Journal of Pure and Applied Algebra 71 (1991) 341–350
North-Holland

F-regular and F-pure normal graded rings

Kei-ichi Watanabe*
Department of Mathematical Sciences, Tokai University, Hiratuka 259-12, Japan

Communicated by J.D. Sally
Received 4 December 1989
Revised 18 September 1990

Dedicated to Professor Hideyuki Matsumura on his sixtieth birthday

Introduction

The notion of tight closure of an ideal introduced by Hochster and Huneke [5] enables us to define classes of singularities corresponding to rational singularities in characteristic 0 for rings of characteristic $p>0$. Among them are the notions of F-regular ring [5] and F-rational ring [3]. It turns out that it is relatively easy to know whether a given ring is F-rational or not comparing the case of F-regularity (cf. [3], §2]). On the other hand, the notion of F-pure ring is defined in [6] and proved to be useful to treat a slightly wider class of rings than F-regular ones.

The aim of this paper is to give a criterion to determine whether a given ring is F-regular (resp. F-pure) or not via the action of Frobenius on the highest local cohomology group of the canonical module. This criterion is especially powerful for the case of normal graded rings. We can understand the difference of F-regularity and F-rationality for normal graded rings using Demazure's expression of normal graded rings. Also, we can completely classify normal F-regular (resp. F-pure) graded rings of dimension 2 for each characteristic.

In this paper all rings are Noetherian with characteristic $p>0$.

1. Preliminaries

Let R be a Noetherian ring of characteristic $p>0$ and $F: R \to R$ be the Frobenius map defined by $F(a)=a^p$. In this paper the letter q always means a power $q=p^e$ of p. For simplicity, we always assume that R is reduced and we will

* Partially supported by the Ishida Foundation.

identify the following three maps:

(i) $F^e : R \to R$,

(ii) the inclusion $R^q \to R$,

(iii) the inclusion $R \to R^{1/q}$.

We sometimes denote $F^e : R \to {}^eR$ to distinguish R of both sides.

For an R-module M, we define the Frobenius action on M by

$$F^e : M = M \otimes_R R \to F^e(M) := M \otimes_R {}^eR\,, \qquad F^e(x) = x \otimes 1\,.$$

For an ideal I of R we denote $I^{[q]} = I \cdot {}^eR$. If $I = (a_1, \ldots, a_n)$, then $I^{[q]} = (a_1^q, \ldots, a_n^q)$. Also, we denote

$$R^0 = \{x \in R \mid x \text{ is not contained in any minimal prime of } R\}\,.$$

The notion of tight closure of an ideal or a submodule is essential in our argument.

1.1. Definition [5]. (i) For an ideal I of R, we define the ideal I^* by $I^* = \{x \in R \mid \text{there exists } c \in R^0 \text{ such that } cx^q \in I^{[q]} \text{ for every } q \gg 0\}$. We call I^* the *tight closure* of I. We say that I is *tightly closed* if $I^* = I$.

(ii) If N is an R-submodule of an R-module M, then $x \in N^*$ if there exists $c \in R^0$ such that for every $q \gg 0$, $c \cdot F^e(x)$ is contained in the image of $F^e(N)$ in $F^e(M)$. We call N^* the *tight closure* of N in M.

Note that if $x \in R$, the condition $x \in I^*$ is equivalent to saying that the image of x in R/I is in $(0)^*$ (the tight closure of (0) in R/I).

1.2. Definition. (i) [5] We say that R is *F-regular* if every ideal of R is tightly closed. (In the terminology of [5], this notion is called 'weakly F-regular'. In this paper, we will always 'forget' the adjective 'weak' since we have no need to distinguish the two notions of F-regularity with and without 'weak'.)

(ii) [3] A local ring R with maximal ideal $\mathfrak{m}$ is called *F-rational* if every ideal generated by a system of parameters of R is tightly closed.

1.3. Definition [6]. We say that R is *F-pure* if the Frobenius morphism $F : R \to {}^1R$ is pure. By definition, this is equivalent to saying that for every R-module M, $F : M \to F(M) = M \otimes_R {}^1R$ is injective.

1.4. Proposition (cf. [5] for (i) and [6] for (ii)). *Assume that R is local with maximal ideal $\mathfrak{m}$. Let $E = E_R(R/\mathfrak{m})$ be the injective envelope of $R/\mathfrak{m}$. Then*

(i) *If $(0)^* = (0)$ in E, then R is (weakly) F-regular.*

(ii) *If $F : E \to F(E)$ is injective, then R is F-pure.*

The converse of both statements is true if R is approximately Gorenstein [4]. □

The above proposition shows that if we can 'describe' the Frobenius action on E, then we can determine whether R is F-regular (resp. F-pure) or not. The condition 'R is approximately Gorenstein' is very weak and is satisfied if R is excellent and reduced. So, we will always assume this condition throughout this paper.

Consequently, the aim of this paper is to describe the Frobenius action on E.

2. The action of Frobenius on $E = E_R(R/\mathfrak{m})$

2.1. In this section, let R be a normal local ring with maximal ideal $\mathfrak{m}$ and let $\dim R = d$. We always assume that R has a canonical module K_R. Let Q be the quotient field of R. Then we can assume that $K_R \subset Q$. As is well known, $E_R(R/\mathfrak{m})$ is isomorphic to the highest local cohomology group $H^d_{\mathfrak{m}}(K_R)$. So, we will analyze the morphism

$$F^e : H^d_{\mathfrak{m}}(K_R) \to H^d_{\mathfrak{m}}(K_R) \otimes_R {}^eR .$$

2.2. Lemma. *Let* $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ *be a finite local homomorphism with* $\dim S = \dim R = d$ *and* M *be a finitely-generated* R*-module. Then,*

$$H^d_{\mathfrak{n}}(M \otimes_R S) \simeq H^d_{\mathfrak{m}}(M) \otimes_R S.$$

Proof. Take a parameter ideal $(\boldsymbol{x}) = (x_1, \dots, x_d)$ of R and let

$$C^{\cdot} = [0 \to C^0 = R \to C^1 = \bigoplus_{i=1}^{d} R_{x_i} \to \cdots \xrightarrow{\delta_d} C_d = R_{x_1 \cdots x_d} \to 0]$$

be the Čech complex of R with respect to $(\boldsymbol{x})$. Then we can write

$$H^d_{\mathfrak{m}}(M) = H^d(M \otimes_R C^{\cdot}) = \mathrm{Coker}(1_M \otimes_R \delta_d) .$$

Since tensoring preserves the cokernel, we have $H^d_{\mathfrak{m}}(M) = M \otimes_R H^d_{\mathfrak{m}}(R)$. On the other hand, by our assumption, $(\boldsymbol{x})$ is also a parameter system for S. Hence we have

$$\begin{aligned} H^d_{\mathfrak{n}}(M \otimes_R S) &= H^d((M \otimes_R S) \otimes_S (C^{\cdot} \otimes_R S)) = H^d((M \otimes_R C^{\cdot}) \otimes_R S) \\ &= H^d(M \otimes_R C^{\cdot}) \otimes_R S = H^d_{\mathfrak{m}}(M) \otimes_R S . \qquad \square \end{aligned}$$

2.3. Let $D = \sum n_{\mathfrak{p}} \cdot V(\mathfrak{p})$ be a divisor of R, where the sum is taken over the prime ideal p of height 1 or R. Then we define

$$\begin{aligned} R(D) &= \{ f \in Q \mid \mathrm{div}_R(f) + D \geq 0 \} \\ &= \{ f \in Q \mid v_{\mathfrak{p}}(f) \geq -n_{\mathfrak{p}} \text{ for every height-1 prime of } R \} . \end{aligned}$$

If $D = V(\mathfrak{p})$, then $R(D) = \mathfrak{p}^{-1}$ and $R(-D) = \mathfrak{p}$. For a divisorial ideal $I = R(D)$ of R, we define $I^{(n)} = R(nD)$. Note that $I^{(n)}$ is the divisorial hull of the nth power I^n of I. (A divisorial ideal of R is an R-submodule of Q which is reflexive as an R-module.)

For a fractional ideal I of R, we denote

$$I^{1/q} = \{x \in Q^{1/q} \mid x^q \in I\} .$$

2.4. Lemma. *Let $I = R(D)$ be a divisorial ideal of R. Then the canonical homomorphism*

$$\alpha : I \otimes_R R^{1/q} \to (I^{(q)})^{1/q} ,$$

defined by $\alpha(a \otimes x^{1/q}) = a \cdot x^{1/q} = (a^q x)^{1/q}$ ($a \in I$, $x \in R$) is an isomorphism in codimension 1. That is, the codimensions of support of Ker(α) *and* Coker(α) *are at least* 2.

Proof. If $I = fR$ is principal, then the left-hand side equals $f \cdot R^{1/q} = (f^q R)^{1/q}$, which is equal to the right-hand side. Now, for every prime ideal $\mathfrak{p}$ of R, $I \cdot R_{\mathfrak{p}}$ is principal. Hence $\alpha \otimes_R R_{\mathfrak{p}}$ is an isomorphism for every prime ideal $\mathfrak{p}$ of height 1. □

Now, since R is normal, K_R is isomorphic to a divisorial ideal. We fix an isomorphism $K_R \simeq R(K)$ for some divisor K of R. By this isomorphism, we define $K_R^{(q)} = R(qK)$.

2.5. Theorem. *We have the canonical isomorphism*

$$\kappa : H^d_{\mathfrak{m}}(K_R) \otimes_R R^{1/q} \to H^d_{\mathfrak{m}}((K_R^{(q)})^{1/q})$$

induced by the canonical homomorphism

$$\alpha : K_R \otimes_R R^{1/q} \to (K_R^{(q)})^{1/q} .$$

Proof. By Lemma 2.2, $H^d_{\mathfrak{m}}(K_R) \otimes_R R^{1/q} \simeq H^d_{\mathfrak{m}}(K_R \otimes_R R^{1/q})$. On the other hand, by Lemma 2.4, the homomorphism α induces an isomorphism

$$H^d_{\mathfrak{m}}(K_R \otimes_R R^{1/q}) \simeq H^d_{\mathfrak{m}}((K_R^{(q)})^{1/q}) . \qquad \square$$

2.6. Remark. Let z be the generator of the socle of $E = E_R(R/\mathfrak{m}) \simeq H^d_{\mathfrak{m}}(K_R)$. Then $F : E \to E \otimes_R {}^1R$ is injective if and only if $F(z) \neq 0$ and $(0)^* = (0)$ in E if and only if $z \not\in (0)^*$. For this reason, it is very important to describe z explicitly. In the next section, we will treat this problem for normal graded rings.

Using Theorem 2.5, we can show that F-regularity is preserved by taking certain 'finite cover'. The same argument is also valid for F-purity, which the author was informed of by V. Srinivas.

2.7. Theorem. *Let $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a finite local homomorphism of normal rings which is etale in codimension* 1. *If R is F-regular, so is S.*

Proof. If $A \to B$ is a finite etale morphism of rings, then we have the isomorphism $B \otimes_A {}^1A \simeq {}^1B$. Hence the canonical map $S \otimes_R {}^1R \to {}^1S$ is an isormorphism after localizing at a prime ideal of height 1 of R by our assumption. From this fact, it follows that the canonical map

$$K_S \otimes_R {}^1R = K_S \otimes_S (S \otimes_R {}^1R) \to K_S \otimes_S {}^1S$$

is an isomorphism in codimension 1. Let $d = \dim R = \dim S$ and apply $H^d_{\mathfrak{m}}(*)$ to both sides. Then, using Lemma 2.2, we get the isomorphism

$$H^d_{\mathfrak{m}}(K_S) \otimes_R {}^1R \simeq H^d_{\mathfrak{m}}(K_S \otimes_R {}^1R) \simeq H^d_{\mathfrak{m}}(K_S \otimes_S {}^1S) \simeq H^d_{\mathfrak{m}}(K_S) \otimes_S {}^1S .$$

Consequently, we can identify the Frobenius action on $H^d_{\mathfrak{n}}(K_S) = H^d_{\mathfrak{m}}(K_S)$ as R-module and the one as S-module. Take $z \in H^d_{\mathfrak{n}}(K_S)$, $z \neq 0$. Since R is F-regular, we have $\bigcap_{e>0} \operatorname{Ann}_R(F^e(z)) = (0)$. Since $R \to S$ is a finite extension of integral domains, we also have $\bigcap_{e>0} \operatorname{Ann}_S(F^e(z)) = (0)$, which shows that S is F-regular, too. □

2.8. Example. Let I be a divisorial ideal of R such that $\mathrm{cl}(I)$ in $\mathrm{Cl}(R)$ is a torsion of order r. (Here, $\mathrm{Cl}(R)$ denotes the divisor class group of R.) Then fixing an isomorphism $I^{(r)} \simeq R$, we can construct an R-algebra

$$S = \bigoplus_{i=0}^{r-1} I^{(i)} .$$

If r is not divisible by p, then the inclusion $R \to S$ is etale in codimension 1. In this case, if R is F-regular, so is S.

2.9. Corollary. *Let I be a divisorial ideal of R and assume that* $\mathrm{cl}(I)$ *in* $\mathrm{Cl}(R)$ *is a torsion of order r with $(r, p) = 1$. If, moreover, R is F-regular, then I is Cohen–Macaulay as an R-module.*

Proof. Let S be as in Example 2.8. Then S is F-regular, which implies that S is a Cohen–Macaulay S-module (resp. R-module) (cf. [5]). Being a direct summand, I is a Cohen–Macaulay R-module, too. □

3. Criterion for F-regularity and F-purity for normal graded rings

3.1. Let $R = \bigoplus_{n \geq 0} R_n$ be a Noetherian normal graded ring with $R_0 = k$, a field with characteristic $p > 0$. Then by Demazure [1], R can be described by a rational coefficient Weil divisor D on $X = \mathrm{Proj}(R)$ such that $\mathrm{N} \cdot \mathrm{D}$ is an ample Cartier divisor for some positive integer N in the form

$$R = R(X, D) = \bigoplus_{n \geq 0} H^0(X, \mathcal{O}_X(nD)) \cdot T^n \subset k(X)[T],$$

where $k(X)$ is the field of rational functions on X and T is a fixed homogeneous element of degree 1 in the quotient field of R.

Throughout this section, we put $\dim R = d + 1$ and $\dim X = d$.

We will write $D = \sum (p_v / q_V)V$, where the sum is taken over irreducible subvarieties V of codimension 1 of X, $q_V > 0$ and $(p_V, q_V) = 1$ for every V. We will denote the fractional part of D by D'. Namely, $D' = \sum ((q_V - 1)/q_V)V$.

3.2. Now, as in [9], we can describe K_R and $K_R^{(q)}$ as

$$K_R = \bigoplus_{n \in \mathbb{Z}} H^0(X, \mathcal{O}_X((K_X + D') + nD)) \cdot T^n \subset k[T, T^{-1}],$$

$$K_R^{(q)} = \bigoplus_{n \in \mathbb{Z}} H^0(X, \mathcal{O}_X(q(K_X + D') + nD)) \cdot T^n \subset k[T, T^{-1}].$$

Also, the highest local cohomology groups are expressed as

$$H_{\mathfrak{m}}^{d+1}(K_R) = \bigoplus_{n \in \mathbb{Z}} H^d(X, \mathcal{O}_X((K_X + D') + nD)) \cdot T^n,$$

$$H_{\mathfrak{m}}^{d+1}(K_R^{(q)}) = \bigoplus_{n \in \mathbb{Z}} H^d(X, \mathcal{O}_X(q(K_X + D') + nD)) \cdot T^n,$$

where K_X is the canonical divisor of X and we define the grading on each module by the power of T.

Now, the socle of $H_{\mathfrak{m}}^{d+1}(K_R)$ is $H^d(X, \mathcal{O}_X(K_X + D')) = H^d(X, \mathcal{O}_X(K_X)) \simeq k$, the component of degree 0. Note that our proof of Theorem 2.5, stated for local rings is also valid for the graded case. Hence we have the following theorem:

3.3. Theorem. *Let $R = R(X, D)$ be a normal graded ring as in* 3.1. *Then*:

(i) *R is F-pure if and only if the Frobenius morphism*

$$F: H^d(X, \mathcal{O}_X(K_X + D')) \to H^d(X, \mathcal{O}_X(p(K_X + D')))$$

is injective.

(ii) *R is F-regular if and only if for every $n > 0$ and for every nonzero element f of $H^0(X, \mathcal{O}_X(nD))$, there exists $e > 0$, such that $f \cdot F^e(z) \neq 0$ in $H^d(X, \mathcal{O}_X(q(K_X + D') + nD))$, where $z \in H^d(X, \mathcal{O}_X(K_X + D'))$, $z \neq 0$.*

Proof. This result is a direct corollary of Theorem 2.5, noting the description of K_R, $K_R^{(q)}$ and the socle of $H_{\mathfrak{m}}^{d+1}(K_R)$. $\square$

Using this criterion, we can show that F-regularity (resp. F-purity) of $R = R(X, D)$ depends only on X and the fractional part of D and not on individual D. For example, if L is an integral divisor on X such that both ND and $N(D + L)$ are ample Cartier divisors for some positive integer N, then $R(X, D)$ is F-regular (resp. F-pure) if and only if so is $R(X, D + L)$.

3.4. Theorem. *Let X be a normal projective variety over k and $D_1, D_2 \in \mathrm{Div}(X, \mathbb{Q})$ such that both ND_1 and ND_2 are ample Cartier divisors for some positive integer N. Assume that D_1 and D_2 have the same fractional part. This is, $(D_1)' = (D_2)'$. Let $R_i = R(X, D_i)$ for $i = 1, 2$. Then R_1 is F-regular (resp. F-pure) if and only if so is R_2.*

Proof. As for the F-purity, our assertion is obvious from Theorem 3.3, since only K_X and D' appear in the criterion.

Now, assume R_1 is F-regular. We will show that R_2 is F-regular. It suffices to show that for every $x \in H^0(X, \mathcal{O}_X(nD_2))$, $x \neq 0$, there exists e such that $x \cdot F^e(z) \neq 0$, where z is the generator of $H^d(X, \mathcal{O}_X(K_X)) = (H_{\mathfrak{m}}^{d+1}(R))_0$.

Our proof is divided into several steps. Note that we may assume that both R_1 and R_2 are F-pure which implies that the Frobenius morphisms are injective.

If $x \cdot F^e(z) \neq 0$, then $F(x \cdot F^e(z)) = x^p F^{e+1}(z) \neq 0$, hence $x \cdot F^{e+1}(z) \neq 0$. That is, $\mathrm{Ann}(F^e(z)) \supset \mathrm{Ann}(F^{e+1}(z))$. Moreover, taking x^N instead of x, if necessary, we may assume that $x \in H^0(X, \mathcal{O}_X(ND_2))$, where ND_2 is an ample Cartier divisor on X.

Now, since $\{\mathrm{Ann}_{R_1}(F^e(z))\}_{e \geq 0}$ forms a decreasing sequence whose intersection is (0) (R_1 is F-regular), for every n there exists e such that the mapping

$$\zeta : H^0(X, \mathcal{O}_X(nD_1)) \to H^d(X, \mathcal{O}_X(q(K_X + D') + nD_1))$$

induced by the multiplication with $F^e(z)$ is injective.

For ND_2 as above, we can take $M > 0$ such that $H^0(X, \mathcal{O}_X(MD_1 - ND_2)) \neq 0$ since D_1 is ample. Any nonzero element in this group induces an injective homomorphism

$$\phi : H^0(X, \mathcal{O}_X(ND_2)) \to H^0(X, \mathcal{O}_X(MD_1)) .$$

Then we have a commutative diagram

$$\begin{array}{ccc} H^0(X, \mathcal{O}_X(ND_2)) & \xrightarrow{\phi} & H_0(X, \mathcal{O}_X(MD_1)) \\ \downarrow{\scriptstyle \zeta'} & & \downarrow{\scriptstyle \zeta} \\ H^d(X, \mathcal{O}_X(q(K_X + D') + ND_2)) & \xrightarrow{H^d(\phi)} & H^d(X, \mathcal{O}_X(q(K_X + D') + MD_1)) , \end{array}$$

where ζ and ζ' are induced by the multiplication with $F^e(z)$. Now, since R_1 is F-regular, we can take e so that ζ is injective and ϕ is also injective. Then the commutativity of the diagram implies injectivity of ζ', which is what we wanted to show. □

4. Classification of normal graded F-regular and F-pure rings in dimension 2

In this section, let $R = R(X, D)$ be a normal graded ring of dimension 2 with $R_0 = k$, an algebraically closed field of characteristic $p > 0$. We will determine the condition for such R to be F-regular or F-pure. By Theorems 3.3 and 3.4, we can state the condition in terms of $K_X + D'$. So, we will list up all pairs of X and D' such that $R(X, D)$ is F-regular (resp. F-pure).

4.1. Since R is normal and $\dim R = 2$, $X = \mathrm{Proj}(R)$ is normal, and $\dim X = 1$. That is, X is a smooth curve. For $D \in \mathrm{Div}(X, \mathbb{Q})$, we denote the degree of D by $\deg(D)$.

Now, assume that R is F-regular. Then Theorem 3.3 shows that $\dim_k H^1(X, \mathcal{O}_X(q(K_X + D')) \to +\infty$ when $q \to +\infty$. This occurs if and only if $\deg(K_X + D') < 0$. Since $\deg(D') > 0$ by definition, if $D' \neq 0$, $\deg(K_X) < 0$ and we know that the genus of X is 0. We may think $X = \mathbb{P}^1$, by our assumption k is algebraically closed.

If R is F-pure, $H^1(X, \mathcal{O}_X(q(K_X + D'))) \neq 0$ for every q. This implies $\deg(K_X + D') \leq 0$. Then we have two cases,

(i) $X = \mathbb{P}^1$ and $\deg(D') \leq 2$,

(ii) X is an elliptic curve and $D' = 0$.

Summarizing these facts, we get the classification.

4.2. Theorem. *Let* $R = R(X, D)$ *with* $\dim R = 2$ *and assume* $k = H^0(X, \mathcal{O}_X)$ *is algebraically closed. If*

$$D' = \sum_{i=1}^{r} \frac{q_i - 1}{q_i} P_i \in \mathrm{Div}(X, \mathbb{Q}) ,$$

we will denote D' *by* $(P_1, \ldots, P_r; q_1, \ldots, q_r)$ *and if* $X = \mathbb{P}^1$ *and* $r \leq 3$, *we will denote* D' *simply by* $(q_1, \ldots, q_r)$ *since there is no need to distinguish the* P_i*'s.*

(1) *If R is F-regular, then* $X = \mathbb{P}^1$, $D' = (q_1, \ldots, q_r)$ *with* $r \leq 3$ *and* D' *is one of those listed in* (a) *to* (d).

(a) *If* $r = 0, 1, 2$, *then the* q_i*'s are arbitrary and R is F-regular in arbitrary characteristic.*

(b) *If* $D' = (2, 2, n)$, *then R is F-regular if and only if* $p \neq 2$.

(c) *If* $D' = (2, 3, 3)$ *or* $(2, 3, 4)$, *then R is F-regular if and only if* $p > 3$.

(d) *If* $D' = (2, 3, 5)$, *then R is F-regular if and only if* $p > 5$.

Moreover, in case (b), (c), (d), *R is not even F-pure for* $p=2$ *in* (b), $p=2,3$ *in* (c) *and* $p=2,3,5$ *in* (c).

(2) *If R is F-pure and not F-regular, then* (X, D') *is one of those listed in* (i) *and* (ii), (e) *to* (h).

(i) *X is an elliptic curve and* $D'=0$. *In this case, R is F-pure if and only if the action of Frobenius on* $H^1(X, \mathcal{O}_X)$ *is injective.*

(ii) $X=\mathbb{P}^1$ *and* $\deg D'=2$. *More explicitly*:

(e) *If* $D'=(3,3,3)$, *R is F-pure if and only if* $p \equiv 1 \pmod 3$.

(f) *If* $D'=(2,4,4)$, *R is F-pure if and only if* $p \equiv 1 \pmod 4$.

(g) *If* $D'=(2,3,6)$, *R is F-pure if and only if* $p \equiv 1 \pmod 3$.

(h) *If* $D'=(\infty, 0, -1, \lambda; 2,2,2,2)$ *with* $\lambda \in k$, $\lambda \neq 0, -1$, *then R is F-pure if and only if* $p=2n+1$ *such that the coefficient of* x^n *in the expansion of* $(x+1)^n (x-\lambda)^n$ *is not zero. (If R is F-pure, then p is odd.)*

Proof. The classification of X and D' is easy to get from the condition on the degree. To know whether R is F-regular (resp. F-pure) or not for fixed p, we examine the Frobenius action

$$F^e : H^1(X, \mathcal{O}_X(K_X)) \to H^1(X, \mathcal{O}_X(q(K_X + D'))) .$$

Remark. The classification of normal 2-dimensional F-pure singularity is done in [10] for the Gorenstein case and in [8] for the general case in terms of resolution of the singularity.

4.3. Remark. (i) For every D' listed in Theorem 4.2(1), we can take D so that R is Gorenstein (for example, take $D=-(K_X+D')$; cf. [9]). If R is Gorenstein, then case (a) (resp. (b); (c); (d)) corresponds to 'rational double points' of type (A_n) (resp. (D_n); (E_6), (E_7); (E_8)). Also, in the case of Theorem 4.2(2)(i), R is always Gorenstein and such R is called a 'simple elliptic singularity'. On the other hand, in the case of Theorem 4.2(2)(ii), R is never Gorenstein (cf. [9]).

(ii) Conversely, all rings listed in Theorem 4.2 are rational singularities except for case (2)(i). Then by [7], the divisor class group $\mathrm{Cl}(R)$ is a finite group and we can construct a canonical cover $S=\bigoplus_{i=0}^{r-1} K_R^{(i)}$ (r is the order of $\mathrm{cl}(K_R)$), which is Gorenstein. Since a rational Gorenstein ring in characteristic 0 and dimension 2 is an invariant subring of a polynomial ring by a finite group (we call such R a 'quotient singularity' for short), our R in Theorem 4.2(i) is also a 'quotient singularity'. Thus we can conclude an F-regular ring of dimension 2 in 'characteristic 0' is a 'quotient singularity'. In characteristic $p>0$, there are F-regular rings which are not invariant subrings.

Now, we know how different the notions of F-regularity and F-rationality are.

4.4 Example. Let $R=R(X, D)$ with $X=\mathbb{P}^1$ and $D>0$ (D is effective). Then R is F-injective in the sense of [2] (cf. [10, (2.3)]). Since $a(R)<0$ in this case (cf. [9]), such R is F-rational by [3, (2.8) and (1.17)]. On the other hand, we can choose $D > 0$ so that $\deg(D') > 2$. Then $R(X, D)$ is not F-regular or F-pure. In particular, there are no implications between F-pure and F-rational. Also the examples in Theorem 4.2(2)(ii) show that F-rationality together with F-purity do not imply F-regularity.

Acknowledgment

This work was initiated when the author was staying at Brandeis University taking leave from Tokai University. The author wishes to express his deep gratitude to both institutes and especially to D. Buchsbaum and D. Eisenbud for their kind hospitality during his stay at Brandeis University.

References

[1] M. Demazure, Anneaux gradués normaux, in: Lê Dûng Tráng, ed., Introduction a la theorie des singularitiés, vol. II (Hermann, Paris, 1988) 35–68.
[2] R. Fedder, F-purity and rational singularity, Trans. Amer. Math. Soc. 278 (1987) 47–62.
[3] R. Fedder and K.-i. Watanabe, A characterization of F-regularity in terms of F-purity, in: Commutative Algebra, Proceedings Microprogram, Berkeley, 1987 (Springer, Berlin, 1989) 227–245.
[4] M. Hochster, Cyclic purity versus purity in excellent Noetherian rings, Trans. Amer. Math. Soc. 231 (1977) 463–488.
[5] M. Hochster and C. Huneke, Tight closure, invariant theory and the Briançon–Skoda theorem, J. Amer. Math. Soc. 3 (1990) 31–116.
[6] M. Hochster and J.L. Roberts, The purity of the Frobenius and local cohomology, Adv. in Math. 21 (1976) 117–172.
[7] J. Lipman, Rational singularities with applications to algebraic surfaces and unique factorization, Publ. I.H.E.S. 36 (1969) 195–280.
[8] V.S. Mehta and V. Srinivas, Normal F-pure surface singularities, Preprint.
[9] K.-i. Watanabe, Some remarks concerning Demazure's construction of normal graded rings, Nagoya Math. J. 83 (1981) 203–211.
[10] K.-i. Watanabe, Study of F-purity in dimension 2, in: H. Hijikata et al., eds., Algebraic Geometry and Commutative Algebra in Honor of M. Nagata, Vol. II (Kinokuniya, Tokyo, 1987) 791–800.