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On closed subsets of M_1-spaces

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Abstract

We show that every closed subset of an M_1-space has a closure-preserving open neighborhood base. This answers a question of Ceder, and gives positive solutions to other problems on adjunction spaces and countable sums of M_1-spaces.

MSC: 54E20

Keywords: M_1-spaces; Closure-preserving; Mosaic; M_3-spaces

1. Introduction

All spaces are assumed to be regular T_1. For a space X, we denote the topology of X by $\tau(X)$ or τ. For a subset A of X, we denote the subspace topology of A by $\tau(A)$. $\mathbb{N}$ always denotes all positive integers. The letters n, k, i are assumed to run through $\mathbb{N}$. For families $\mathcal{U}, \mathcal{V}$ of subsets of X, the operators $\mathcal{U} \wedge \mathcal{V}$ and $\mathcal{U} \vee \mathcal{V}$ are families $\{U \cap V \mid U \in \mathcal{U},\ V \in \mathcal{V}\}$ and $\{U \cup V \mid U \in \mathcal{U},\ V \in \mathcal{V}\}$, respectively. For the case $\mathcal{V} = \{V\}$, we simply write $\mathcal{U}|V$ in place of $\mathcal{U} \wedge \mathcal{V}$. For brevity, let "CP" stand for the term "*closure-preserving*". In 1961, Ceder [1] introduced M_i-spaces $(i = 1, 2, 3)$ as generalized metric spaces and proposed the following problems on M_1-spaces:

(1) Does any closed subset of an M_1-space have a CP open neighborhood base?
(2) Is any adjunction space of M_1-spaces M_1?
(Strictly speaking, Ceder himself proposed weaker problems than (1) and (2), but essentially (1) and (2) are better to pose as open problems.) In this paper, we give a positive answer to (1), which implies a positive answer to (2) as well as the following problem of Gruenhage [3]:
(3) If an M_3-space X is a countable union of closed M_1-spaces, is X M_1?

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Finally, we simply recall the definitions of M_i-spaces. A space X is called an M_1-*space* if there exists a σ-CP base for X, an M_2-*space* if there exists a σ-CP quasi-base $\mathcal{B}$ for X, where $\mathcal{B}$ is a quasi-base whenever $x \in U$ with U open in X, there exists $B \in \mathcal{B}$ such that $x \in \operatorname{Int} B \subset B \subset U$, and an M_3-*space* if there exists a σ-cushioned pair-base or equivalently there exists a stratification for X. For more detail and the properties of these spaces, it is better to refer to [4]. It is worthy to be noted that M_2-spaces and M_3-spaces coincide with each other by the famous independent study of Junnila and Gruenhage (see Theorem 5.27 in [4]).

2. The class $\mathcal{P}$ of M_1-spaces

Let $\mathcal{P}$ be the class of M_1-spaces whose every closed subset has a CP open neighborhood base. Then Ceder's problem (1) above is nothing but whether every M_1-space belongs to $\mathcal{P}$. As for the properties of $\mathcal{P}$, Mizokami showed the following result:

(i) Every adjunction spaces of spaces in $\mathcal{P}$ belongs to $\mathcal{P}$ [7, Corollary 3].
(ii) If an M_3-spaces X is the countable union of closed subspaces in $\mathcal{P}$, then $X \in \mathcal{P}$ [8, Theorem 3.16].

These results are used later to give the corollaries. Previously, Ito have obtained two important results about $\mathcal{P}$ as follows:

(1) Every M_3-space whose every point has a CP open neighborhood base belongs to $\mathcal{P}$ [6].
(2) Every hereditarily M_1-space belongs to $\mathcal{P}$ [5].

On the other hand, it is well known that every M_1-space with $\operatorname{Ind} \leqslant 0$ belongs to $\mathcal{P}$. And now, we come to the final stage. To prepare for the proof, we list up some known facts and prove two lemmas.

Fact 1 [9, Fact 4]. *Let $\mathcal{B}$ be a CP family of closed subsets of an M_3-space X. Then there exists a pair $\langle \mathcal{F}, \mathcal{V} \rangle$ of families of subsets of X satisfying the following*:

(i) *$\mathcal{F}$ is a σ-discrete closed cover of X and $\mathcal{V} = \{V(F) \mid F \in \mathcal{F}\}$ is a point-finite σ-discrete open cover of X such that $F \subset V(F)$ for each $F \in \mathcal{F}$,*
(ii) *for each $F \in \mathcal{F}$ and $B \in \mathcal{B}$, $F \cap B \neq \emptyset$ if and only if $F \subset B$ and if $F \cap B = \emptyset$, then $V(F) \cap B = \emptyset$. (We call $\mathcal{F}$ the mosaic of $\mathcal{B}$ and $\mathcal{V}$ the frill of $\mathcal{F}$.)*

Fact 2 [11]. *Let $(\mathcal{B}_i)$ be a sequence of CP families of closed subsets of an M_3-space X. Then there is a weaker metric topology τ_m of $\tau(X)$ such that for each i, $\mathcal{B}_i$ is a CP family of closed subsets of (X, τ_m).*

Lemma 2.1. *Let M be a closed subset of an M_3-space X. Then there exists a family $\mathcal{B} = \{B(O) \mid O \in \tau(X)\}$ satisfying the following*:

(i) $\mathcal{B}|(X \setminus M)$ *is a CP family of closed subsets of* $X \setminus M$;
(ii) *for each* $O \in \tau(X)$,

$$O \cap M = B(O) \cap M \subset \operatorname{Int} B(O) \subset B(O) \subset O;$$

(iii) *for each* $O \in \tau(X)$, $\overline{B(O)} \cap M \subset \overline{O \cap M}$;
(iv) *for each* $O_1, O_2 \in \tau(X)$ *with* $O_1 \cap O_2 \cap M = \emptyset$, *then* $B(O_1) \cap B(O_2) = \emptyset$.

Proof. By [8, Lemma 3.1] there exists $\mathcal{B}_1 = \{B_1(O) \mid O \in \tau(X)\}$ satisfying the following:

(1) $\mathcal{B}_1|(X \setminus M)$ is a CP family of closed subsets of $X \setminus M$;
(2) for each $O \in \tau(X)$,

$$O \cap M = B_1(O) \cap M \subset \operatorname{Int} B_1(O) \subset B_1(O) \subset O.$$

By [2, Theorem 2.2], there exists a function $\kappa : \tau(M) \to \tau(X)$ satisfying the following:
(3) $\kappa(O) \cap M = O \cap M$, $O \in \tau(M)$;
(4) if $O_1, O_2 \in \tau(M)$ and $O_1 \cap O_2 = \emptyset$, then $\kappa(O_1) \cap \kappa(O_2) = \emptyset$.

For each $O \in \tau(X)$, take

$$B(O) = B_1\bigl(\operatorname{Int} B_1(O) \cap \kappa(O \cap M)\bigr) \in \mathcal{B}_1.$$

Then it is easy to see that $\mathcal{B} = \{B(O) \mid O \in \tau(X)\}$ satisfies the required conditions. □

We call $\mathcal{B}$ the L1-*extension* of $\tau(M)$ in X. Let $\mathcal{U}$ be a family of subsets of a space X and let $x \in X$. We call that $\mathcal{U}$ is CP *at* x *in* X if whenever $x \in \overline{\bigcup \mathcal{U}_0}$, $\mathcal{U}_0 \subset \mathcal{U}$, $x \in \overline{U}$ for some $U \in \mathcal{U}_0$, equivalently, if whenever $x \notin \overline{U}$ for each $U \in \mathcal{U}_0$, where $\mathcal{U}_0 \subset \mathcal{U}$, there exists an open neighborhood O of x in X such that $O \cap (\bigcup \mathcal{U}_0) = \emptyset$.

In the proof of the main theorem later, we use the fact that if $\mathcal{O} \subset \tau(X)$ and $\mathcal{O}|M$ is CP in M, then $\{B(O) \mid O \in \mathcal{O}\}$ is CP in X. This follows from the above lemma as a corollary:

Corollary 2.2. *Let* M *be a closed subset of an* M_3*-space* X. *Let* $\mathcal{V}$ *be a CP family of open subsets of* M *and let* $\mathcal{B}$ *be the* L1*-extension of* $\tau(M)$ *in* X. *Then*

$$\mathcal{B}(\mathcal{V}) = \{B \in \mathcal{B} \mid B \cap M = V \text{ for some } V \in \mathcal{V}\}$$

is a CP family of open subsets of X *such that for each* $V \in \mathcal{V}$, $\mathcal{B}(\mathcal{V})$ *is an open neighborhood base of* V *in* X.

Proof. For each $V \in \mathcal{V}$, let

$$\mathcal{O}(V) = \bigl\{O \in \tau(X) \mid O \cap M = V\bigr\},$$

and let

$$\mathcal{B}(V) = \bigl\{B(O) \mid O \in \mathcal{O}(V)\bigr\}.$$

Then $\mathcal{B}(V)$ is an open neighborhood base of V in X, and obviously

$$\mathcal{B}(\mathcal{V}) = \bigcup \bigl\{\mathcal{B}(V) \mid V \in \mathcal{V}\bigr\}.$$

We show that $\mathcal{B}(\mathcal{V})$ is CP in X. It is obvious that $\mathcal{B}(\mathcal{V})$ is CP at each point of $X \setminus M$ in X by virtue of the property (i) of L1-extension $\mathcal{B}$. So, it remains to show that $\mathcal{B}(\mathcal{V})$ is CP at each point of M in X. To this end, let $p \in M$ and suppose $p \notin \bigcup \overline{\mathcal{B}_0}$, where

$$\mathcal{B}_0 = \bigcup \{\mathcal{B}_0(V) \mid V \in \mathcal{V}_0\},$$

$\mathcal{B}_0(V) \subset \mathcal{B}(V)$ for each $V \in \mathcal{V}_0$ and $\mathcal{V}_0 \subset \mathcal{V}$. Since $\mathcal{V}$ is CP at p in X, there exists an open neighborhood O of p in X such that $O \cap (\bigcup \mathcal{V}_0) = \emptyset$. Hence by the property (iv) above, $B(O)$ is an open neighborhood of p in X such that

$$B(O) \cap \left(\bigcup \mathcal{B}_0\right) = \emptyset.$$

This proves that $\mathcal{B}(\mathcal{V})$ is CP at p in X. □

Fact 3 [10]. *Let M be a closed subset of a metric space. Then there exists a family $\mathcal{V}$ of open subsets of X satisfying the following*:

(i) *$\{M\} \vee \mathcal{V}$ is CP in X;*
(ii) *for each $O \in \tau(X)$, there exists $V \in \mathcal{V}$ such that*

$$V \cap M = O \cap M \subset V \subset O, \qquad \overline{V} \cap (X \setminus M) \subset O.$$

Lemma 2.3. *Let $\mathcal{B}$ be a CP family of closed subsets of a metric space X. Then there exists families $\{\mathcal{V}(B) \mid B \in \mathcal{B}\}$ of open subsets of X satisfying the following*:

(i) *$\bigcup \{\{B\} \vee \mathcal{V}(B) \mid B \in \mathcal{B}\}$ is CP in X;*
(ii) *for each $O \in \tau(X)$ and $B \in \mathcal{B}$, there exists $V \in \mathcal{V}(B)$ such that*

$$O \cap B = V \cap B \subset V \subset O, \qquad \overline{V} \cap (X \setminus B) \subset O.$$

Proof. By Fact 1, there exists a pair $\langle \mathcal{F}, \mathcal{V} \rangle$ of the mosaic $\mathcal{F}$ of $\mathcal{B}$ and its frill $\mathcal{V}$ in X. Let $\mathcal{F} = \bigcup_n \mathcal{F}_n$, where each $\mathcal{F}_n$ is discrete in X. Assume that for each $F \in \mathcal{F}_n$, $F \subset V(F) \subset \{x \in X \mid d(x, F) < \frac{1}{n}\}$, where d is a metric of X. Since X is a metric space, by Fact 3, for each $F \in \mathcal{F}_n, n \in \mathbb{N}$, there exists a family $\mathcal{V}(F)$ of open subsets of X satisfying the following:

(1) $\{F\} \vee \mathcal{V}(F)$ is CP in X and $\bigcup \mathcal{V}(F) \subset V(F)$;
(2) for each $O \in \tau(X)$, there exists $V \in \mathcal{V}(F)$ such that

$$F \cap O = V \cap F \subset V \subset O, \qquad \overline{V} \cap (X \setminus F) \subset O.$$

For each $B \in \mathcal{B}$, let $\mathcal{F}(B) = \{F \in \mathcal{F} \mid F \subset B\}$.

For each

$$\delta = \langle V(F) \rangle_{F \in \mathcal{F}(B)} \in \prod \{\mathcal{V}(F) \mid F \in \mathcal{F}(B)\},$$

define

$$W(\delta) = \bigcup \{V(F) \mid F \in \mathcal{F}(B)\}.$$

Let

$$\mathcal{V}(B) = \left\{ W(\delta) \mid \delta \in \prod \left\{ \mathcal{V}(F) \mid F \in \mathcal{F}(B) \right\} \right\}$$

for each $B \in \mathcal{B}$. Then it is easily checked that $\{\mathcal{V}(B) \mid B \in \mathcal{B}\}$ are the required families. □

We call $\{\mathcal{V}(B) \mid B \in \mathcal{B}\}$ the L2-*extension* of $\mathcal{B}$ in X.

Theorem 2.4. *If X is an M_1-space, then $X \in \mathcal{P}$.*

Proof. By [6], it suffices to show that for each $p \in X$ there exists a CP neighborhood base of p consisting of regular closed subsets of X. There exists a sequence (O_m) of open neighborhoods of p in X such that

$$\{p\} = \bigcap_m O_m, \quad \overline{O_{m+1}} \subset O_m, \quad m \in \mathbb{N}.$$

Let $\bigcup \{\mathcal{B}_m \mid m \in \mathbb{N}\}$ be a quasi-base for X such that for each m, $\mathcal{B}_m \subset \mathcal{B}_{m+1}$ and $\mathcal{B}_m$ is a CP family of regular closed subsets of X. Let $m \in \mathbb{N}$ be fixed for a while, and let

$$\mathcal{B}'_m = \{B \in \mathcal{B}_m \mid p \in \operatorname{Int} B\} = \{B_\alpha \mid \alpha \in A_m\}.$$

We shall construct a CP family $\{G(\alpha) \mid \alpha \in A_m\}$ of regular closed neighborhoods of p in X such that $G(\alpha) \subset O_m \cap B_\alpha$ for each $\alpha \in A_m$. Since X is an M_3-space, there exists a CP closed neighborhood base $\mathcal{B}_0$ of p in X. By Fact 1, there exists a pair $\langle \mathcal{F}, \mathcal{V} \rangle$, where $\mathcal{F}$ is a mosaic of the CP family

$$\mathcal{B}' = \mathcal{B}_0 \cup \mathcal{B}'_m \cup \{X \setminus O_m\}$$

and $\mathcal{V}$ is its frill. Let $\mathcal{F} = \bigcup_n \mathcal{F}_n$, where each $\mathcal{F}_n$ is discrete in X. Let

$$X(n) = \bigcup \mathcal{F}_n, \qquad Y(n) = X(1) \cup \cdots \cup X(n),$$

$$Z(n) = Y(n) \setminus Y(n-1), \quad Y(0) = \emptyset, \quad n \in \mathbb{N}.$$

For each n, there exists the L1-extension $\mathcal{B}(n)$ of $\tau(Y(n))$ in X. By Fact 2, there exists a weaker metric topology τ_m of $\tau(X)$ satisfying the following:

(M_1) $\{X(n) \mid n \in \mathbb{N}\}$ is a closed cover of (X, τ_m);
(M_2) for each k,

$$\mathcal{B}[k] = \mathcal{B}(k) \vee \left\{ Y(k) \right\}$$

is a CP family of closed subsets of (X, τ_m);
(M_3) $\mathcal{B}_0$ is a CP family of closed subsets of (X, τ_m).

For each $\alpha \in A_m$, let

$$N(\alpha) = \left\{ k \in \mathbb{N} \mid B_\alpha \cap Z(k) \neq \emptyset \right\}.$$

Then obviously

$$B_\alpha \subset \bigcup \left\{ Z(k) \mid k \in N(\alpha) \right\},$$

$$B_\alpha \cap \Bigl(\bigcup \bigl\{Z(k) \mid k \in \mathbb{N} \setminus N(\alpha)\bigr\}\Bigr) = \emptyset.$$

Let $N(\alpha) = \{n(i) \mid i \in \mathbb{N}\}$ with $n(1) < n(2) < \cdots$. Let $\Delta[k]$ be the totality of finite unions of members of

$$\mathcal{B}_0 \cup \Bigl(\bigcup \bigl\{\mathcal{B}[i] \mid 1 \leqslant i \leqslant k\bigr\}\Bigr).$$

Obviously, by (M_1), (M_2), (M_3), $\Delta[k]$ is a CP family of closed subsets of (X, τ_m). Since $(Y(k), \tau_m(Y(k)))$ is a metric subspace, there exists the L2-extension

$$\bigl\{\mathcal{V}_k\bigl(B \cap Y(k)\bigr) \mid B \in \Delta[k-1]\bigr\}$$

of $\Delta[k-1] \mid Y(k)$ in $(Y(k), \tau_m(Y(k)))$. To construct a subset $G(\alpha)$ of B_α, let us fix $\alpha \in A_m$. Take $B(\alpha, n(0)) \in \mathcal{B}_0$ such that

$$B\bigl(\alpha, n(0)\bigr) \cap (X \setminus O_m) = \emptyset, \qquad B\bigl(\alpha, n(0)\bigr) \subset \operatorname{Int} B_\alpha.$$

Define

$$V\bigl(\alpha, n(1)\bigr) = B\bigl(\alpha, n(0)\bigr) \cap Z\bigl(n(1)\bigr). \tag{$*$}$$

Note that

$$V\bigl(\alpha, n(1)\bigr) = B\bigl(\alpha, n(0)\bigr) \cap Y\bigl(n(1)\bigr).$$

Since $\mathcal{F}$ is the mosaic of $\mathcal{B}'$, $V(\alpha, n(1))$ is clopen in $Y(n(1))$. Since $\mathcal{B}(n(1))$ is the L1-extension of $\tau(Y(n(1)))$ in X, there exists $B(\alpha, n(1)) \in \mathcal{B}(n(1))$ such that

$$B\bigl(\alpha, n(1)\bigr) \cap Y\bigl(n(1)\bigr) = V\bigl(\alpha, n(1)\bigr) \subset \operatorname{Int} B\bigl(\alpha, n(1)\bigr),$$

$$\overline{B\bigl(\alpha, n(1)\bigr)} \cap Y\bigl(n(1)\bigr) = V\bigl(\alpha, n(1)\bigr),$$

$$B\bigl(\alpha, n(1)\bigr) \cap (X \setminus O_m) = \emptyset.$$

Let

$$C\bigl(\alpha, n(1)\bigr) = \bigcup \Bigl\{ B \in \bigcup \bigl\{\mathcal{B}(i) \mid n(1) \leqslant i < n(2)\bigr\} \mid B \cap \bigl(B\bigl(\alpha, n(0)\bigr) \cup B\bigl(\alpha, n(1)\bigr)\bigr) = \emptyset\Bigr\}$$

and take

$$V\bigl(\alpha, n(2)\bigr) \in \mathcal{V}_{n(2)}\bigl(B\bigl(\alpha, n(0)\bigr) \cup B\bigl(\alpha, n(1)\bigr) \cup Y\bigl(n(1)\bigr)\bigr) \cap Y\bigl(n(2)\bigr)$$

such that

$$\overline{V\bigl(\alpha, n(2)\bigr)} \cap \bigl(C\bigl(\alpha, n(1)\bigr) \cup (X \setminus O_m)\bigr) = \emptyset,$$

$$V\bigl(\alpha, n(2)\bigr) \cap \bigl(B\bigl(\alpha, n(0)\bigr) \cup B\bigl(\alpha, n(1)\bigr) \cup Y\bigl(n(1)\bigr)\bigr) = \bigl(B\bigl(\alpha, n(0)\bigr) \cup B\bigl(\alpha, n(1)\bigr)\bigr) \cap Z\bigl(n(2)\bigr).$$

Assume that $\{B(\alpha, n(i)) \mid i \leqslant k\}$, $\{V(\alpha, n(i)) \mid i \leqslant k\}$ and $\{C(\alpha, n(i)) \mid i \leqslant k\}$ have been chosen. Choose

$$V\bigl(\alpha, n(k+1)\bigr) \in \mathcal{V}_{n(k+1)}\Biggl(\Biggl(\bigcup_{i=0}^{k} B\bigl(\alpha, n(i)\bigr)\Biggr)\Biggr) \cup Y\bigl(n(k)\bigr) \cap Y\bigl(n(k+1)\bigr)$$

such that

$$\overline{V\big(\alpha, n(k+1)\big)} \cap \left(\bigcup_{i=1}^{k} C\big(\alpha, n(i)\big) \cup (X \setminus O_m) \right) = \emptyset,$$

$$V\big(\alpha, n(k+1)\big) \cap \left(\bigcup_{i=0}^{k} B\big(\alpha, n(i)\big) \cup Y\big(n(k)\big) \right)$$
$$= \bigcup_{i=0}^{k} B\big(\alpha, n(i)\big) \cap Z\big(n(k+1)\big).$$

Take

$$B\big(\alpha, n(k+1)\big) \in \mathcal{B}\big(n(k+1)\big)$$

such that

$$B\big(\alpha, n(k+1)\big) \cap Y\big(n(k+1)\big) = V\big(\alpha, n(k+1)\big) \subset \operatorname{Int} B\big(\alpha, n(k+1)\big),$$
$$\overline{B\big(\alpha, n(k+1)\big)} \cap Y\big(n(k+1)\big) = \overline{V\big(\alpha, n(k+1)\big)},$$
$$B\big(\alpha, n(k+1)\big) \cap \left(\bigcup_{i=1}^{k} C\big(\alpha, n(i)\big) \cup (X \setminus O_m) \right) = \emptyset.$$

Let

$$C\big(\alpha, n(k+1)\big)$$
$$= \bigcup \left\{ B \in \bigcup \big\{ \mathcal{B}(i) \mid n(k+1) \leqslant i < n(k+2) \big\} \;\middle|\; B \cap \left(\bigcup_{i=0}^{k+1} B\big(\alpha, n(i)\big) \right) = \emptyset \right\}.$$

In this way, we can obtain three sequences

$$\big(B\big(\alpha, n(i)\big)\big)_i, \qquad \big(V\big(\alpha, n(i)\big)\big)_i, \qquad \big(C\big(\alpha, n(i)\big)\big)_i.$$

Define $G(\alpha)$ as follows:

$$G(\alpha) = \left(\bigcup_{i=1}^{\infty} \overline{V\big(\alpha, n(i)\big)} \right) \cap B_\alpha.$$

Claim 1. *$G(\alpha)$ is a regular closed neighborhood of p in X.*

Proof. Since

$$B\big(\alpha, n(0)\big) \cap B_\alpha \subset G(\alpha),$$

and both $B(\alpha, n(0))$ and B_α are neighborhoods of p in X, $G(\alpha)$ is a neighborhood of p in X. To see that $G(\alpha)$ is closed in B_α, let $x \in B_\alpha \setminus G(\alpha)$. There exists $n(k) \in N(\alpha)$ such that $x \in Z(n(k))$. Since $x \notin \overline{V(\alpha, n(k))}$, there exists an open neighborhood O of x in the subspace $Z(n(k))$ such that $O \cap V(\alpha, n(k)) = \emptyset$. Since $\mathcal{B}(n(k))$ is the L1-extension of

$\tau(Y(n(k)))$ in X, there exists a $B \in \mathcal{B}(n(k))$ such that B is a neighborhood of x in X such that

$$B \cap Y\big(n(k)\big) = O \subset \operatorname{Int} B, \qquad B \cap \left(\bigcup_{i=0}^{k} B\big(\alpha, n(i)\big)\right) = \emptyset.$$

Note that this B is contained in $C(\alpha, n(k))$. Therefore by the choice of $\{V(\alpha, n(i))\}$, we have

$$B \cap \left(\bigcup_{i \geqslant k} V\big(\alpha, n(i)\big)\right) = \emptyset.$$

Hence $B \cap B_\alpha$ is a neighborhood of x in B_α missing $G(\alpha)$, proving that $G(\alpha)$ is closed in B_α.

Next, we show that $G(\alpha)$ is regular closed in X. For brevity, let $Z(\alpha) = \bigcup\{Z(n(i)) \mid i \in \mathbb{N}\}$, where $N(\alpha) = \{n(i) \mid i \in \mathbb{N}\}$. By the same discussion as above, we can observe that

$$\overline{\bigcup\big\{V\big(\alpha, n(i)\big) \mid i \in \mathbb{N}\big\}} \cap Z(\alpha) = \bigcup\big\{\overline{V\big(\alpha, n(i)\big)} \cap Z\big(n(i)\big) \mid i \in \mathbb{N}\big\},$$
$$\overline{\bigcup_{i>k} V\big(\alpha, n(i)\big)} \cap Z\big(n(k)\big) \subset \overline{V\big(\alpha, n(k)\big)}, \quad k \in \mathbb{N}. \tag{1}$$

Since $\mathcal{F}$ is the mosaic of $\mathcal{B}'$, for each i, $B_\alpha \cap Z(n(i))$ is clopen in the subspace $Z(n(i))$, we have

$$\overline{V\big(\alpha, n(i)\big)} \cap B_\alpha \cap Z\big(n(i)\big) = \overline{V\big(\alpha, n(i)\big) \cap B_\alpha} \cap Z\big(n(i)\big). \tag{2}$$

Then we can show the following:

$$\overline{\bigcup_{i=1}^{\infty} V\big(\alpha, n(i)\big) \cap B_\alpha} = \bigcup_{i=1}^{\infty} \overline{V\big(\alpha, n(i)\big) \cap B_\alpha} \cap Z\big(n(i)\big). \tag{3}$$

For, if x is an arbitrary point of the left term of (3), then by (1) and by the fact $B_\alpha \subset Z(\alpha)$, there exists $i \in \mathbb{N}$ such that

$$x \in \overline{V\big(\alpha, n(i)\big)} \cap Z\big(n(i)\big).$$

Hence by (2) we have

$$x \in \overline{V\big(\alpha, n(i)\big) \cap B_\alpha} \cap Z\big(n(i)\big),$$

proving that the left term is contained in the right. Since the reverse inclusion is trivial, we have the equality (3). Using (1)–(3) we have the following expression:

$$\begin{aligned} G(\alpha) &= \left(\bigcup_{i=1}^{\infty} \overline{V\big(\alpha, n(i)\big)}\right) \cap B_\alpha \\ &= \bigcup_{i=1}^{\infty} \big(\overline{V\big(\alpha, n(i)\big)} \cap B_\alpha \cap Z\big(n(i)\big)\big) \quad \text{(by (1))} \end{aligned}$$

$$= \bigcup_{i=1}^{\infty}\left(\overline{V\left(\alpha, n(i)\right) \cap B_\alpha} \cap Z\left(n(i)\right)\right) \quad \text{(by (2))}$$

$$= \overline{\left(\bigcup_{i=1}^{\infty} V\left(\alpha, n(i)\right)\right) \cap B_\alpha} \quad \text{(by (3))}.$$

Since for each i

$$V\left(\alpha, n(i)\right) \subset \operatorname{Int} B\left(\alpha, n(i)\right),$$

we have

$$\left(\bigcup_{i=1}^{\infty} V\left(\alpha, n(i)\right)\right) \cap Z(\alpha) = \left(\bigcup_{i=0}^{\infty} \operatorname{Int} B\left(\alpha, n(i)\right)\right) \cap Z(\alpha).$$

These imply that

$$G(\alpha) = \overline{\left(\bigcup_{i=0}^{\infty} \operatorname{Int} B\left(\alpha, n(i)\right)\right) \cap B_\alpha}.$$

Therefore $G(\alpha)$ is regular closed in B_α. Since B_α is regular closed in X, so is $G(\alpha)$ in X. □

Claim 2.

$$\mathcal{G}(m) = \left\{G(\alpha) \mid \alpha \in A_m\right\}$$

is CP in X.

Proof. Suppose $\Omega \subset A_m$ and

$$x \in X \setminus \bigcup\left\{G(\alpha) \mid \alpha \in \Omega\right\}.$$

Let Ω be divided into Ω_1 and Ω_2 as

$$\Omega_1 = \{\alpha \in \Omega \mid x \notin B_\alpha\}, \qquad \Omega_2 = \{\alpha \in \Omega \mid x \in B_\alpha\}.$$

Since $\mathcal{B}_m$ is CP in X, there exists a neighborhood O of p in X such that

$$O \cap \left(\bigcup\left\{G(\alpha) \mid \alpha \in \Omega_1\right\}\right) = \emptyset.$$

There exists a unique $n \in \mathbb{N}$ such that $x \in Z(n)$. It follows that for each $\alpha \in \Omega_2$, $n \in N(\alpha)$ and $n = n(k_\alpha)$ for some $k_\alpha \in \mathbb{N}$. By the above discussion to see the regular closedness of $G(\alpha)$, we can easily observe the following: If $\alpha \in \Omega_2$ and $n = n(k_\alpha)$ with $k_\alpha \in N$, then

$$G(\alpha) \cap Z(n) = \overline{V\left(\alpha, n(k_\alpha)\right)} \cap Z(n).$$

If we recall the definition $(*)$, then it follows that

$$\left\{\overline{V\left(\alpha, n(k_\alpha)\right)} \mid n(k_\alpha) = \min N(\alpha), \alpha \in \Omega_2\right\}$$

is CP at x in X. If we recall that

$$\left\{\mathcal{V}_n\left(B \cap Y(n)\right) \mid B \in \Delta[n-1]\right\}$$

is the L2-extension of $\Delta[n-1]|Y(n)$ in $(Y(k), \tau_m(Y(n))$, then it follows that

$$\left\{\overline{V\left(\alpha, n(k_\alpha)\right)} \cap Z(n) \mid \alpha \in \Omega_2\right\}$$

is CP at x in X. Hence there exists an open neighborhood P of x in X such that $P \cap Y(n-1) = \emptyset$ and

$$P \cap \overline{V\left(\alpha, n(k_\alpha)\right)} = \emptyset, \quad \alpha \in \Omega_2.$$

Since $\mathcal{B}(n)$ is the L1-extension of $\tau(Y(n))$ in X, there exists $C \in \mathcal{B}(n)$ such that

$$C \cap Y(n) = P \cap Y(n) \subset \operatorname{Int} C \subset C \subset P.$$

Note that this C is contained in $C(\alpha, n(k_\alpha))$ for each $\alpha \in \Omega_2$. This implies

$$C \cap \left(\bigcup\left\{G(\alpha) \mid \alpha \in \Omega_2\right\}\right) = \emptyset.$$

Hence $O \cap C$ is a neighborhood of x in X missing all $G(\alpha)$, $\alpha \in \Omega$, which proves that $\mathcal{G}(m)$ is CP in X. □

Set

$$\mathcal{G} = \bigcup\left\{\mathcal{G}(m) \mid m \in \mathbb{N}\right\}.$$

Then it is easy to see that $\mathcal{G}$ is a CP neighborhood base of p in X consisting of regular closed subsets of X. □

Since we have no difference between M_1-spaces and $\mathcal{P}$, the next corollaries follow from the known results stated in the beginning of this section.

Corollary 2.5 [7, Corollary 3]. *Every adjunction space of M_1-spaces is M_1.*

Corollary 2.6 [8, Theorem 3.16]. *If an M_3-space X is the countable union of closed M_1-spaces, then X is an M_1-space.*

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