Abstract

In this paper, we will show how to kill the obstructions to Lie algebra deformations via a method which essentially embeds a Lie algebra into a strong homotopy Lie algebra or $L_\infty$ algebra such that the sh-Lie algebra is the same homotopy type as the original Lie algebra.

MSC: 22E60

1. Introduction

In the last two decades, deformations of various types of structures have assumed an ever increasing role in mathematics and physics. For each such deformation problem, a goal is to determine whether all related deformation obstructions vanish; many beautiful techniques have been developed to determine when this is so. Sometimes genuine deformation obstructions arise and occasionally that closes mathematical development in such cases, but in physics such problems are dealt with by introducing new auxiliary fields to kill such obstructions. This idea suggests that one might deal with deformation problems by enlarging the relevant category to a new category obtained by appending additional algebraic structures to the old category. In this paper we focus on the problem of deforming Lie algebras. It turns out that the relevant enlarged category in which obstructions to deformations of Lie algebras are removed is the category of sh-Lie algebras. Many physicists were introduced to sh-Lie algebras by Lada and Stasheff [5]. Our work here was more directly influenced by [6] and [7]. For completeness we review basic facts on Lie algebra deformations; more detail may be found in the book edited by Hazewinkel and Gerstenhaber [4].

2. Deformation theory, sh-Lie algebras

First of all, we need to recall standard deformation theory [3]. Let $A$ be a $k$-algebra and $\alpha$ be its multiplication, i.e., $\alpha$ is a $k$-bilinear map $A \times A \rightarrow A$ defined by $\alpha(a, b) = ab$. A deformation of $A$ may be defined to be a formal power series $\alpha_t = \alpha + t\alpha_1 + t^2\alpha_2 + \cdots$ where each $\alpha_i : A \times A \rightarrow A$ is a $k$-bilinear map and the “multiplication” $\alpha_t$ is formally of the same “kind” as $\alpha$, e.g., it is associative or Lie or whatever is required. One technique used to set up a deformation problem is to extend a $k$-bilinear mapping $\alpha_t : A \times A \rightarrow A[[t]]$ to a $k[[t]]$-bilinear mapping.
α : A[[t]] × A[[t]] → A[[t]]. A mapping α : A[[t]] × A[[t]] → A[[t]] obtained in this manner is necessarily uniquely determined by its values on A × A. In fact we would not regard the mapping α : A[[t]] × A[[t]] → A[[t]] as being a deformation of A unless it is determined by its values on A × A.

From this point on, we assume that (A, α) is a Lie algebra, i.e., we assume that α(α(α(a, b), c) + α(α(b, c), a) + α(α(c, a), b)) = 0. The problem of deforming a Lie algebra A is equivalent to the problem of finding a mapping α : A × A → A[[t]] such that α(α(a, b), c) + α(α(b, c), a) + α(α(c, a), b) = 0. If we set α0 = α and expand this Jacobi identity by making the substitution αt = α + tα1 + t2α2 + · · · , we get the equation

\[ \sum_{i, j=0}^{\infty} [\alpha_j(\alpha_i(a, b), c) + \alpha_j(\alpha_i(b, c), a) + \alpha_j(\alpha_i(c, a), b)]t^{i+j} = 0 \]  

and consequently a sequence of deformation equations:

\[ \sum_{i, j \geq 0, i+j=n} [\alpha_j(\alpha_i(a, b), c) + \alpha_j(\alpha_i(b, c), a) + \alpha_j(\alpha_i(c, a), b)] = 0. \]

The first two equations are

\[ \alpha_0(\alpha_0(a, b), c) + \alpha_0(\alpha_0(b, c), a) + \alpha_0(\alpha_0(c, a), b) = 0 \]  

\[ \alpha_0(\alpha_1(a, b), c) + \alpha_0(\alpha_1(b, c), a) + \alpha_0(\alpha_1(c, a), b) + \alpha_1(\alpha_0(a, b), c) + \alpha_1(\alpha_0(b, c), a) + \alpha_1(\alpha_0(c, a), b) = 0. \]

We can reformulate the discussion above in a slightly more compact form. Given a sequence \( \alpha_n : A \times A \to A \) of bilinear maps, we define “compositions” of various of the \( \alpha_n \) as follows:

\[ \alpha_n \alpha_i : A \times A \times A \to A \]

is defined by

\[ (\alpha_n \alpha_i)(x_1, x_2, x_3) = \sum_{\sigma \in unsh(2,1)} (-1)^\sigma \alpha_i(\alpha_j(x_{\sigma(1)}, x_{\sigma(2)}), x_{\sigma(3)}) \]

for arbitrary \( x_1, x_2, x_3 \in A \), where \( unsh(2,1) \) denotes the set of all (2, 1) unshuffle permutations.

Thus the deformation equations are equivalent to the following equations:

\[ \alpha_0^2 = 0 \]  

\[ \alpha_0 \alpha_1 + \alpha_1 \alpha_0 = 0 \]  

\[ \alpha_1^2 + \alpha_0 \alpha_2 + \alpha_2 \alpha_0 = 0 \]  

\[ \alpha_i \alpha_j = 0 \]  

\[ \sum_{i+j=n} i+j=n \]

Define a bracket on the sequence \( \{\alpha_i\} \) of mappings by \[ [\alpha_i, \alpha_j] = \alpha_i \alpha_j + \alpha_j \alpha_i \] and a “differential” \( d \) by \( d = ad_{\alpha_0} = [\alpha_0, \cdot] \), the “adjoint representation” relative to \( \alpha_0 \). Notice that the second equation in the list above is equivalent to the statement that \( \alpha_1 \) defines a cocycle \( [\alpha_1] \in Z^2(A, A) \) in the Lie algebra cohomology of \( A \). Moreover it is known that the second cohomology group \( H^2(A, A) \) classifies the equivalence class of infinitesimal deformations of \( A \) \([4]\). This being the case we refer to the triple \( (A, \alpha_0, \alpha_1) \) as being initial conditions for deforming the Lie algebra \( (A, \alpha_0) \). Notice that the third equation in the above list can be rewritten as

\[ [\alpha_1, \alpha_1] = -[\alpha_0, \alpha_2] = -d \alpha_2. \]

When this equation holds one has then that \( [\alpha_1, \alpha_1] \) is a coboundary and so defines the trivial element of \( H^3(A, A) \) for any given deformation \( \alpha_t \). Thus if \( [\alpha_1, \alpha_1] \) is not a coboundary, then we may regard \( [\alpha_1, \alpha_1] \) as the first obstruction to a possible deformation and in this case we cannot deform \( A \) at second order. In general, to say that there exists
a deformation of \((A, \alpha_0, \alpha_1)\) up to order \(n-1\) means that there exists a sequence of maps \(\alpha_0, \ldots, \alpha_{n-1}\) such that

\[
\sum_{\sigma \in \text{sh}
(2,1)}(-1)^{\sigma} \alpha_l(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}) = 0 \quad \text{(mod } l^n)\.
\]

Let \(\rho_n = -\sum_{j+i=n, i,j>0} \alpha_{l} \alpha_{l} \). If \([\rho_n]\) is a nontrivial element of \(H^3(A, A)\), it is the obstruction at \(n\)th order. In principle, it is possible that one could return to the beginning and select different terms for the \(\alpha_i\) but when this fails what can one say? This is the issue in the remainder of this section.

The required sh-Lie structure lives on a graded vector space \(X_s\) which we define below. This space in degree zero is given by \(X_0 = A[1]\) = \(\langle \{\sum a_i t^i \mid a_i \in A\}\rangle\). The spaces \(B = (t^2) = A[1] \cdot t^2 = \langle \sum_{i \geq 2} a_i t^i \mid a_i \in A\rangle\) and \(F = X_0/B\) are also relevant to our construction. Notice that \(\mathcal{F}\) is isomorphic to \(\{a_0 + a_1 t \mid a_0, a_1 \in A\}\) as a linear space and that \(X_0, B\) are both \(k[1]\)-modules while \(\mathcal{F}\) is a \(k[1]/(t^2)\) module (recall that \(k\) is the underlying field of \(A\)). To summarize, we have the following short exact sequence:

\[0 \rightarrow B \rightarrow X_0 \rightarrow \mathcal{F} \rightarrow 0.\]

Suppose that the initial Lie structure of \(A\) is given by \(\alpha_0 : A \times A \rightarrow A\) and denote a fixed infinitesimal deformation by \([\alpha_1] \in H^2(A, A)\). One of the structure mappings of our sh-Lie structure will be determined by the mapping

\[\tilde{l}_2 : X_0 \times X_0 \rightarrow \mathcal{F}\]

and extend it to \(X_0\) by requiring that it be \(k[1]\)-bilinear. Obviously, \(\tilde{l}_2\) induces a Lie bracket \([\cdot, \cdot]\) on \(\mathcal{F}\), but if the obstruction \([\alpha_1, \alpha_1]\) is not zero, then \(\tilde{l}_2^2 \neq 0\) and consequently \(\tilde{l}_2\) cannot be a Lie bracket on \(X_0\) (since it does not satisfy the Jacobi identity).

To deal with this obstruction we will show that we can use \(\alpha_0, \alpha_1\) to construct an sh-Lie structure with at most three nontrivial structure maps \(l_1, l_2, l_3\) such that the value of \(l_3\) on \(A \times A \times A\) is the same as that of \([\alpha_1, \alpha_1]\). In particular, \(l_3\) will vanish if and only if the obstruction \([\alpha_1, \alpha_1]\) vanishes. Thus the sh-Lie algebra encodes the obstruction to deformation of the Lie algebra \((A, \alpha_0)\).

The required sh-Lie algebra lives on a certain homological resolution \((X_s, l_1)\) of \(\mathcal{F}\), so our first task is to construct this resolution space for \(\mathcal{F}\). To do this let us introduce a “superpartner set of \(A\)”, denoted by \(A[1]\), as follows: for each \(a \in A\), introduce \(a^*\) such that \(a^* \leftrightarrow a\) is a one to one correspondence and define \(\epsilon(a^*) = \epsilon(a) + 1\), where \(\epsilon\) defines the parity function of the graded space \(X_s\). Let \(X_1 = A[1][1][t]t^2\) and define a map \(l_1 : X_1 \rightarrow X_0\) by

\[l_1(x) = \sum_{i \geq 2} a_i t^i \in X_0, \quad x = \sum_{i \geq 2} a_i^* t^i \in X_1.\]

Indeed the central point of this section is to show that when there is an obstruction to the deformation of a Lie algebra, one can use the obstruction itself to define one of the structure mappings of an sh-Lie algebra. Without loss of generality, we consider a deformation problem which has a first order obstruction.

Notice that the construction of the map \(l_1\) is just the \(k[1]\) extension of the \(a^* \leftrightarrow a\) map. Since \(l_1\) is injective, we obtain a homological resolution \(X_s = X_0 \oplus X_1\) due to the fact that the complex defined by

\[0 \rightarrow X_1 \xrightarrow{l_1} X_0 \rightarrow 0\]

has the obvious property that \(H(X_s) = H_0(X_s) \simeq \mathcal{F}\).

The sh-Lie algebra being constructed will have the property that \(l_n = 0\), \(n \geq 4\). Generally sh-Lie algebras can have any number of nontrivial structure maps. The fact that all the structure mappings of our sh-Lie algebra are zero with the exception of \(l_1, l_2, l_3\) is an immediate consequence of the fact that we are able to produce a resolution of the space \(\mathcal{F}\) such that \(X_k = 0\) for \(k \geq 2\). In general such resolutions do not exist and so one does not have \(l_n = 0\) for \(n \geq 4\).

In order to finish the preliminaries, we now construct a contracting homotopy \(s\) such that following commutative diagram holds:

\[
\begin{array}{ccc}
0 & \rightarrow & X_1 \\
\xrightarrow{l_1} & & \xleftarrow{s} \xrightarrow{\lambda} X_0 \\
& & \xrightarrow{\eta} \\
0 & \rightarrow & 0 \\
\end{array}
\]

\[
\begin{array}{ccc}
0 & \rightarrow & \mathcal{F} \\
& & \xrightarrow{\eta} \\
0 & \rightarrow & 0
\end{array}
\]
Clearly the linear space \( X_0 \) is the direct sum of \( \mathcal{B} \) and a complementary subspace which is isomorphic to \( \mathcal{F} \); consequently we have \( X_0 \cong \mathcal{B} \oplus \mathcal{F} \). Define \( \eta = \text{proj}_{\mathcal{F}}, \lambda = i_{\mathcal{F}} \rightarrow X_0 \) and a contracting homotopy \( s : X_0 \rightarrow X_1 \) as follows: write \( X_0 = \mathcal{B} \oplus \mathcal{F} \), set \( s|_{\mathcal{F}} = 0 \), and let \( s(x) = -x^* \) for all \( x \in \mathcal{B} \). It is easy to show that \( \lambda \circ \eta - 1_{X_1} = l_1 \circ s + s \circ l_1 \). In order to obtain the sh-Lie algebra referred to above, we apply a theorem of [2]. The hypothesis of this theorem requires the existence of a bilinear mapping \( \tilde{l}_2 \) from \( X_0 \times X_0 \) to \( X_0 \) with the properties that for \( c, c_1, c_2, c_3 \in X_0 \) and \( b \in \mathcal{B} \)

(i) \( \tilde{l}_2(c, b) \in \mathcal{B} \) \hspace{1cm} (16)
(ii) \( \tilde{l}_2^2(c_1, c_2, c_3) \in \mathcal{B} \). \hspace{1cm} (17)

To see that (i) holds notice that if \( p(t), q(t) \in X_0 = A[[t]] \), then \( \tilde{l}_2(p(t), q(t)t^2) = r(t)t^2 \) for some \( r(t) \in A[[t]] = X_0 \). Also note that the fact that \( \tilde{l}_2 \) induces a Lie bracket on \( \mathcal{F} = X_0/\mathcal{B} \) implies that \( \tilde{l}_2^2 \) is zero modulo \( \mathcal{B} \) and (ii) follows. Thus \( X_0 \) supports an sh-Lie structure with only three nonzero structure maps \( l_1, l_2, l_3 \) (see the remark at the end of [2]).

**Theorem 1.** Given a Lie algebra \( A \) with Lie bracket \( \alpha_0 \) and an infinitesimal deformation \( [\alpha_1] \in H^2(A, A) \) to deforming \( (A, \alpha_0) \), there is a graded space \((X_0, l_1)\) defined in (14) and (15) and a sh-Lie algebra on \( X_0 \) with structure maps \( l_1, l_2, l_3 \).

**Remark.** In the theorem above, the mappings \( l_2 \) and \( l_3 \) have the following precise description: The mapping \( l_2 \) is determined by its values on the set of \( X_0 \times X_0 \) and \( X_1 \times X_0 \). On \( X_0 \times X_0, l_2 \) is determined by \( \tilde{l}_2 = \alpha_0 + \alpha_1 t \) which has been defined on the subset \( A \times A \subset X_0 \times X_0 \). On \( X_1 \times X_0, l_2 \) is determined by \( l_2(a^*t^2, b) = t^2(\alpha_0(a, b)^* + \alpha_1(a, b)*t) \) for \( a^* \in A[1], b \in A \). Finally, \( l_3 \) has nonzero values only on the set of \( X_0 \times X_0 \times X_0 \) and is uniquely determined by its values on \( A \times A \subset X_0 \times X_0 \times X_0 : l_3(a_1, a_2, a_3) = -\frac{1}{2} t^2[\alpha_1, \alpha_1](a_1, a_2, a_3), a_i \in A \). We should notice that above expression contains the obstruction to the deformation of \( (A, \alpha_0) \).

**Proof.** First of all, we examine the mapping \( l_2 : X_0 \times X_0 \rightarrow X_0 \). Now \( l_2 : A \times A \rightarrow X_0 \) is uniquely determined by \( l_2 : X_0 \times X_0 \rightarrow X_0 \), and consequently we need only consider the restricted mapping

\[
l_2 : X_1 \times X_0 \rightarrow X_1.
\] (18)

Moreover, since \( X_0 \) is a module over \( k[[t]] \), \( X_1 \) is a module over \( k[[t]]t^2 \), and \( \tilde{l}_2 \) respects these structures we need only consider its values on pairs \((a^*t^2, b)\) with \( a^*t^2 \in X_1, b \in X_0 \). By Theorem 2.2 of [1], we have

\[
l_2(a^*t^2, b) = -s[l_2[(a^*t^2) \otimes b]] = -s[l_2[l_2[(a^*t^2) \otimes b] + (-1)^s(a^*)((a^*t^2) \otimes l_1(b)) = -s[l_2[(a^*t^2) \otimes b]] = -s[l_2[a_0(a, b) + \alpha_1(a, b)t]] = -s[a_0(a, b)t^2 + \alpha_1(a, b)t^3] = a_0(a, b)t^2 + \alpha_1(a, b)^*t^3 = t^2(a_0(a, b)^* + \alpha_1(a, b)^*t). \hspace{1cm} (19)
\]

The next mapping we examine is the mapping

\[
l_3 : X_0 \times X_0 \times X_0 \rightarrow X_1.
\] (20)

Since \( l_3 \) is \( k[[t]] \)-linear, we need only consider mappings of the type:

\[
l_3(x_1, x_2, x_3) = s[l_2^2(x_1, x_2, x_3) = \sum_{\sigma \in unsh(2, 1)} (-1)^{\sigma} s[l_2(l_2[(x_{\sigma(1)}, x_{\sigma(2)}), x_{\sigma(3)})} = \sum_{\sigma \in unsh(2, 1)} (-1)^{\sigma} s[l_2(\alpha_0(x_{\sigma(1)}, x_{\sigma(2)}) + \alpha_1(x_{\sigma(1)}, x_{\sigma(2)})t, x_{\sigma(3)})]
\]

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\]
The sh-Lie structure maps of the above theorem are given by Theorem 7 of [2].

Remark. The sh-Lie structure maps of the above theorem are given by Theorem 7 of [2]. The fact that \( l_n = 0, n \geq 4 \) is an observation of Markl which was proved by Barnich (see the remark at the end of [2]).

A direct calculation shows that \( l_1, l_2, l_3 \) satisfy the following sh-Lie algebra conditions:

\[
\begin{align*}
l_1l_2 - l_1l_2 &= 0, \\
l_2^2 + l_1l_3 + l_3l_1 &= 0, \\
l_3^2 &= 0, \\
l_2l_3 + l_3l_2 &= 0. \quad \square
\end{align*}
\]

(21)

Or \( l_3(x_1, x_2, x_3) = -\frac{1}{2}l_2([\alpha_1, \alpha_1](x_1, x_2, x_3))^* \) which is precisely the “first deformation obstruction class”.

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l_3^2 &= 0, \\
l_2l_3 + l_3l_2 &= 0. \quad \square
\end{align*}
\]

(22)

(23)

(24)

(25)

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References