that based on Examples 1 and 2 in [12], one can easily give two-dimensional examples to demonstrate that Corollary 2 is not valid for the cases \( p = 1 \) and \( p = \infty \).

As has been observed in [14], the inequality (1) can also be interpreted as saying that the norm of the inverse of the operator \( \Phi_S \) is \( \frac{1}{2} \). Related invertibility results have been also considered in [14].

3. COMPARISON OF NORMS \( \| \Phi_S(T) \|_p \) AND \( \| \psi_S(T) \|_p \)

The comparison of the norms \( \| \Phi_S(T) \|_p \) and \( \| \psi_S(T) \|_p \) depends on the relation between the quantities \( \| AX + XA \|_p \) and \( \| AX - XA \|_p \), where \( A \) is a positive operator in \( B(H) \). In fact, for \( S \) invertible and self-adjoint, letting \( A = S^2 \) and \( X = S^{-1}TS^{-1} \), we see that

\[
\Phi_S(T) = AX + XA
\]

and

\[
\psi_S(T) = AX - XA.
\]

The following estimate (see [2] and [7]) in the Schatten \( p \)-norms with \( 1 < p < \infty \) makes it possible for us to compare \( \| \Phi_S(T) \|_p \) and \( \| \psi_S(T) \|_p \).

THEOREM 3. If \( A, B, \) and \( X \) are operators in \( B(H) \) such that \( A \) and \( B \) are positive and if \( 1 < p < \infty \), then

\[
\| AX - XB \|_p \leq \gamma_p \| AX + XB \|_p,
\]

where \( \gamma_p \) is a constant depending only on \( p \).
Before applying Theorem 3 we would like to make two comments about it. The first one is that the original form of Theorem 3, which appeared in [7], assumes that $A$ and $B$ are positive elements in the corresponding Schatten class, and the form which appeared in [2] assumes that $A$ and $B$ are positive diagonal operators. However, by appealing to the Weyl-von Neumann-Kuroda perturbation theorem (see [10]), Theorem 3 is still valid under the mere positivity assumption on $A$ and $B$. The second comment is that Theorem 3 fails to hold for the cases $p = 1$ and $p = \infty$ (see [7]). For upper and lower bounds on $\gamma_p$ the reader is referred to [1] and references therein.

As an application of Theorem 3 we now have the following general result on the operators $\Phi_S$ and $\psi_S$.

**Corollary 3.** If $R$, $S$, and $T$ are operators in $B(H)$ such that $R$ and $S$ are invertible and if $1 < p < \infty$, then

$$\|R^*TS^{-1} - R^{-1}TS^*\|_p \leq \gamma_p \|R^*TS^{-1} + R^{-1}TS^*\|_p.$$  

(10)

In particular we have

$$\|\psi_S(T)\|_p \leq \gamma_p \|\Phi_S(T)\|_p.$$  

(11)

**Proof.** The inequality (10) follows from (9) by letting $A = R^*R$, $B = SS^*$, and $X = R^{-1}TS^{-1}$. The particular inequality (11) follows from (10) by assuming $R = S = S^*$. □

We conclude this section with the following remarks.

**Remark 1.** It has been shown in [11] that in Theorem 3 the constant $\gamma_2$, which corresponds to the Hilbert-Schmidt norm, is equal to one. Consequently we have

$$\|R^*TS^{-1} - R^{-1}TS^*\|_2 \leq \|R^*TS^{-1} + R^{-1}TS^*\|_2.$$  

(12)

In particular we have

$$\|\psi_S(T)\|_2 \leq \|\Phi_S(T)\|_2.$$  

(13)

Using a trace argument, one can easily show that equality holds in (12) [and hence in (13)] if and only if $T = 0$. 

REMARK 2. The class of unitarily invariant norms for which Theorem 3 holds has been recently characterized by Kosaki [13]. Thus in these norms it is still possible to compare $\Phi_3(T)$ and $\psi_2(T)$. The characterization of such norms given in [13] involves important phenomena such as the Lipschitz continuity of the absolute-value map, the validity of the Matsaev theorem on Volterra operators, and interpolation spaces.

REMARK 3. For $2 \times 2$ matrices $A$ and $X$ with $A$ positive, it is not hard to see that for any unitarily invariant norm we have

$$\| AX - XA \| \leq \| AX + XA \|,$$

and so in the two-dimensional case we have

$$\| \psi_2(T) \| \leq \| \Phi_3(T) \|,$$

which is a generalization of (4).

Note that by the unitary invariance of $\| \|$, it is sufficient to prove (14) for the case that $A$ is diagonal. So let

$$A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \quad X = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix},$$

where $\lambda_1 \geq \lambda_2 > 0$. Thus we have

$$AX - XA = \begin{bmatrix} 0 & (\lambda_1 - \lambda_2)x_{12} \\ (\lambda_2 - \lambda_1)x_{21} & 0 \end{bmatrix},$$

$$AX + XA = \begin{bmatrix} 2\lambda_1 x_{11} & (\lambda_1 + \lambda_2)x_{12} \\ (\lambda_1 + \lambda_2)x_{21} & 2\lambda_2 x_{22} \end{bmatrix}.$$
Remark 4. Using Theorem 14 in [7] and Proposition 8 in [13], one can conclude that if $A$, $B$, and $X$ are $n \times n$ matrices such that $A$ and $B$ are positive, then

$$
\|AX - XB\| \leq (\kappa \log 2n)\|AX + XB\| \quad (16)
$$

and

$$
\|AX - XB\|_1 \leq (\kappa \log 2n)\|AX + XB\|_1, \quad (17)
$$

where $\kappa > 0$ is an absolute constant. Thus in spite of the failure of the inequality (11) for the cases $p = 1$ and $p = \infty$, and the failure of the inequality (15) when the dimension of $H$ is greater than two, it is still possible to have $n$-dimensional estimates of the form

$$
\|\psi_s(T)\| \leq (\kappa \log 2n)\|\Phi_s(T)\| \quad (18)
$$

and

$$
\|\psi_s(T)\|_1 \leq (\kappa \log 2n)\|\Phi_s(T)\|_1. \quad (19)
$$

It should be pointed out here that (16) can also be obtained as a neat application of an estimate of Pokrzywa (see [16] and references therein), which says that if $Z$ is an $n \times n$ complex matrix with real spectrum, then

$$
\|Z - Z^*\| \leq \beta_n\|Z + Z^*\|, \quad (20)
$$

where

$$
\beta_n = \frac{2}{n} \sum_{j=1}^{[n/2]} \cot \frac{(2j - 1)\pi}{2n}
$$

with $[n/2]$ denoting the integral part of $n/2$. To see how it is possible to conclude (16) from (20), we apply (20) to the $2n \times 2n$ matrix

$$
Z = \begin{bmatrix} 0 & AX \\ BX^* & 0 \end{bmatrix}.
$$

Note that $Z$ has a real spectrum. In fact if

$$
T = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & X \\ X^* & 0 \end{bmatrix},
$$

then
then $Z = TY$ and so the spectrum of $Z$ is the same as the spectrum of the self-adjoint matrix $T^{1/2}YT^{1/2}$. Now

$$Z - Z^* = \begin{bmatrix} 0 & AX - XB \\ BX^* - X^*A & 0 \end{bmatrix},$$

$$Z + Z^* = \begin{bmatrix} 0 & AX + XB \\ BX^* + X^*A & 0 \end{bmatrix}.$$

Thus by (20) we have

$$\left\| \begin{bmatrix} 0 & AX - XB \\ BX^* - X^*A & 0 \end{bmatrix} \right\| \leq \beta_{2n} \left\| \begin{bmatrix} 0 & AX + XB \\ BX^* + X^*A & 0 \end{bmatrix} \right\|.$$

Since

$$\left\| \begin{bmatrix} 0 & W \\ W^* & 0 \end{bmatrix} \right\| = \left\| \begin{bmatrix} 0 & -W \\ -W^* & 0 \end{bmatrix} \right\| = \|W\|$$

for any $n \times n$ matrix $W$, we obtain that

$$\|AX - XB\| \leq \beta_{2n} \|AX + XB\|.$$

This, together with the fact that

$$\beta_{2n} = \frac{1}{n} \sum_{j=1}^{n} \cot\left(\frac{2j-1}{4n} \pi \right) \sim \log 2n \quad \text{as} \quad n \to \infty,$$

yields (16).

**Remark 5.** Concerning the example given at the end of [6], one can use Lemma 15 in [7] to show that if $n \geq 1$, then there exist $n \times n$ self-adjoint matrices $S$ and $T$ with $S$ invertible such that

$$\|\psi_S(T)\| \geq (\kappa \log n)\|\Phi_S(T)\|,$$  

(21)

where $\kappa > 0$ is an absolute constant.
4. ON A RESULT OF BERBERIAN

As another application of Theorem 3 we have the following estimate, which is motivated by Berberian's result mentioned in Section 1. In fact our estimate can be considered as a quantitative version of a generalized form of Berberian's result.

**Theorem 4.** If \( A, B, \) and \( X \) are operators in \( B(H) \) such that \( A \) and \( B \) are positive and if \( 1 < p < \infty \), then

\[
\|AXB + X\|_p \geq \frac{2}{1 + \gamma_p} \|X\|_p. \tag{22}
\]

**Proof.** For every \( \epsilon > 0 \), let \( A_\epsilon = A + \epsilon \). Then \( A_\epsilon \) is invertible for each \( \epsilon \). By Theorem 3 we have

\[
\|X - A_\epsilon XB\|_p = \|A_\epsilon^{-1}(A_\epsilon X) - (A_\epsilon X)B\|_p
\]

\[
\leq \gamma_p \|A_\epsilon^{-1}(A_\epsilon X) + (A_\epsilon X)B\|_p
\]

\[
= \gamma_p \|X + A_\epsilon XB\|_p.
\]

Now

\[
2\|X\|_p \leq \|X + A_\epsilon XB\|_p + \|X - A_\epsilon XB\|_p
\]

\[
\leq \|X + A_\epsilon XB\|_p + \gamma_p \|X + A_\epsilon XB\|_p
\]

\[
= (1 + \gamma_p)\|X + A_\epsilon XB\|_p.
\]

Hence

\[
\|A_\epsilon XB + X\|_p \geq \frac{2}{1 + \gamma_p} \|X\|_p.
\]

The inequality (22) now follows by letting \( \epsilon \to 0 \).

Finally, we comment that it is straightforward to obtain remarks about Theorem 4 that are analogues of the remarks given in Section 3.
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