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Optimal control of fractional diffusion equation

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ABSTRACT

In this paper we apply the classical control theory to a fractional diffusion equation in a bounded domain. The fractional time derivative is considered in a Riemann–Liouville sense. We first study the existence and the uniqueness of the solution of the fractional diffusion equation in a Hilbert space. Then we show that the considered optimal control problem has a unique solution. Interpreting the Euler–Lagrange first order optimality condition with an adjoint problem defined by means of right fractional Caputo derivative, we obtain an optimality system for the optimal control.

1. Introduction

Let $N \in \mathbb{N}^*$ and Ω be a bounded open subset of $\mathbb{R}^N$ with boundary $\partial\Omega$ of class $\mathcal{C}^2$. For a time $T > 0$, we set $Q = \Omega \times (0, T)$ and $\Sigma = \partial\Omega \times (0, T)$ and we consider the fractional diffusion equation:

$$\begin{cases} D_+^{\alpha} y - \Delta y = v & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ I_+^{1-\alpha} y(0^+) = y^0 & \text{in } \Omega \end{cases} \tag{1}$$

where $0 < \alpha < 1$, $y^0 \in H^2(\Omega) \cap H_1^0(\Omega)$, the control v belongs to $L^2(Q)$. The fractional integral $I_+^{1-\alpha}$ and derivative D_+^{α} are understood here in the Riemann–Liouville sense, $I_+^{1-\alpha} y(0^+) = \lim_{t\to 0^+} I_+^{1-\alpha} y(t)$.

A strong motivation for studying and investigating the solution and the properties for fractional diffusion equations comes from the fact that they describe efficiently anomalous diffusion on fractals (physical objects of fractional dimension, like some amorphous semiconductors or strongly porous materials; see [1,2] and references therein), fractional random walk, etc. In [3], Oldham and Spanier discuss the relation between a regular diffusion equation and a fractional diffusion equation that contains a first order derivative in space and a half order derivative in time. Mainardi [4], Mainardi and Paradisi [5] and Mainardi and Pagnini [6] generalized the diffusion equation by replacing the first time derivative with a fractional derivative of order α. These authors proved that the process changes from slow diffusion to classical diffusion, then to diffusion-wave and finally to classical wave when α increases from 0 to 2. The fundamental solutions of the Cauchy problems associated to these generalized diffusion equation ($0 < \alpha \leq 2$) are studied in [6,7]. By means of Fourier–Laplace transforms, the authors expressed these solutions in term of Wright-type functions that can be interpreted as spatial probability density functions evolving in time with similarity properties. Agrawal [8] studied the solutions for a fractional diffusion wave equation defined in a bounded domain when the fractional time derivative is described in the Caputo sense. Using Laplace transform and finite sine transform technique, the author obtained the general solution in terms of Mittag-Leffler functions. Note also that the formulation of Mainardi et al. is extended to a fractional wave equation that contains a fourth order space derivative term by Agrawal [9,10]. Wyss in [11] used Mellin transform theory to obtain a closed form

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solution of the fractional diffusion equation in terms of Fox's H-function. In [12], Metzler and Klafter used the method of images and the Fourier–Laplace transform technique to solve the fractional diffusion equation for different boundary value problems.

In the area of calculus of variations and optimal control of fractional differential equations little has been done compared to a differential equation with integer time derivative. In [13], Agrawal presented a general formulation and solution scheme for the fractional optimal control problem. That is an optimal control problem in which either the performance index or the differential equations governing the dynamics of the system or both contain at least one fractional derivative term. In that paper, the fractional derivative was defined in the Riemann–Liouville sense and the formulation was obtained by means of the fractional variation principle [14] and the Lagrange multiplier technique. Following the same technique, Frederico Gastao and Torres Delfim [15] obtained a Noether-like theorem for the fractional optimal control problem in the sense of Caputo. Recently, Agrawal [16] presented an eigenfunction expansion approach for a class of distributed systems whose dynamics are defined in the Caputo sense. Following the same approach as Agrawal, in [17] Özdemir investigated the fractional optimal control problem of a distributed system in cylindrical coordinates whose dynamics are defined in the Riemann–Liouville sense. Note that, for computational purposes, only a finite number of eigenfunctions are considered in both papers.

In this paper we are concerned with the following optimal control problem: *find the control $u = u(x, t) \in L^2(Q)$ that minimizes the cost function*

$$J(v) = \|y(v) - z_d\|^2_{L^2(Q)} + N\|v\|^2_{L^2(Q)}, \quad z_d \in L^2(Q) \text{ and } N > 0$$

subject to the system (1).

To solve this problem, we prove that problem (1) has a unique solution in $L^2(Q)$. Then we show that the optimal control has a unique solution. Finally interpreting the Euler–Lagrange first order optimality condition with an adjoint problem defined by means of a right fractional Caputo derivative, we obtain an optimality system for the optimal control. As far as we know, the result presented here is new in fractional calculus since we give a complete theoretical study of the considered optimal control and a way to compute this control.

The rest of the paper is organized as follows. Section 2 is devoted to some definitions and preliminary results. In Section 3 we prove the existence and uniqueness of the solution of (1). In Section 4 we show that our optimal control problem holds and gives the optimality system for the optimal control. Concluding remarks are made in Section 5.

2. Preliminaries

Definition 2.1. Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a continuous function on $\mathbb{R}^+$ and $\alpha > 0$. Then the expression

$$I^\alpha_+ f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\mathrm{d}s, \quad t > 0$$

is called the Riemann–Liouville integral of order α.

Definition 2.2 (*[18]*)**.** Let $f : \mathbb{R}_+ \to \mathbb{R}$. The Riemann–Liouville fractional derivative of order α of f is defined by

$$D^\alpha_+ f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_0^t (t-s)^{n-\alpha-1} f(s)\mathrm{d}s, \quad t > 0,$$

where $\alpha \in (n-1, n)$, $n \in \mathbb{N}$.

Definition 2.3 (*[18]*)**.** Let $f : \mathbb{R}_+ \to \mathbb{R}$. The (left) Caputo fractional derivative of order α of f is defined by

$$D^\alpha_0 f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)\mathrm{d}s, \quad t > 0,$$

where $\alpha \in (n-1, n)$, $n \in \mathbb{N}$.

The Caputo fractional derivative is a sort of regularization in the time origin for the Riemann–Liouville fractional derivative.

Lemma 2.4 (*[18,19]*)**.** *Let $T > 0$, $u \in C^m([0; T])$, $p \in (m-1; m)$, $m \in \mathbb{N}$ and $v \in C^1([0; T])$. Then for $t \in [0; T]$, the following properties hold:*

$$D_+{}^p v(t) = \frac{\mathrm{d}}{\mathrm{d}t} I^{1-p}_+ v(t), \quad m = 1, \tag{2}$$

$$D^p_+ I^p_+ v(t) = v(t); \tag{3}$$

$$I^p_+ D^p_0 u(t) = u(t) - \sum_{k=0}^{m-1} \frac{t^k}{k!} u^{(k)}(0); \tag{4}$$

$$\lim_{t\to 0^+} D^p_0 u(t) = \lim_{t\to 0^+} I^p_+ u(t) = 0. \tag{5}$$

From now on we set:

$$\mathcal{D}^{\alpha} f(t)=\frac{1}{\Gamma(1-\alpha)} \int_{t}^{T}(s-t)^{-\alpha} f'(s) \mathrm{d} s. \tag{6}$$

Remark 2.5. $-\mathcal{D}^{\alpha} f(t)$ is the so-called right fractional Caputo derivative. It represents the future state of $f(t)$. For more details on this derivative we refer to [18,19]. Note also that when $T=+\infty$, $\mathcal{D}^{\alpha} f(t)$ is the Weyl fractional integral of order α of f' [20].

Lemma 2.6. *For any $\varphi \in \mathcal{C}^{\infty}(\overline{Q})$, we have*

$$\begin{aligned}
\int_{0}^{T} \int_{\Omega}\left(D_{+}^{\alpha} y(x, t)-\Delta y(x, t)\right) \varphi(x, t) \mathrm{d} x \mathrm{d} t = & \int_{\Omega} \varphi(x, T) I_{+}^{1-\alpha} y(x, T) \mathrm{d} x-\int_{\Omega} \varphi(x, 0) I_{+}^{1-\alpha} y(x, 0^{+}) \mathrm{d} x \\
& +\int_{0}^{T} \int_{\partial \Omega} y \frac{\partial \varphi}{\partial \nu} \mathrm{d} \sigma \mathrm{d} t-\int_{0}^{T} \int_{\partial \Omega} \frac{\partial y}{\partial \nu} \varphi \mathrm{d} \sigma \mathrm{d} t \\
& +\int_{\Omega} \int_{0}^{T} y(x, t)\left(-\mathcal{D}^{\alpha} \varphi(x, t)-\Delta \varphi(x, t)\right) \mathrm{d} x \mathrm{d} t.
\end{aligned} \tag{7}$$

Proof. Let $\varphi \in \mathcal{C}^{\infty}(\overline{Q})$. We have

$$\int_{0}^{T} \int_{\Omega}\left(D_{+}^{\alpha} y(x, t)-\Delta y(x, t)\right) \varphi(x, t) \mathrm{d} x \mathrm{d} t=\int_{0}^{T} \int_{\Omega} D_{+}^{\alpha} y(x, t) \varphi(x, t) \mathrm{d} x \mathrm{d} t-\int_{0}^{T} \int_{\Omega} \Delta y(x, t) \varphi(x, t) \mathrm{d} x \mathrm{d} t.$$

We have

$$-\int_{0}^{T} \int_{\Omega} \Delta y(x, t) \varphi(x, t) \mathrm{d} x \mathrm{d} t=-\int_{0}^{T} \int_{\partial \Omega} \frac{\partial y}{\partial \nu} \varphi \mathrm{d} \sigma \mathrm{d} t+\int_{0}^{T} \int_{\partial \Omega} y \frac{\partial \varphi}{\partial \nu} \mathrm{d} \sigma \mathrm{d} t-\int_{0}^{T} \int_{\Omega} y(x, t) \Delta \varphi(x, t) \mathrm{d} x \mathrm{d} t. \tag{8}$$

Using the notations above and (2), we have

$$\begin{aligned}
\int_{0}^{T} \int_{\Omega} D_{+}^{\alpha} y(x, t) \varphi(x, t) \mathrm{d} x \mathrm{d} t & =\int_{\Omega}\left[\int_{0}^{T} \varphi(x, t) \frac{\mathrm{d}}{\mathrm{d} t}\left(I_{+}^{1-\alpha} y(x, t)\right) \mathrm{d} t\right] \mathrm{d} x \\
& =\int_{\Omega} \varphi(x, T) I_{+}^{1-\alpha} y(x, T) \mathrm{d} x-\int_{\Omega} \varphi(x, 0) I_{+}^{1-\alpha} y(x, 0^{+}) \mathrm{d} x \\
& \quad -\int_{\Omega}\left[\int_{0}^{T} \varphi'(x, t) I_{+}^{1-\alpha} y(x, t) \mathrm{d} t\right] \mathrm{d} x.
\end{aligned}$$

Since

$$\begin{aligned}
-\int_{\Omega}\left[\int_{0}^{T} \varphi'(x, t) I_{+}^{1-\alpha} y(x, t) \mathrm{d} t\right] \mathrm{d} x & =-\int_{\Omega}\left[\int_{0}^{T} \varphi'(x, t)\left(\frac{1}{\Gamma(1-\alpha)} \int_{0}^{t}(t-s)^{-\alpha} y(x, s) \mathrm{d} s\right) \mathrm{d} t\right] \mathrm{d} x \\
& =-\int_{\Omega}\left[\int_{0}^{T} y(x, s)\left(\frac{1}{\Gamma(1-\alpha)} \int_{s}^{T}(t-s)^{-\alpha} \varphi'(x, t) \mathrm{d} t\right) \mathrm{d} s\right] \mathrm{d} x \\
& =-\int_{\Omega}\left[\int_{0}^{T} y(x, s) \mathcal{D}^{\alpha} \varphi(x, s) \mathrm{d} s\right] \mathrm{d} x
\end{aligned}$$

where $\mathcal{D}^{\alpha} \varphi(x, t)$ is given by (6), we deduce that

$$\begin{aligned}
\int_{0}^{T} \int_{\Omega} D_{+}^{\alpha} y(x, t) \varphi(x, t) \mathrm{d} x \mathrm{d} t = & \int_{\Omega} \varphi(x, T) I_{+}^{1-\alpha} y(x, T) \mathrm{d} x-\int_{\Omega} \varphi(x, 0) I_{+}^{1-\alpha} y(x, 0^{+}) \mathrm{d} x \\
& -\int_{\Omega}\left[\int_{0}^{T} y(x, t) \mathcal{D}^{\alpha} \varphi(x, t) \mathrm{d} t\right] \mathrm{d} x.
\end{aligned} \tag{9}$$

Hence adding (9) to (8), we obtain (7). □

Lemma 2.7. *Let y be the solution of (1). Then for any $\varphi \in \mathcal{C}^{\infty}(\overline{Q})$ such that $\varphi(x, T)=0$ in Ω and $\varphi=0$ on Σ, we have*

$$\int_0^T \int_\Omega \left(D_+^\alpha y(x,t) - \Delta y(x,t)\right)\varphi(x,t)\mathrm{d}x\mathrm{d}t = -\int_\Omega \varphi(x,0) I_+^{1-\alpha} y(x,0^+)\mathrm{d}x - \int_0^T \int_{\partial\Omega} \frac{\partial y}{\partial \nu}\varphi \mathrm{d}\sigma \mathrm{d}t + \int_0^T \int_{\partial\Omega} y \frac{\partial \varphi}{\partial \nu}\mathrm{d}\sigma \mathrm{d}t + \int_0^T \int_\Omega y(x,t)\left(-\mathcal{D}^\alpha \varphi(x,t) - \Delta\varphi(x,t)\right)\mathrm{d}x\mathrm{d}t.$$

Proof. It is an immediate consequence of Lemma 2.6. □

Lemma 2.8 ([19]). *Let $0 < \alpha < 1$. Let $g \in L^p(0,T)$, $1 \le p \le \infty$ and $\phi :]0,T] \to \mathbb{R}_+$ be the function defined by:*

$$\phi(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}.$$

Then for almost every $t \in [0,T]$, the function $s \mapsto \phi(t-s)g(s)$ is integrable on $[0,T]$. Set

$$\phi \star g(t) = \int_0^t \phi(t-s)g(s)\mathrm{d}s.$$

Then $\phi \star g \in L^p(0,T)$ and

$$\|\phi \star g\|_{L^p(0,T)} \le \|\phi\|_{L^1(0,T)}\|g\|_{L^p(0,T)}.$$

For more details on fractional integrals and derivatives with their applications see also [21].

Recall the so-called Mainardi function which is a particular Wright function [6,18,22]:

$$\Phi_\alpha(z) = \sum_{n=0}^{+\infty} \frac{(-z)^n}{n!\Gamma(-\alpha n + 1 - \alpha)} = \frac{1}{2i\pi}\int_G \lambda^{\alpha-1} e^{(\lambda - z\lambda^\alpha)}\mathrm{d}\lambda, \quad 0 < \alpha < 1 \tag{10}$$

where G is a contour which starts and ends at $-\infty$ and encircles the origin once clockwise. We have the following relation between the Wright function and the Mittag-Leffler function:

$$E_\alpha(z) = \int_0^\infty \Phi_\alpha(t) e^{zt}\mathrm{d}t, \quad 0 < \alpha < 1.$$

This means that $E_\alpha(-z)$ is the Laplace transform of Φ_α in the whole complex plane. Therefore Φ_α is a probability density function, i.e.:

$$\Phi_\alpha(t) \ge 0 \quad \text{for all } t > 0,$$
$$\int_0^\infty \Phi_\alpha(t)\mathrm{d}t = 1.$$

3. Existence and uniqueness of the solution of (1)

Consider the following abstract fractional differential equation in a Banach space $\mathbb{X}$:

$$\begin{cases} D_+^\alpha y(t) = Ay(t) + f(t), & t \in [0,T] \\ I_+^{1-\alpha} y(0) = y^0, \end{cases} \tag{11}$$

where $(\mathbb{X}, \|.\|_{\mathbb{X}})$ is a Banach space, $0 < \alpha < 1$, the operator $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ is a linear closed operator defined on a dense set $D(A)$ of the Banach space $\mathbb{X}$, $y^0 \in D(A)$ and $f \in L^2((0,T);\mathbb{X})$.

There have been many papers on such an inhomogeneous fractional differential equation.

For instance, considering the following fractional differential equation:

$$\begin{cases} D_0^\alpha y(t) = Ay(t) + f(t), & t \in [0,T] \\ y(0) = y^0. \end{cases} \tag{12}$$

Baeumer et al. [23] developed an analytical formula for the solution when $f = I_0^{1-\alpha} r$, $f(0) = 0$, r being a given function, assuming that A is the generator of C_0-semigroup. Then they proved the existence and uniqueness of the solution considering an equivalent equation of convolution type. Assuming that f satisfies a uniform Hölder condition, El-Boraï in [24] proved the existence and uniqueness of the solution of (12) when A is the generator of an analytic semigroup. We also refer to [25–29] etc. for more literature on fractional differential equations.

In this paper, by a Laplace transform of vector-valued functions, we prove the existence and uniqueness of the solution to (11) assuming that A is the generator of a uniformly bounded C_0-semigroup $(Q(t))_{t\geq 0}$.

From now on we assume that A is the generator of a uniformly bounded C_0-semigroup $(Q(t))_{t\geq 0}$. That is there exists $K > 0$ such that

$$\sup_{t\geq 0} \|Q(t)\|_{B(\mathbb{X})} \leq K, \tag{13}$$

where $(B(\mathbb{X}), \|.\|_{B(\mathbb{X})})$ is the Banach space of all linear bounded operators on $\mathbb{X}$.

Remark 3.1. Note that if $(A, D(A)) = (\Delta, H^2(\Omega) \cap H_0^1(\Omega))$ where Δ is the Laplacian operator then A is the generator of a contraction semigroup [30]. So (13) is satisfied.

Theorem 3.2. *Let $1/2 < \alpha < 1$. Assume that $f \in L^2((0, T); \mathbb{X})$ and A is the generator of a C_0-semigroup $(Q(t))_{t\geq 0}$ on a Banach space $\mathbb{X}$ satisfying (13). Then for any $y^0 \in D(A)$, problem (11) has a unique solution $y \in L^2((0, T); \mathbb{X})$ given by:*

$$y(t) = \mathbb{P}_\alpha y^0 + \int_0^t \mathbb{P}_\alpha(t-\tau) f(\tau)\mathrm{d}\tau$$

where

$$\mathbb{P}_\alpha(t) = \alpha \int_0^\infty \theta t^{\alpha-1} \Phi_\alpha(\theta) Q(t^\alpha \theta)\mathrm{d}\theta \tag{14}$$

with Φ_α defined as in (10). Moreover

$$\|y\|_{L^2((0,T);\mathbb{X})} \leq \frac{K}{\Gamma(\alpha)} \left(\sqrt{\frac{2T^{2\alpha-1}}{(2\alpha-1)}} \|y^0\|_{\mathbb{X}} + \sqrt{\frac{2T^{3\alpha}}{\alpha^3}} \|f\|_{L^2((0,T);\mathbb{X})} \right). \tag{15}$$

Proof. Using the fact that the Laplace transform of $D_+^\alpha y$ is given by

$$\widehat{(D_+^\alpha y)}(\lambda) = \lambda^\alpha \widehat{y}(\lambda) - I_+^{1-\alpha} y(0^+)$$

we deduce that the Laplace transform of the solution of (11)

$$\widehat{y}(\lambda) = (\lambda^\alpha I - A)^{-1} y^0 + (\lambda^\alpha I - A)^{-1} \widehat{f}(\lambda). \tag{16}$$

Observing that for any $h \in \mathbb{X}$, $(\lambda I - A)^{-1} h = \int_0^\infty \mathrm{e}^{-\lambda s} Q(s) h \mathrm{d}s$, and using (10) we get

$$\begin{aligned}
(\lambda^\alpha I - A)^{-1} h &= \int_0^\infty \mathrm{e}^{(-\lambda^\alpha s)} Q(s) h \mathrm{d}s \\
&= \int_0^\infty \int_0^\infty \mathrm{e}^{-\lambda s^{1/\alpha} u} \rho_\alpha(u) Q(s) h \mathrm{d}u \mathrm{d}s \\
&= \int_0^\infty \mathrm{e}^{-\lambda t} \left(\int_0^\infty s^{-1/\alpha} \rho_\alpha(t s^{-1/\alpha}) Q(s) h \mathrm{d}s \right) \mathrm{d}t
\end{aligned}$$

where ρ_α represents the one side stable probability density whose Laplace transform is given by [24]:

$$\int_0^\infty \mathrm{e}^{-\lambda \tau} \rho_\alpha(\tau)\mathrm{d}\tau = \mathrm{e}^{(-\lambda^\alpha)} \tag{17}$$

and which satisfies

$$\alpha \Phi_\alpha(\theta) = \theta^{-1-1/\alpha} \rho_\alpha(\theta^{-1/\alpha}). \tag{18}$$

Therefore using (18), we obtain

$$\begin{aligned}
(\lambda^\alpha I - A)^{-1} h &= \int_0^\infty \mathrm{e}^{-\lambda t} \left(\int_0^\infty s^{-1/\alpha} \rho_\alpha(t s^{-1/\alpha}) Q(s) h \mathrm{d}s \right) \mathrm{d}t \\
&= \int_0^\infty \mathrm{e}^{-\lambda t} \left(\int_0^\infty \alpha s t^{\alpha-1} \Phi_\alpha(t^{-\alpha} s) Q(s) h \mathrm{d}s \right) \mathrm{d}t \\
&= \int_0^\infty \mathrm{e}^{-\lambda t} \left(\int_0^\infty \alpha \theta t^{\alpha-1} \Phi_\alpha(\theta) Q(t^\alpha \theta) h \mathrm{d}\theta \right) \mathrm{d}t.
\end{aligned}$$

Hence we deduce that

$$(\lambda^\alpha I - A)^{-1} h = \widehat{\mathbb{P}}(\lambda)h \tag{19}$$

with

$$\mathbb{P}_\alpha(t) = \int_0^\infty \alpha t^{\alpha-1}\theta\Phi_\alpha(\theta)Q(t^\alpha\theta)\mathrm{d}\theta.$$

It follows from (16) that

$$y(t) = \mathbb{P}_\alpha(t)y^0 + \int_0^t \mathbb{P}_\alpha(t-s)f(s)\mathrm{d}s.$$

Now we prove that (15) holds. Using (13) we have

$$\begin{aligned}\|y(t)\|_{\mathbb{X}} &\le \int_0^\infty \alpha t^{\alpha-1}\theta\Phi_\alpha(\theta)\|Q(t^\alpha\theta)y^0\|_{\mathbb{X}}\mathrm{d}\theta + \int_0^t\int_0^\infty \alpha(t-s)^{\alpha-1}\theta\Phi_\alpha(\theta)\|Q((t-s)^\alpha\theta)f(s)\|_{\mathbb{X}}\mathrm{d}\theta\mathrm{d}s \\ &\le \frac{K}{\Gamma(\alpha)}t^{\alpha-1}\|y^0\|_{\mathbb{X}} + \frac{K}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|f(s)\|_{\mathbb{X}}\mathrm{d}s\end{aligned}$$

since $\int_0^\infty \theta\Phi_\alpha(\theta)\mathrm{d}\theta = \frac{1}{\Gamma(1+\alpha)}$ (see [7] for instance). Consequently using the fact that $(t-s)^{\alpha-1} \in L^1(0,T)$ and $f \in L^2((0,T);\mathbb{X})$, we obtain

$$\begin{aligned}\|y(t)\|_{\mathbb{X}} &\le \frac{K}{\Gamma(\alpha)}t^{\alpha-1}\|y^0\|_{\mathbb{X}} + \frac{K}{\Gamma(\alpha)}\int_0^t \left[(t-s)^{\alpha-1}\right]^{1/2}\left[(t-s)^{\alpha-1}\|f(s)\|_{\mathbb{X}}^2\right]^{1/2}\mathrm{d}s \\ &\le \frac{K}{\Gamma(\alpha)}t^{\alpha-1}\|y^0\|_{\mathbb{X}} + \frac{KT^\alpha}{\Gamma(1+\alpha)}\left(\int_0^t (t-s)^{\alpha-1}\|f(s)\|_{\mathbb{X}}^2\mathrm{d}s\right)^{1/2},\end{aligned}$$

which implies that

$$\|y(t)\|_{\mathbb{X}}^2 \le \frac{2K^2}{(\Gamma(\alpha))^2}t^{2\alpha-2}\|y^0\|_{\mathbb{X}}^2 + \frac{2K^2T^{2\alpha}}{(\Gamma(1+\alpha))^2}\int_0^t (t-s)^{\alpha-1}\|f(s)\|_{\mathbb{X}}^2\mathrm{d}s.$$

Thus

$$\begin{aligned}\int_0^T \|y(t)\|_{\mathbb{X}}^2\mathrm{d}t &\le \frac{2K^2}{(\Gamma(\alpha))^2}\|y^0\|_{\mathbb{X}}^2\int_0^T t^{2\alpha-2}\mathrm{d}t + \frac{2K^2T^{2\alpha}}{(\Gamma(1+\alpha))^2}\int_0^T\int_0^t (t-s)^{\alpha-1}\|f(s)\|_{\mathbb{X}}^2\mathrm{d}s\mathrm{d}t \\ &\le \frac{2K^2T^{2\alpha-1}}{(2\alpha-1)(\Gamma(\alpha))^2}\|y^0\|_{\mathbb{X}}^2 + \frac{2K^2T^{2\alpha}}{(\Gamma(1+\alpha))^2}\int_0^T \|f(s)\|_{\mathbb{X}}^2\int_s^T (t-s)^{\alpha-1}\mathrm{d}t\mathrm{d}s \\ &\le \frac{2K^2T^{2\alpha-1}}{(2\alpha-1)(\Gamma(\alpha))^2}\|y^0\|_{\mathbb{X}}^2 + \frac{2K^2T^{3\alpha}}{\alpha^3(\Gamma(\alpha))^2}\|f\|_{L^2((0,T);\mathbb{X})}^2\end{aligned}$$

and we deduce (15). □

Corollary 3.3. *Let $0 < \alpha < 1$ and $y^0 \equiv 0$. Assume that A is the generator of a C_0-semigroup $(Q(t))_{t\ge0}$ on a Banach space $\mathbb{X}$ satisfying (13). Then problem (11) has a unique solution $y \in L^2((0,T);\mathbb{X})$ given by:*

$$y(t) = \int_0^t \mathbb{P}_\alpha(t-\tau)f(\tau)\mathrm{d}\tau. \tag{20}$$

Moreover

$$\|y\|_{L^2((0,T);\mathbb{X})} \le \frac{K}{\Gamma(\alpha)}\sqrt{\frac{T^{3\alpha}}{\alpha^3}}\|f\|_{L^2((0,T);\mathbb{X})}. \tag{21}$$

Theorem 3.4. *Let $1/2 < \alpha < 1$, $y^0 \in H^2(\Omega)\cap H_0^1(\Omega)$ and $v \in L^2(Q)$. Then (1) has a unique solution in $L^2(Q)$. Moreover*

$$\|y\|_{L^2(Q)} \le \frac{1}{\Gamma(\alpha)}\left(\sqrt{\frac{2T^{2\alpha-1}}{(2\alpha-1)}}\|y^0\|_{L^2(\Omega)} + \sqrt{\frac{2T^{3\alpha}}{\alpha^3}}\|v\|_{L^2(Q)}\right). \tag{22}$$

Proof. We apply Theorem 3.2 with $A = \Delta$, $D(A) = H^2(\Omega)\cap H_0^1(\Omega)$, $\mathbb{X} = L^2(\Omega)$ and $f = v$. Note that (22) is obtained from (15) by taking $K < 1$ since Δ generates a semigroup of contractions. □

In view of Corollary 3.3, we have this other result:

Corollary 3.5. *Let $0 < \alpha < 1$, $y^0 \equiv 0$ and $v \in L^2(Q)$. Then (1) has a unique solution in $L^2(Q)$. Moreover*

$$\|y\|_{L^2(Q)} \leq \frac{1}{\Gamma(\alpha)}\sqrt{\frac{T^{3\alpha}}{\alpha^3}}\|v\|_{L^2(Q)}.$$

Now, consider the following fractional differential equation:

$$\begin{cases} -\mathcal{D}^\alpha p(t) - Ap(t) = g(t), & t \in [0, T] \\ p(T) = 0. \end{cases} \tag{23}$$

where $0 < \alpha < 1$, $g \in L^2((0, T); \mathbb{X})$ and A is the generator of a C_0-semigroup $(Q(t))_{t\geq 0}$ on a Banach space $\mathbb{X}$.

Proposition 3.6. *Assume that $0 < \alpha < 1$, A is the generator of a C_0-semigroup $(Q(t))_{t\geq 0}$ on a Banach space $\mathbb{X}$ satisfying (13) and $g \in L^2((0, T); \mathbb{X})$. Then problem (23) has a unique solution $p \in L^2(Q)$ given by:*

$$p(t) = \int_0^t \mathbb{P}_\alpha(t-\tau)g(\tau)\mathrm{d}\tau \tag{24}$$

where $\mathbb{P}(t)$ is the operator defined by (14). Moreover

$$\|p\|_{L^2((0,T);\mathbb{X})} \leq \frac{K}{\Gamma(\alpha)}\sqrt{\frac{T^{3\alpha}}{\alpha^3}}\|g\|_{L^2((0,T);\mathbb{X})}. \tag{25}$$

Proof. We proceed in two steps.

Step 1. We prove that (23) is a backward fractional diffusion equation defined with a Captuto derivative.

Set

$$\mathcal{T}_T p(t) = p(T-t), \quad t \in [0, T]. \tag{26}$$

Then $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{T}_T p(t) = -p'(T-t) = -\mathcal{T}_T p'(t)$.

Next, making the change of variable $t \to T - t$ in

$$\mathcal{D}^\alpha p(t) = \frac{1}{\Gamma(1-\alpha)}\int_t^T (s-t)^{-\alpha}p'(s)\mathrm{d}s,$$

we have

$$\begin{aligned} \mathcal{D}^\alpha p(T-t) &= \frac{1}{\Gamma(1-\alpha)}\int_{T-t}^T (s-(T-t))^{-\alpha}p'(s)\mathrm{d}s \\ &= \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-u)^{-\alpha}p'(T-u)\mathrm{d}u \end{aligned}$$

which according to the notation (26) can be rewritten as

$$\mathcal{D}^\alpha \mathcal{T}_T p(t) = -\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-u)^{-\alpha}(\mathcal{T}_T p)'(u)\mathrm{d}u.$$

This means that

$$\mathcal{D}^\alpha \mathcal{T}_T p(t) = -D_0^\alpha \mathcal{T}_T p(t).$$

Finally, making the change of variable $t \to T - t$ in (23), we obtain

$$\begin{cases} D_0^\alpha \mathcal{T}_T p(t) - A\mathcal{T}_T p(t) = \mathcal{T}_T g(t), & T - t \in [0, T] \\ p(0) = 0 \end{cases}$$

That is

$$\begin{cases} D_0^\alpha p(\tau) - Ap(\tau) = g(\tau), & \tau \in [0, T] \\ p(0) = 0 \end{cases} \tag{27}$$

Step 2. We show that (24) and (25) hold.

Using the fact that

$$\widehat{(D_0^\alpha p)}(\lambda) = \lambda^\alpha \widehat{p}(\lambda) - \lambda^{\alpha-1}P(0)$$

we deduce that the Laplace transform of the solution of (27)

$$\widehat{p}(\lambda) = (\lambda^\alpha I - A)^{-1}\widehat{g}(\lambda).$$

Therefore using (19), we deduce that

$$p(t) = \int_0^t \mathbb{P}_\alpha(t-\tau)g(\tau)\mathrm{d}\tau.$$

Hence proceeding as for the proof of (22) in Theorem 3.2 we obtain

$$\|p\|_{L^2((0,T);\mathbb{X})} \leq \frac{K}{\Gamma(\alpha)}\sqrt{\frac{T^{3\alpha}}{\alpha^3}}\|g\|_{L^2((0,T);\mathbb{X})}. \quad \square$$

Remark 3.7. The Laplace transform of vector-valued functions plays a key role in the proof of the above proposition as well as in the proof of Theorem 3.2. For more information on the basic theory of the Laplace transform of vector-valued functions and their nice applications to evolution equations, please see, e.g., [31–38].

4. Optimal control

In this section, we want to control system (1). More precisely, we want to approach the state $y(v)$ of (1) by a desired state z_d in controlling v.

Let $v \in L^2(Q)$. Then in view of the results of Section 3 we know that the solution $y = y(v)$ of (1) belongs to $L^2(Q)$. Thus we can define the functional

$$J(v) = \|y(v) - z_d\|^2_{L^2(Q)} + N\|v\|^2_{L^2(Q)} \tag{28}$$

where $z_d \in L^2(Q)$ and $N > 0$. The optimal control problem consists in finding $u \in L^2(Q)$ such that

$$J(u) = \inf_{v \in L^2(Q)} J(v). \tag{29}$$

Proposition 4.1. *Assume that the state of the system is given by (1). Then there exists a unique optimal control u such that (29) holds.*

Proof. Let $v_n \in L^2(Q)$ be a minimizing sequence such that

$$J(v_n) \to \inf_{v \in L^2(Q)} J(v). \tag{30}$$

Then $y_n = y(v_n)$ is a solution of (1). This means that y_n satisfies:

$$D^\alpha_+ y_n - \Delta y_n = v_n \quad \text{in } Q, \tag{31a}$$

$$y_n = 0 \quad \text{on } \Sigma, \tag{31b}$$

$$I^{1-\alpha}_+ y_n(x,0) = y^0 \quad \text{in } \Omega. \tag{31c}$$

Moreover, in view of (30), there exists $C > 0$ independent of n such that

$$\|v_n\|_{L^2(Q)} \leq C,$$
$$\|y_n\|_{L^2(Q)} \leq C$$

and it follows from (31a) that

$$\|D^\alpha_+ y_n - \Delta y_n\|_{L^2(Q)} \leq C. \tag{32}$$

Hence there exists u, y, δ in $L^2(Q)$ and a subsequence extracted from (v_n) and (y_n) (still called (v_n) and (y_n)) such that

$$v_n \rightharpoonup u \quad \text{weakly in } L^2(Q), \tag{33}$$

$$y_n \rightharpoonup y \quad \text{weakly in } L^2(Q), \tag{34}$$

$$D^\alpha_+ y_n - \Delta y_n \rightharpoonup \delta \quad \text{weakly in } L^2(Q). \tag{35}$$

We set

$$\mathbb{D}(Q) = \{\varphi \in \mathcal{C}^\infty(Q) \text{ such that } \varphi|_{\partial\Omega} = 0, \varphi(x,0) = \varphi(x,T) = 0 \text{ in } \Omega\}$$

and we denote by $\mathbb{D}'(Q)$ the dual of $\mathbb{D}(Q)$.

In view of Lemma 2.7, we have

$$\int_0^T\int_\Omega \left(D_+^\alpha y_n(x,t)-\Delta y_n(x,t)\right)\varphi(x,t)\mathrm{d}x\mathrm{d}t=\int_0^T\int_\Omega y_n(x,t)\left(-\mathcal{D}^\alpha\varphi(x,t)-\Delta\varphi(x,t)\right)\mathrm{d}x\mathrm{d}t,\quad \forall\varphi\in\mathbb{D}(Q).$$

Therefore in view of (34), we obtain for $\varphi\in\mathbb{D}(Q)$,

$$\begin{aligned}\lim_{n\to\infty}\int_0^T\int_\Omega \left(D_+^\alpha y_n(x,t)-\Delta y_n(x,t)\right)\varphi(x,t)\mathrm{d}x\mathrm{d}t &=\int_0^T\int_\Omega y(x,t)\left(-\mathcal{D}^\alpha\varphi(x,t)-\Delta\varphi(x,t)\right)\mathrm{d}x\mathrm{d}t\\ &=\int_0^T\int_\Omega\left(D_+^\alpha y(x,t)-\Delta y(x,t)\right)\varphi(x,t)\mathrm{d}x\mathrm{d}t.\end{aligned}$$

This means that

$$D_+^\alpha y_n-\Delta y_n\rightharpoonup D_+^\alpha y-\Delta y\quad\text{weakly in }\mathbb{D}'(Q),$$

and we deduce that

$$D_+^\alpha y-\Delta y=\delta\in L^2(Q).\tag{36}$$

Hence, passing to the limit in (31a) while using (35), (33) and (36), we deduce that

$$D_+^\alpha y-\Delta y=u\quad\text{in }Q.\tag{37}$$

$$D_+^\alpha y_n-\Delta y_n\rightharpoonup D_+^\alpha y-\Delta y\quad\text{weakly in }L^2(Q).\tag{38}$$

If $y\in L^2(Q)$, then in view of Lemma 2.8, $I_+^{1-\alpha}y(x,t)\in L^2(Q)$. Therefore, on the one hand we have $D_+^\alpha y(x,t)=\frac{\mathrm{d}}{\mathrm{d}t}I_+^{1-\alpha}y(x,t)\in H^{-1}((0,T);L^2(\Omega))$ and then, $\Delta y\in H^{-1}((0,T);L^2(\Omega))$ since (36) holds. Thus $y(t)\in L^2(\Omega)$ and $\Delta y(t)\in L^2(\Omega)$. Hence, we deduce that $y|_{\partial\Omega}$ exists and belongs to $H^{-1/2}(\partial\Omega)$ (see [39]).

On the other hand, we have $\Delta y\in L^2((0,T);H^{-2}(\Omega))$ and then, $D_+^\alpha y(x,t)=\frac{\mathrm{d}}{\mathrm{d}t}I_+^{1-\alpha}y(x,t)\in L^2((0,T);H^{-2}(\Omega))$ since (36) holds. Thus $I_+^{1-\alpha}y(x,t)\in L^2(Q)$ and $\frac{\mathrm{d}}{\mathrm{d}t}I_+^{1-\alpha}y(x,t)\in L^2((0,T);H^{-2}(\Omega))$. Consequently $I_+^{1-\alpha}y$ belongs to $C([0,T],H^{-1}(\Omega))$ (see [40]). This means that $I_+^{1-\alpha}y(x,0)$ exist and belongs to $H^{-1}(\Omega)$.

Now, multiplying (31a) by $\varphi\in\mathcal{C}^\infty(\overline{Q})$ with $\varphi|_{\partial\Omega}=0$ and $\varphi(T,x)=0$ on Ω, and integrating by parts over Q, we obtain by using Lemma 2.7,

$$\int_0^T\int_\Omega\left(D_+^\alpha y_n(x,t)-\Delta y_n(x,t)\right)\varphi(x,t)\mathrm{d}x\mathrm{d}t=-\int_\Omega\varphi(x,0)y^0\mathrm{d}x+\int_0^T\int_\Omega y_n(x,t)\left(-\mathcal{D}^\alpha\varphi(x,t)-\Delta\varphi(x,t)\right)\mathrm{d}x\mathrm{d}t.$$

Passing this latter identity to the limit when $n\to\infty$ while using (38) and (34),

$$\int_0^T\int_\Omega\left(D_+^\alpha y(x,t)-\Delta y(x,t)\right)\varphi(x,t)\mathrm{d}x\mathrm{d}t+\int_\Omega\varphi(x,0)y^0\mathrm{d}x=\int_0^T\int_\Omega y(x,t)\left(-\mathcal{D}^\alpha\varphi(x,t)-\Delta\varphi(x,t)\right)\mathrm{d}x\mathrm{d}t.\tag{39}$$

Integrating by parts the right side of (39) while using Lemma 2.6, we obtain

$$\begin{aligned}&\int_0^T\int_\Omega\left(D_+^\alpha y(x,t)-\Delta y(x,t)\right)\varphi(x,t)\mathrm{d}x\mathrm{d}t+\int_\Omega\varphi(x,0)y^0\mathrm{d}x=+\left\langle\varphi(x,0),I_+^{1-\alpha}y(x,0^+)\right\rangle_{H_0^1(\Omega),H^{-1}(\Omega)}\\ &\quad-\int_0^T\left\langle y,\frac{\partial\varphi}{\partial\nu}\right\rangle_{H^{-1/2}(\partial\Omega),H^{1/2}(\partial\Omega)}\mathrm{d}t+\int_0^T\int_\Omega\left(D_+^\alpha y(x,t)-\Delta y(x,t)\right)\varphi(x,t)\mathrm{d}x\mathrm{d}t,\\ &\quad\text{for all }\varphi\in\mathcal{C}^\infty(\overline{Q})\text{ with }\varphi|_{\partial\Omega}=0\text{ and }\varphi(x,T)=0\text{ on }\Omega\end{aligned}\tag{40}$$

where $\langle .,.\rangle_{Y,Y'}$ represents the duality bracket between the spaces Y and Y'.

Hence, (40) yields

$$\begin{aligned}&\int_\Omega\varphi(x,0)y^0\mathrm{d}x=+\left\langle\varphi(x,0),I_+^{1-\alpha}y(x,0^+)\right\rangle_{H_0^1(\Omega),H^{-1}(\Omega)}-\int_0^T\left\langle y,\frac{\partial\varphi}{\partial\nu}\right\rangle_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)}\mathrm{d}t,\\ &\quad\text{for all }\varphi\in\mathcal{C}^\infty(\overline{Q})\text{ with }\varphi|_{\partial\Omega}=0\text{ and }\varphi(x,T)=0\text{ on }\Omega.\end{aligned}$$

There for taking in this latter identity φ such that $\frac{\partial\varphi}{\partial\nu}=0$ on $\partial\Omega$, we obtain

$$I_+^{1-\alpha}y(x,0^+)=y^0(x)\quad\text{in }\Omega\tag{41}$$

and then,

$$y=0\quad\text{on }\partial\Omega.\tag{42}$$

In view of (37), (41) and (42), we deduce that $y = y(u)$ is a solution of (1). From weak lower semi-continuity of the function $v \to J(v)$ we deduce

$$\liminf_{n\to\infty} J(v_n) \geq J(u).$$

Hence according to (30), we deduce that

$$J(u) \leq \inf_{v\in L^2(Q)} J(v)$$

which implies that

$$J(u) = \inf_{v\in L^2(Q)} J(v).$$

The uniqueness of u is straightforward from the strict convexity of J. □

Theorem 4.2. *If u is a solution of (29), then there exist $p \in L^2(Q)$ such that (u, y, p) satisfies the following optimality system:*

$$\begin{cases} D_+^\alpha y - \Delta y = u & \text{in } Q, \\ y = 0, & \text{on } \Sigma, \\ I_+^{1-\alpha} y(x, 0^+) = y^0 & \text{in } \Omega \end{cases} \tag{43}$$

$$\begin{cases} -\mathcal{D}^\alpha p - \Delta p = y - z_d & \text{in } Q, \\ p = 0 & \text{on } \Sigma, \\ p(T) = 0 & \text{in } \Omega \end{cases} \tag{44}$$

$$u = -\frac{p}{N} \quad \text{in } Q. \tag{45}$$

Proof. Relations (37), (41) and (42) give (43). To prove (44) and (45), we express the Euler–Lagrange optimality conditions which characterize the optimal control u:

$$\frac{\mathrm{d}}{\mathrm{d}\mu} J(u + \mu\varphi)|_{\mu=0} = 0, \quad \text{for all } \varphi \in L^2(Q). \tag{46}$$

The state $z(\varphi)$ associated to the control $\varphi \in L^2(Q)$ is a solution of

$$\begin{aligned} & D_+^\alpha z - \Delta z = \varphi \quad \text{in } Q, \\ & z = 0, \quad \text{on } \Sigma, \\ & I_+^{1-\alpha} z(x, 0^+) = 0 \quad \text{in } \Omega. \end{aligned} \tag{47}$$

After calculations, (46) gives

$$\int_0^T \int_\Omega z(y(u) - z_d)\mathrm{d}x\mathrm{d}t + N \int_0^T \int_\Omega u\varphi \mathrm{d}x\mathrm{d}t = 0 \quad \forall \varphi \in L^2(Q). \tag{48}$$

To interpret (48), we consider the adjoint state equation:

$$\begin{aligned} & -\mathcal{D}^\alpha p - \Delta p = y(u) - z_d \quad \text{in } Q, \\ & p = 0 \quad \text{on } \Sigma, \\ & p(T) = 0 \quad \text{in } \Omega. \end{aligned} \tag{49}$$

Since $y(u) - z_d \in L^2(Q)$, applying Proposition 3.6 with $(A, D(A)) = (\Delta, H^2(\Omega) \cap H_0^1(\Omega))$, we deduce that problem (49) has a unique solution in $L^2(Q)$. Thus, multiplying (47) by the p solution of (49), we obtain by using Lemma 2.7,

$$\begin{aligned} \int_0^T \int_\Omega \left(D_+^\alpha z - \Delta z\right) p \mathrm{d}x\mathrm{d}t &= \int_0^T \int_\Omega (-\mathcal{D}^\alpha p - \Delta p)\, z \mathrm{d}x\mathrm{d}t \\ &= \int_0^T \int_\Omega (y(u) - z_d) z \mathrm{d}x\mathrm{d}t. \end{aligned}$$

Hence, in view of (47) and (48), we deduce that

$$\int_0^T \int_\Omega \varphi p \mathrm{d}x\mathrm{d}t = -N \int_0^T \int_\Omega \varphi u \mathrm{d}x\mathrm{d}t \quad \forall \varphi \in L^2(Q).$$

Consequently,

$$u = -\frac{p}{N} \quad \text{in } Q. \quad \square$$

5. Concluding remarks

We have proved that if $1/2 < \alpha < 1$ or $0 < \alpha < 1$ with $y^0 \equiv 0$ then (1) has a unique solution in $L^2(Q)$. Moreover, we show that one can approach the state $y(v)$ of (1) by a desired state z_d by controlling v and compute the control v using the algorithm given by the optimality system. Note that two cases are considered here because we need the L^2 estimate of the solution for control purposes.

Let us also mention that by proceeding as in Section 3, one can easily prove that problem (1) with a Caputo derivative has a unique solution in $L^2(Q)$ and obtain the continuity of the solution with respect to the data. However, it will not be easy to give a sense to the initial condition while establishing the optimality system condition.

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