A common fixed point theorem with applications to vector equilibrium problems

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ABSTRACT

In this paper, using the Brouwer fixed point theorem, we establish a common fixed point theorem for a family of set-valued mappings. As applications of this result we obtain existence theorems for the solutions of two types of vector equilibrium problems, a Ky Fan-type minimax inequality and a generalization of a known result due to Iohvidov.

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1. Introduction and preliminaries

Let $X$ be a nonempty set and $Z$ be a convex subset of a vector space. Recall that a set-valued mapping $T : X \rightrightarrows Z$ is said to be generalized KKM (see [1]) if for any nonempty finite subset $\{x_1, \ldots, x_n\}$ of $X$ there is $\{z_1, \ldots, z_n\} \subseteq Z$ such that $\text{co}\{z_i : i \in I\} \subseteq \bigcup_{i \in I} T(x_i)$, for each nonempty subset $I$ of $\{1, \ldots, n\}$. Inspired by this definition we introduce a new concept as follows:

**Definition 1.** Let $X$ be a nonempty set, $Z$ be a convex subset of a vector space and $\mathcal{T}$ be a family of set-valued mappings with nonempty values from $X$ into $Z$. We say that $\mathcal{T}$ is generalized equi-KKM if for any nonempty finite subset $\{x_1, \ldots, x_n\}$ of $X$ there is $\{z_1, \ldots, z_n\} \subseteq Z$ such that $\text{co}\{z_i : i \in I\} \subseteq \bigcup_{i \in I} T(x_i)$, for each nonempty subset $I$ of $\{1, \ldots, n\}$ and for all $T \in \mathcal{T}$.

**Remark 1.** If $Z$ is a convex subset of a topological vector space and $\mathcal{T}$ is generalized equi-KKM then, according to Lemma 3.3 in [2], for each $T \in \mathcal{T}$, $\bigcap_{T \in \mathcal{T}} T(x)$ has the finite intersection property.

**Example 1.** Let $X = Z = [0, 1]$ and $\mathcal{T} = \{T_y : [0, 1] \rightrightarrows [0, 1]\}_{y \in [0, 1]}$, where

$$T_y(x) = \begin{cases} \left[\frac{xy}{2}, 1\right] & \text{if } x \in \left[0, \frac{1}{2}\right] \\ \left[0, \frac{x+y}{2}\right] & \text{if } x \in \left[\frac{1}{2}, 1\right]. \end{cases}$$
We show that the family $\mathcal{T}$ is generalized equi-KKM. One can easily check that for all $y \in [0, 1]$ we have $[\frac{1}{2}, 1] \subseteq T_y(x)$ for any $x \leq \frac{1}{2}$ and respectively, $[0, \frac{1}{2}] \subseteq T_y(x)$ if $x > \frac{1}{2}$. For any $\{x_1, \ldots, x_n\} \subseteq [0, 1]$, put $z_i = \frac{1-x_i}{2}, 1 \leq i \leq n$. We claim that

$$\text{co}\{z_i : i \in I\} \subseteq \bigcup_{i \in I} T_y(x_i),$$

for each nonempty subset $I$ of $\{1, \ldots, n\}$ and for all $y \in [0, 1]$. Suppose $x_{i_1} = \min_{i \in I} x_i$ and $x_{i_2} = \max_{i \in I} x_i$. If $x_{i_2} \leq \frac{1}{2}$, then $\text{co}\{z_i : i \in I\} \subseteq [\frac{1}{2}, 1] \subseteq \bigcup_{i \in I} T_y(x_i)$. If $x_{i_1} > \frac{1}{2}$, then $\text{co}\{z_i : i \in I\} \subseteq [0, \frac{1}{2}] \subseteq \bigcup_{i \in I} T_y(x_i)$. If $x_{i_1} \leq \frac{1}{2} < x_{i_2}$, then

$$\text{co}\{z_i : i \in I\} \subseteq [0, 1] = T_y(x_{i_1}) \cup T_y(x_{i_2}) = \bigcup_{i \in I} T_y(x_i).$$

If $X$ and $Y$ are topological spaces a set-valued mapping $T : X \to Y$ is said to be: (i) upper semicontinuous (in short, u.s.c.) (respectively, lower semicontinuous (in short, l.s.c.)) if for every closed subset $B$ of $Y$ the set $\{x \in X : T(x) \cap B \neq \emptyset\}$ (respectively, $\{x \in X : T(x) \subseteq B\}$) is closed; (ii) closed if its graph is a closed subset of $X \times Y$; (iii) compact if $T(X)$ is a compact subset of $Y$.

The following lemma collects known facts about u.s.c. or l.s.c. set-valued mappings (see for instance [3] for assertion (i) and [4] for assertion (ii)).

**Lemma 1.** Let $X$ and $Y$ be topological spaces and $T : X \to Y$ be a set-valued mapping.

(i) If $Y$ is regular and $T$ is u.s.c. with closed values, then $T$ is closed.

(ii) $T$ is l.s.c. if and only if for any $x \in X, y \in T(x)$ and any net $\{x_i\}$ converging to $x$, there exists a net $\{y_i\}$ converging to $y$, with $y_i \in T(x_i)$ for each $i$.

**Definition 2 (of [5]).** Let $X$ and $Y$ be two nonempty convex subsets of two vector spaces and $V$ be a vector space. Let $F : X \times Y \to V$ and $C : X \to V$ be two set-valued mappings such that for each $x \in X$, $C(x)$ is a convex cone. We say that:

(i) $F$ is $C(x)$-quasiconvex if for all $x \in X, y_1, y_2 \in Y$ and $y \in \text{co}\{y_1, y_2\}$, we have either $F(x, y_1) \subseteq F(x, y) + C(x)$, or $F(x, y_2) \subseteq F(x, y) - C(x)$.

(ii) $F$ is $C(x)$-quasiconvex-like if for any $x \in X, y_1, y_2 \in Y$ and $y \in \text{co}\{y_1, y_2\}$, we have either $F(x, y) \subseteq F(x, y_1) - C(x)$, or $F(x, y) \subseteq F(x, y_2) - C(x)$.

It is worth mentioning that the concepts introduced above are special cases of many recent general and relaxed notions (see e.g. [6, Def. 2.5], [7, p. 1271], [8, p. 58] and [9, Def. 4.1]). By induction one can prove the following lemma (see [5] for assertion (i), respectively [10] for assertion (ii)).

**Lemma 2.** Let $X$ and $Y$ be two nonempty convex subsets of two vector spaces and $V$ be a vector space. Let $F : X \times Y \to V$ and $C : X \to V$ be two set-valued mappings such that for each $x \in X$, $C(x)$ is a convex cone.

(i) $F$ is $C(x)$-quasiconvex if and only if for any $x \in X, y_i \in Y, 1 \leq i \leq n$, $y \in \text{co}\{y_1 : 1 \leq i \leq n\}$ there exists $1 \leq j \leq n$ such that $F(x, y_j) \subseteq F(x, y) + C(x)$.

(ii) $F$ is $C(x)$-quasiconvex-like if and only if for any $x \in X, y_i \in Y, 1 \leq i \leq n$, $y \in \text{co}\{y_1 : 1 \leq i \leq n\}$ there exists $1 \leq j \leq n$ such that $F(x, y) \subseteq F(x, y_j) - C(x)$.

2. A common fixed point theorem

**Theorem 1.** Let $X$ be a nonempty convex subset of a topological vector space, $Y$ be a nonempty set and $T : X \times Y \to X$ a compact set-valued mapping satisfying the following conditions:

(i) for each $y \in Y$, the set $\{x \in X : x \in T(x, y)\}$ is closed;

(ii) the family of set-valued mappings $\{T(\cdot, y)\}_{y \in Y}$ is generalized equi-KKM on $Y$.

Then the family of set-valued mappings $\{T(\cdot, y)\}_{y \in Y}$ has a common fixed point, that is, there exists $x_0 \in X$ such that $x_0 \in \bigcap_{y \in Y} T(x_0, y)$.

**Proof.** For each $y \in Y$, put $G(y) = \{x \in X : x \notin T(x, y)\}$. Suppose that the conclusion is not true. Then $X = \bigcup_{y \in Y} G(y)$. Since the sets $G(y)$ are all open and $\overline{T(X \times Y)}$ is compact there exists a finite set $\{y_1, \ldots, y_n\} \subseteq Y$ such that $\overline{T(X \times Y)} \subseteq \bigcup_{i=1}^n G(y_i)$. Moreover, for each $x \in X \setminus \overline{T(X \times Y)}$ and $i \in \{1, \ldots, n\}$, $x \in G(y_i)$, hence $X = \bigcup_{i=1}^n G(y_i)$. By (ii), there exists $\{z_1, \ldots, z_n\} \subseteq X$ such that $\text{co}\{z_i : i \in I\} \subseteq \bigcup_{i \in I} T(x, y_i)$, for each nonempty subset $I$ of $\{1, \ldots, n\}$ and for all $x \in X$. Set $K = \text{co}\{z_1, \ldots, z_n\}$. Then $G(y) \cap K_{1 \leq i \leq n}$ is an open cover of $K$. Consider a partition of unity on $K$, $\{\alpha_1, \ldots, \alpha_n\}$, subordinated to this open cover. Recall that this means that

$$\alpha_i : K \to [0, 1] \text{ is continuous, for each } i \in \{1, \ldots, n\};$$

$$\alpha_i(x) > 0 \Rightarrow x \in G(y_i) \cap K;$$

$$\sum_{i=1}^n \alpha_i(x) = 1 \text{ for each } x \in K.$$
Define the function \( p : K \rightarrow K \) by \( p(x) = \sum_{i=1}^{m} \alpha_i(x)z_i \). Since \( p \) is continuous function, by the Brouwer fixed point theorem there exists \( x_0 \in K \) such that \( x_0 = p(x_0) \). Let \( I = \{ i \in \{ 1, \ldots, n \} : \alpha_i(x_0) > 0 \} \). Then \( x_0 = p(x_0) \in \text{co}(\{z_i : i \in I\}) \subseteq \bigcup_{i \in I} T(x_0, y_i) \).

On the other hand, for each \( i \in I \), since \( x_0 \in G(y_i), x_0 \notin T(x_0, y_i) \), hence \( x_0 \notin \bigcup_{i \in I} T(x_0, y_i) \). The obtained contradiction completes the proof. \( \square \)

3. Applications

Let \( X \) be a nonempty subset of a topological vector space and \( f : X \times X \rightarrow \mathbb{R} \) be a function with \( f(x, x) \geq 0 \) for all \( x \in X \). Then the scalar equilibrium problem, in the sense of Blum and Oettli \([11]\), is to find \( x_0 \in X \) such that \( f(x_0, y) \geq 0 \) for all \( y \in X \). In the last years the scalar equilibrium problem was extensively generalized in several ways to vector equilibrium for set-valued mappings. In this paper we fix our attention on two types of vector equilibrium problems described below:

Let \( X \) be a nonempty compact convex subset of a topological vector space, \( Y \) be a nonempty set and \( V \) be a topological vector space. Let \( F : X \times Y \rightharpoonup V, G : X \times X \rightharpoonup V \) and \( C : X \rightharpoonup V \) be three set-valued mappings. Suppose that for each \( x \in X, C(x) \) is a nonempty convex set. Moreover, in case of problem (1), suppose that \( \text{int} C(x) \neq \emptyset \), for all \( x \in X \). We are interested in finding a \( x_0 \in X \) such that:

\[
F(x_0, y) \nsubseteq -\text{int} C(x_0) \quad \text{for all} \quad y \in Y,
\]

respectively

\[
F(x_0, y) \subseteq C(x_0) \quad \text{for all} \quad y \in Y.
\]

These problems, or more general equilibrium problems, are studied in many papers (see, for instance, \([5, 7, 8, 12-17]\)) when either \( X = Y \) or \( X \) and \( Y \) are distinct convex sets, each in a topological vector space. In this section existence theorems for the solutions of these problems will be obtained when \( Y \) is an arbitrary nonempty set without any algebraic or topological structure.

**Theorem 2.** Suppose that the set-valued mappings \( F, G \) and \( C \) satisfy the following conditions:

(i) for each \( x \in X, G(x, x) \nsubseteq -\text{int} C(x) \);
(ii) for any \( y \in Y \) there exists \( z \in X \) such that \( G(x, z) \subseteq F(x, y) \), for all \( x \in X \);
(iii) \( G \) is l.s.c. on \( \Delta X = \{(x, x) : x \in X \} \) and for each \( y \in Y \) the set-valued mapping \( x \rightarrow F(x, y) - C(x) \) is closed;
(iv) \( G \) is \( (x) \)-quasiconvex-like.

Then there exists \( x_0 \in X \) such that \( F(x_0, y) \nsubseteq -\text{int} C(x_0) \) for all \( y \in Y \).

**Proof.** Define \( T : X \times Y \rightharpoonup X \) by

\[
T(x, y) = \{ z \in X : G(x, z) \subseteq F(x, y) - C(x) \}.
\]

For an arbitrary \( y \in Y \) denote

\[
M = \{ x \in X : T(x, y) \} = \{ x \in X : G(x, x) \subseteq F(x, y) - C(x) \}.
\]

Let \( x \in \overline{M} \) and \( \{x_1\} \) a net in \( M \) converging to \( x \). Since \( G \) is l.s.c. on \( \Delta X \), for each \( v \in G(x, x) \) there exists a net \( \{v_i\} \) such that \( v_1 \rightarrow v \) and \( v_i \in G(x, x_i) \) for all \( i \). Since \( x_i \in M \), we have \( v_i \in F(x_i, y) - C(x_i) \). Since the mapping \( x \rightarrow F(x, y) - C(x) \) is closed, it follows that \( v \in F(x, y) - C(x) \). Thus, \( x \in M \), hence \( M \) is closed.

Let \( \{y_1, \ldots, y_n\} \) be a finite subset of \( Y \). By (ii), there exists \( \{z_1, \ldots, z_n\} \subseteq X \) such that \( G(x, z_i) \subseteq F(x, y_i) \) for each \( i \in \{1, \ldots, n\} \) and all \( x \in X \). Let \( I \subseteq \{1, \ldots, n\} \) and \( z = \text{co}\{z_i : i \in I\} \). By (iv), for each \( x \in X \) there exists \( i \in I \) such that \( G(x, z) \subseteq G(x, z_i) - C(x) \subseteq F(x, y_i) - C(x) \). Thus, \( \text{co}\{z_i : i \in I\} \subseteq \bigcup_{i \in I} T(x, y_i) \), for all \( x \in X \).

By **Theorem 1**, there exists \( x_0 \in X \) such that \( x_0 \in \bigcap_{y \in Y} T(x_0, y) \). If for some \( y \in Y \) we would have \( F(x_0, y) \nsubseteq -\text{int} C(x_0) \) then, since \( x_0 \in T(x_0, y) \), we would obtain

\[
G(x_0, x_0) \subseteq F(x_0, y) - C(x_0) \nsubseteq -\text{int} C(x_0) - C(x_0) = -\text{int} C(x_0),
\]

which contradicts (i). Thus, \( F(x_0, y) \nsubseteq -\text{int} C(x_0) \), for all \( y \in Y \). \( \square \)

**Remark 2.** If we take into account Lemma 3.2 in \([18]\), the set-valued mapping \( x \rightarrow F(x, y) - C(x) \) is closed whenever \( F(\cdot, y) \) is u.s.c. with nonempty compact values and \( C \) is closed.

**Theorem 3.** Suppose that the set-valued mappings \( F, G \) and \( C \) satisfy the following conditions:

(i) for each \( x \in X, G(x, x) \subseteq C(x) \);
(ii) for any \( y \in Y \) there exists \( z \in X \) such that \( F(x, y) \subseteq G(x, z) \), for all \( x \in X \);
(iii) the set-valued mapping \( x \rightarrow G(x, x) + C(x) \) is closed and for each \( y \in Y, F(\cdot, y) \) is l.s.c.;
(iv) \( G \) is \( (x) \)-quasiconvex-like.

Then there exists \( x_0 \in X \) such that \( F(x_0, y) \subseteq C(x_0) \) for all \( y \in Y \).
Proof. Define \( T : X \times Y \to X \) by

\[
T(x, y) = \{ z \in X : F(x, y) \subseteq G(x, z) + C(x) \}.
\]

We prove that \( T \) satisfies the requirements of Theorem 1. For an arbitrary \( y \in Y \) denote

\[
M = \{ x \in X : x \in T(x, y) \} = \{ x \in X : F(x, y) \subseteq G(x, x) + C(x) \}.
\]

Let \( x \in \overline{M} \) and \( \{x_i\} \) a net in \( M \) converging to \( x \). Since \( F(., y) \) is l.s.c., for any \( v \in F(x, y) \) there exists a net \( \{v_i\} \) such that \( v_i \to v \) and \( v_i \in F(x_i, y) \) for all \( i \). Since \( x_i \in M \), we have \( v_i \in G(x_i, x_i) + C(x_i) \). Since the mapping \( x \to G(x, x) + C(x) \) is closed, it follows that \( v \in G(x, x) + C(x) \). Thus, \( x \in M \), hence \( M \) is closed.

Let \( \{y_1, \ldots, y_n\} \) be a finite subset of \( Y \). By (ii), there exists \( \{z_1, \ldots, z_n\} \subseteq X \) such that \( F(x, y_i) \subseteq G(x, z_i) \) for each \( i \in \{1, \ldots, n\} \) and all \( x \in X \). Let \( I \subseteq \{1, \ldots, n\} \) and \( z = \text{co} \{z_i : i \in I\} \). By (iv), for each \( x \in X \) there exists \( i_x \in I \) such that \( G(x, z_{i_x}) \subseteq G(x, x) + C(x) \). Hence, \( F(x, y_i) \subseteq G(x, z_{i_x}) \subseteq G(x, z) + C(x) \). This implies \( z \in T(x, y_i) \subseteq \bigcup_{i \in I} T(x, y_i) \), hence \( \text{co} \{z_i : i \in I\} \subseteq \bigcup_{i \in I} T(x, y_i) \).

By Theorem 1, there exists \( x_0 \in X \) such that \( x_0 \in \bigcap_{y \in Y} T(x_0, y) \). Then, for each \( y \in Y \) we have

\[
F(x_0, y) \subseteq G(x_0, x_0) + C(x_0) \subseteq G(x_0) + C(x_0) = C(x_0).
\]

Theorems 2 and 3 are different from other close results from [5,7,12,14,15,17] by conditions (ii) and (iii), proof techniques and, especially, by the fact that \( Y \) is a set without any algebraic or topological structure. The applicability of Theorem 3 is put into evidence by the following example.

Example 2. Let \( Y \) be a nonempty subset of the interval \([1, \infty)\), \( X = [0, 1] \), \( V = \mathbb{R} \),

\[
G(x, z) = \begin{cases} (\infty, z - x) & \text{if } z \leq 1, \\ (\infty, -x - z) & \text{if } z > 1, \end{cases}
\]

\[
F(x, y) = (-\infty, 1 - xy) \quad \text{and} \quad C(x) = (-\infty, 0].
\]

Condition (ii) in Theorem 3 is fulfilled since, for each \( y \in Y \), \( F(x, y) \subseteq G(x, 1) \), for all \( x \in [0, 1] \). One can readily verify that all the other requirements of the same theorem are satisfied. By direct checking one can see that \( x_0 = 1 \) satisfy the conclusion of Theorem 3. Since the set \( Y \) is not necessarily convex any other known result is not applicable.

Theorem 4. Let \( X \) be a nonempty compact convex subset of a topological vector space and \( Y \) be a nonempty set. Let \( f : X \times Y \to \mathbb{R} \), \( g : X \times X \to \mathbb{R} \) be two functions and \( a \in \mathbb{R} \). Suppose that:

(i) \( g(x, x) \leq a \) for all \( x \in X \);
(ii) for each \( y \in Y \) there is \( z \in X \) such that \( f(x, y) \leq g(x, z) \) for all \( x \in X \);
(iii) \( f \) is l.s.c. in the first variable and \( g \) is u.s.c. on \( A_X \);
(iv) \( g \) is quasiconcave in the second variable.

Then there exists \( x_0 \in X \) such that \( f(x_0, y) \leq a \) for all \( y \in Y \).

Proof. Take in the previous theorem \( V = \mathbb{R} \),

\[
F(x, y) = (-\infty, f(x, y) - a], \quad G(x, z) = (-\infty, g(x, z) - a], \quad C(x) = (-\infty, 0].
\]

It can be readily shown that if \( h : X \to \mathbb{R} \) is a u.s.c. (respectively, l.s.c.) function, then the set-valued mapping \( H : X \to \mathbb{R} \), defined by \( H(x) = (\infty, h(x)) \), is u.s.c. (respectively, l.s.c.). Consequently, under condition (iii), \( G|_{A_X} \) is u.s.c. and for each \( y \in Y \), \( F(., y) \) is l.s.c.. Moreover, since a t.v.s. is regular, by Lemma 1(i), the set-valued mapping \( x \to G(x, x) = G(x, x) + C(x) \) is closed. Thus, condition (iii) in Theorem 3 holds. We check condition (iv) from the same theorem. Let \( x \in X \), \( z_1, z_2 \in X \) and \( z \in \text{co} \{z_1, z_2\} \). Since \( g \) is quasiconcave in the second variable, for each \( x \in X \), \( g(x, z) - a \geq \min \{g(x, z_1) - a, g(x, z_2) - a\} \). Thus, for each \( x \in X \) there is an index \( i \in \{1, 2\} \) such that \( G(x, z_1) \subseteq G(x, z) = G(x, z) + C(x) \).

It is easy to verify that all the other conditions of Theorem 3 are satisfied and the desired conclusion follows from this theorem. \( \square \)

By Theorem 4 we derive the following Ky Fan-type minimax inequality:

Theorem 5. If conditions (ii), (iii) and (iv) in Theorem 4 hold, then

\[
\inf_{x \in X} \sup_{y \in Y} f(x, y) \leq \max_{x \in X} g(x, x).
\]

Proof. Take in the previous theorem \( a = \max_{x \in X} g(x, x) \). \( \square \)

Definition 3. Let \( X \) be a convex set in a vector space and \( V \) be a vector space. A function \( f : X \to V \) is said to be almost affine if for each \( x_1, x_2 \in X \) and \( \mu \in [0, 1] \) there exists \( \lambda \in [0, 1] \) such that \( f(\mu x_1 + (1 - \mu)x_2) = \lambda f(x_1) + (1 - \lambda)f(x_2) \).
Remark 3. (a) If \( I \) is a real interval any monotone function \( f : I \to \mathbb{R} \) is almost affine.
(b) If \( f : X \to V \) is almost affine, then one can easily prove that for any \( x_1, \ldots, x_n \in X \) and \( x \in \text{co}\{x_1, \ldots, x_n\} \) there exist \( \lambda_1, \ldots, \lambda_n \geq 0 \) with \( \sum_{i=1}^{n} \lambda_i = 1 \) such that \( f(x) = \sum_{i=1}^{n} \lambda_i f(x_i) \).

As an application of Theorem 1, we give a slight generalization of a known result, due to Iohvidov [19].

**Theorem 6.** Let \( X \) be a nonempty compact convex subset of locally convex Hausdorff topological vector space \( E \) and \( g : X \times X \to E \) be a continuous function. Suppose that:

(i) for each \( x \in X \), \( g(x, \cdot) \) is almost affine;
(ii) for each \( x \in X \) there exists \( y \in X \) such that \( g(x, y) = 0 \).

Then there exists \( x_0 \in X \) such that \( g(x_0, x_0) = 0 \).

**Proof.** Let \( \mathcal{P} \) be a sufficient family of continuous seminorms on \( E \) generating the topology of \( E \). Denote
\[
F_p = \{ x \in X : p(g(x, x)) = 0 \}, \quad p \in \mathcal{P}.
\]
Since the family \( \mathcal{P} \) is sufficient, a point \( x_0 \in X \) satisfies the conclusion of the theorem if \( x_0 \in \bigcap_{p \in \mathcal{P}} F_p \). Since \( X \) is compact and \( F_p \) are closed sets, it suffices to prove that for any nonzero finite subset \( \{ p_1, \ldots, p_k \} \) of \( \mathcal{P} \), \( \bigcap_{j=1}^{k} F_{p_j} \neq \emptyset \). Define the set-valued mapping \( T : X \times X \to X \) by
\[
T(x, y) = \left\{ z \in X : \sum_{j=1}^{k} p_j(g(x, z)) \leq \sum_{j=1}^{k} p_j(g(x, y)) \right\}.
\]
It is clear that for each \( y \in X \), the set \( \{ x \in X : x \in T(x, y) \} \) is closed. We show that for each \( x \in X \), \( T(x, \cdot) \) is a KKM mapping. Let \( \{ y_1, \ldots, y_n \} \subseteq X, I \subseteq \{ 1, \ldots, n \} \) and \( y \in \text{co}\{ y_i : i \in I \} \). Then, by Remark 3(b), there exists \( \lambda_i \geq 0 \) (depending on \( x \)), with \( \sum_{i \in I} \lambda_i = 1 \), such that \( g(x, y) = \sum_{i \in I} \lambda_i g(x, y_i) \). We have
\[
\sum_{j=1}^{k} p_j(g(x, y)) = \sum_{j=1}^{k} p_j \left( \sum_{i \in I} \lambda_i g(x, y_i) \right) \leq \sum_{j=1}^{k} \lambda_i p_j(g(x, y_i)) = \sum_{i \in I} \lambda_i \left( \sum_{j=1}^{k} p_j(g(x, y_i)) \right) \leq \max_{i \in I} \sum_{j=1}^{k} p_j(g(x, y_i)).
\]
Hence \( \text{co}\{ y_i : i \in I \} \subseteq \bigcup_{i \in I} T(x, y_i) \). By Theorem 1 there exists \( x_0 \in X \) such that \( x_0 \in T(x_0, y) \), for all \( y \in X \). By (ii), there is \( y_0 \in X \) such that \( g(x_0, y_0) = 0 \). Since \( x_0 \in T(x_0, y_0) \), \( \sum_{j=1}^{k} p_j(g(x_0, x_0)) \leq 0 \), \( x_0 \in \bigcap_{j=1}^{k} F_{p_j} \). \( \square \)

**Remark 4.** If \( f : X \to X \) is a continuous function, taking \( g(x, y) = f(x) - y \), hence Theorem 6 reduces to the well-known Tychonoff fixed point theorem.

**References**