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Eigenvalues and edge-connectivity of regular graphs

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A R T I C L E   I N F O

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In this paper, we show that if the second largest eigenvalue of a d-regular graph is less than $d - \frac{2(k-1)}{d+1}$, then the graph is k-edge-connected. When k is 2 or 3, we prove stronger results. Let $\rho(d)$ denote the largest root of $x^3 - (d-3)x^2 - (3d-2)x - 2 = 0$. We show that if the second largest eigenvalue of a d-regular graph G is less than $\rho(d)$, then G is 2-edge-connected and we prove that if the second largest eigenvalue of G is less than $\frac{d-3+\sqrt{(d+3)^2-16}}{2}$, then G is 3-edge-connected.

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1. Introduction

Let $\kappa(G)$ and $\kappa'(G)$ denote the vertex- and edge-connectivity of a connected graph G. If δ is the minimum degree of G, then $1 \leqslant \kappa(G) \leqslant \kappa'(G) \leqslant \delta$. Let $L = D - A$ be the Laplacian matrix of G, where D is the diagonal degree matrix and A is the adjacency matrix of G. We denote by $0 = \mu_1 < \mu_2 \leqslant \cdots \leqslant \mu_n$ the eigenvalues of the Laplacian of G. The complement of G is denoted by $\overline{G}$. A graph is called disconnected if it is not connected. The join $G_1 \vee G_2$ of two vertex-disjoint graphs G_1 and G_2 is the graph formed from the union of G_1 and G_2 by joining each vertex of G_1 to each vertex of G_2.

A classical result in spectral graph theory due to Fiedler [7] states that

$$\kappa(G) \geqslant \mu_2(G) \tag{1}$$

for any non-complete graph G. Fiedler called $\mu_2(G)$ the algebraic connectivity of G and his work stimulated a large amount of research in spectral graph theory over the last forty years (see [1,11,13,15]). In [12], Kirkland, Molitierno, Neumann and Shader characterize the equality case in Fiedler's inequality (1).

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Theorem 1.1 (Kirkland–Molitierno–Neumann–Shader [12]). *Let G be a non-complete connected graph on n vertices. Then $\kappa(G) = \mu_2(G)$ if and only if $G = G_1 \vee G_2$ where G_1 is a disconnected graph on $n - \kappa(G)$ vertices and G_2 is a graph on $\kappa(G)$ vertices with $\mu_2(G_2) \geqslant 2\kappa(G) - n$.*

Eigenvalue techniques have been also used recently by Brouwer and Koolen [5] to show that the vertex-connectivity of a distance-regular graph equals its degree (see also [2,4] for related results).

In this paper, we study the relations between the edge-connectivity and the second largest eigenvalue of a d-regular graph. Let $\lambda_1(G) \geqslant \lambda_2(G) \geqslant \ldots \geqslant \lambda_n(G)$ denote the eigenvalues of the adjacency matrix of G. If G is d-regular graph, then $\lambda_i(G) = d - \mu_i(G)$ for any $1 \leqslant i \leqslant n$. Thus, $\lambda_1(G) = d$ and G is connected if and only if $\lambda_2(G) < d$.

Chandran [6] proved that if G is a d-regular graph of order n and $\lambda_2(G) < d - 1 - \frac{d}{n-d}$, then $\kappa'(G) = d$ and the only disconnecting edge-cuts are trivial, i.e., d edges adjacent to the same vertex. Krivelevich and Sudakov [13] showed that $\lambda_2(G) \leqslant d - 2$ implies $\kappa'(G) = d$. If G is a d-regular graph on n vertices and $n \leqslant 2d + 1$, then $\kappa'(G) = d$ regardless of the eigenvalues of G (see also the proof of Lemma 1.3). Both results are based on the following well-known lemma (see [15] for a short proof).

Lemma 1.2. *If $G = (V, E)$ is a connected graph of order n and S is a subset of vertices of G, then*

$$e(S, V \setminus S) \geqslant \frac{\mu_2 |S|(n - |S|)}{n},$$

where $e(S, V \setminus S)$ denotes the number of edges between S and $V \setminus S$.

We extend and improve the previous results as follows. We prove the following sufficient condition for the k-edge-connectivity of a d-regular graph for any $2 \leqslant k \leqslant d$.

Theorem 1.3. *If $d \geqslant k \geqslant 2$ are two integers and G is a d-regular graph such that $\lambda_2(G) \leqslant d - \frac{(k-1)n}{(d+1)(n-d-1)}$, then $\kappa'(G) \geqslant k$.*

When $k = d$, this result states that if $\lambda_2(G) \leqslant d - \frac{(d-1)n}{(d+1)(n-d-1)}$, then $\kappa'(G) = d$. We get a small improvement for d even because $\kappa'(G)$ must be even in this case (see Lemma 3.1). When d is even, Theorem 1.3 shows that $\lambda_2(G) \leqslant d - \frac{(d-2)n}{(d+1)(n-d-1)}$ implies $\kappa'(G) = d$. A simple calculation reveals that these bounds improve the previous result of Chandran.

When $n \geqslant 2d + 2$, Theorem 1.3 implies that if G is a d-regular graph with $\lambda_2(G) \leqslant d - 2 + \frac{4}{d+1}$, then $\kappa'(G) = d$. When d is even, the right hand-side of the previous inequality can be replaced by $d - 2 + \frac{6}{d+1}$. These results improve the previous bound of Krivelevich and Sudakov. Note that there are many d-regular graphs G with $d - 2 < \lambda_2(G) \leqslant d - 2 + \frac{4}{d+1}$. An example is presented in Fig. 1.

For $k \in \{2, 3\}$ we further improve these results as shown below. Note that the edge-connectivity of a connected, regular graph of even degree is even (see Lemma 3.1).

Theorem 1.4. *Let $d \geqslant 3$ be an odd integer and $\rho(d)$ denote the largest root of $x^3 - (d-3)x^2 - (3d-2)x - 2 = 0$. If G is a d-regular graph such that $\lambda_2(G) < \rho(d)$, then $\kappa'(G) \geqslant 2$.*

The value of $\rho(d)$ is about $d - \frac{2}{d+5}$.

Theorem 1.5. *Let $d \geqslant 3$ be any integer. If G is a d-regular graph such that*

$$\lambda_2(G) < \frac{d - 3 + \sqrt{(d+3)^2 - 16}}{2},$$

then $\kappa'(G) \geqslant 3$.

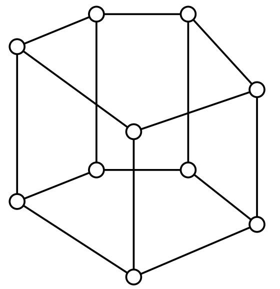

Fig. 1. A 3-regular graph with $3-2<\lambda_2=\frac{1+\sqrt{5}}{2}<3-2+\frac{4}{3+1}$.

The value of $\frac{d-3+\sqrt{(d+3)^2-16}}{2}$ is about $d-\frac{4}{d+3}$.

We show that Theorems 1.4 and 1.5 are best possible in the sense that for each odd $d \geqslant 3$, there exists a d-regular graph X_d such that $\lambda_2(X_d)=\rho(d)$ and $\kappa'(X_d)=1$. Also, for each $d \geqslant 3$, there exists a d-regular graph Y_d such that $\lambda_2(Y_d)=\frac{d-3+\sqrt{(d+3)^2-16}}{2}$ and $\kappa'(Y_d)=2$.

The following result is an immediate consequence of Theorems 1.4 and 1.5.

Corollary 1.6. *If G is a* 3*-regular graph and* $\lambda_2(G)<\sqrt{5}\approx 2.23$, *then* $\kappa'(G)=3$.
If G is 4*-regular graph and* $\lambda_2<\frac{1+\sqrt{33}}{2}\approx 3.37$, *then* $\kappa'(G)=4$.

Our results imply that if G is a 3-regular graph with $\lambda_2(G)<\sqrt{5}$, then $\kappa(G)=3$. This is because for 3-regular graphs the vertex- and the edge-connectivity are the same. It is very likely that the eigenvalues results for $\kappa(G)$ and $\kappa'(G)$ can be quite different. We comment on this in the last section of the paper.

2. Proof of Theorem 1.3

In this section, we give a short proof of Theorem 1.3. Recall that a partition $V_1\cup\cdots\cup V_l=V(G)$ of the vertex set of a graph G is called equitable if for all $1\leqslant i,j\leqslant l$, the number of neighbours in V_j of a vertex v in V_i is a constant b_{ij} independent of v (see Section 9.3 of [8] or [9] for more details on equitable partitions).

Proof of Theorem 1.3. We prove the contrapositive, namely we show that if G is a d-regular graph such that $\kappa'(G)\leqslant k-1$, then $\lambda_2(G)>d-\frac{(k-1)n}{(d+1)(n-d-1)}$. Let $V_1\cup V_2=V(G)$ be a partition of G such that $r=e(V_1,V_2)\leqslant k-1\leqslant d-1$. Note that if $|V_1|\leqslant d$, then $d-1\geqslant e(V_1,V_2)\geqslant |V_1|(d-|V_1|+1)\geqslant d$ which is a contradiction. Thus, $n_i:=|V_i|\geqslant d+1$ for each $i=1,2$. Since $n_1+n_2=n$, this implies that $n_1n_2\geqslant(d+1)(n-d-1)$. The quotient matrix of the partition $V_1\cup V_2=V(G)$ (see [8,9,10]) is

$$A_2=\begin{bmatrix} d-\frac{r}{n_1} & \frac{r}{n_1} \\ \frac{r}{n_2} & d-\frac{r}{n_2}\end{bmatrix}. \tag{2}$$

The eigenvalues of A_2 are d and $d-\frac{r}{n_1}-\frac{r}{n_2}$. Eigenvalue interlacing (see [8,9,10]), $r\leqslant k-1$ and $n_1n_2\geqslant(d+1)(n-d-1)$ imply that

$$\lambda_2(G)\geqslant d-\frac{r}{n_1}-\frac{r}{n_2}\geqslant d-\frac{(k-1)n}{n_1n_2}\geqslant d-\frac{(k-1)n}{(d+1)(n-d-1)}. \tag{3}$$

We actually have strict inequality here. Otherwise, the partition $V_1\cup V_2$ would be equitable. This would mean that each vertex of V_1 has the same number of neighbours in V_2 which is impossible since there are vertices in V_1 without a neighbour in V_2. The proof is finished.

3. Proof of Theorem 1.4

In this section, we present the proof of Theorem 1.4. First, we show that any connected, regular graph of even degree has edge-connectivity larger than two. This follows from the next lemma.

Lemma 3.1. *Let G be a connected d-regular graph. If d is even, then $\kappa'(G)$ is even.*

Proof. Let $V = A \cup B$ be a partition of the vertex set of G such that $e(A,B) = \kappa'(G)$, where $e(A,B)$ denotes the number of edges with one endpoint in A and one endpoint in B. Summing up the degrees of the vertices in A, we obtain that $d|A| = 2e(A) + e(A,B) = 2e(A) + \kappa'(G)$, where $e(A)$ denotes the number of edges with both endpoints in A. Since d is even, this implies $\kappa'(G)$ is even which finishes the proof. □

For the rest of this section, we assume that d is an odd integer with $d \geqslant 3$. We describe first the d-regular graph X_d having $\kappa'(X_d) = 1$ and $\lambda_2(X_d) = \rho(d)$.

If $H_1, \ldots, H_k$ are pairwise vertex-disjoint graphs, then we define the product $H_1 \vee H_2 \vee \cdots \vee H_k$ recursively as follows. If $k = 1$, the product is H_1. If $k = 2$, then $H_1 \vee H_2$ is the usual join of H_1 and H_2, i.e. the graph formed from the union of H_1 and H_2 by joining each vertex of H_1 to each vertex of H_2. If $k \geqslant 3$, then $H_1 \vee H_2 \vee \cdots \vee H_k$ is the graph obtained by taking the union of $H_1 \vee H_2 \vee \cdots \vee H_{k-1}$ and H_k and joining each vertex of H_{k-1} to each vertex of H_k.

The extremal graph X_d is defined as:

$$X_d = K_2 \vee \overline{M_{d-1}} \vee K_1 \vee K_1 \vee \overline{M_{d-1}} \vee K_2, \tag{4}$$

where $\overline{M_{d-1}}$ is the unique $(d-3)$-regular graph on $d-1$ vertices (the complement of a perfect matching M_{d-1} on $d-1$ vertices). Fig. 2 shows X_3 and Fig. 3 describes X_5.

The graph X_d is d-regular, has $2d+4$ vertices and its edge-connectivity is 1. It has an equitable partition of its vertices into six parts which is described by the following quotient matrix $\widetilde{X_d}$. The sizes of the parts in this equitable partition are $2, d-1, 1, 1, d-1, 2$.

$$\widetilde{X_d} = \begin{bmatrix} 1 & d-1 & 0 & 0 & 0 & 0 \\ 2 & d-3 & 1 & 0 & 0 & 0 \\ 0 & d-1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & d-1 & 0 \\ 0 & 0 & 0 & 1 & d-3 & 2 \\ 0 & 0 & 0 & 0 & d-1 & 1 \end{bmatrix}. \tag{5}$$

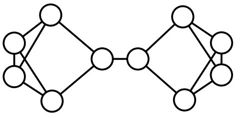

Fig. 2. X_3 is 3-regular with $\lambda_2(X_3) = \rho(3) \approx 2.7784$.

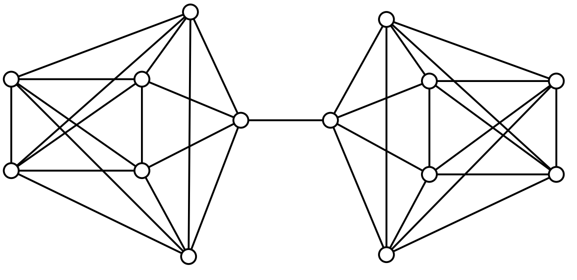

Fig. 3. X_5 is 5-regular with $\lambda_2(X_5) = \rho(5) \approx 4.7969$.

Next, we show that the second largest eigenvalue of X_d equals the second largest eigenvalue of $\widetilde{X_d}$. This will enable us to get sharp estimates for $\lambda_2(X_d)$ which will be used later in this section.

Lemma 3.2. *For each odd integer $d \geqslant 3$, we have that $\lambda_2(X_d) = \lambda_2(\widetilde{X_d}) = \rho(d)$.*

Proof. Since the partition with quotient matrix $\widetilde{X_d}$ is equitable, it follows that the spectrum of X_d contains the spectrum of $\widetilde{X_d}$. Using Maple, we determine the characteristic polynomial of $\widetilde{X_d}$:

$$P_{\widetilde{X_d}}(x) = (x-d)(x-1)(x+2)(x^3 - (d-3)x^2 - (3d-2)x - 2). \tag{6}$$

The roots of $P_{\widetilde{X_d}}$ are $d, 1, -2$ and the three roots of $Q(x) = x^3 - (d-3)x^2 - (3d-2)x - 2$. Denote by $\tilde{\lambda}_2 \geqslant \tilde{\lambda}_3 \geqslant \tilde{\lambda}_4$ the roots of $Q(x)$. Since these roots sum up to $d - 3 \geqslant 0$ and their product is 2, it follows that $\tilde{\lambda}_2$ is positive and both $\tilde{\lambda}_3$ and $\tilde{\lambda}_4$ are negative. Because $Q(1) = -4d + 4 < 0$, we deduce that $\tilde{\lambda}_2 > 1$. Thus, $\lambda_2(\widetilde{X_d}) = \tilde{\lambda}_2 = \rho(d)$.

Let $W \subset \mathbb{R}^{2d+4}$ be the subspace of vectors which are constant on each part of the six part equitable partition. The lifted eigenvectors corresponding to the six roots of $P_{\widetilde{X_d}}$ form a basis for W. The remaining eigenvectors in a basis of eigenvectors for X_d can be chosen to be perpendicular to the vectors in W. Thus, they may be chosen to be perpendicular to the characteristic vectors of the parts in the six-part equitable partition since these characteristic vectors form a basis for W. This implies that these eigenvectors will correspond to the non-trivial eigenvalues of the graph obtained as a disjoint union of $2K_2$ and $2\overline{M_{d-1}}$. The corresponding eigenvalues will be $(-2)^{(d-3)}, (-1)^{(2)}, 0^{(d-1)}$, where the exponent denotes the multiplicity of each eigenvalue. These $2d - 2$ eigenvalues together with the six roots of $P_{\widetilde{X_d}}$ form the spectrum of the graph X_d. Thus, $\lambda_2(X_d) = \lambda_2(\widetilde{X_d}) = \rho(d)$. □

Using Maple, we find that $\rho(3) \approx 2.7784$ and $\rho(5) \approx 4.7969$. A simple algebraic manipulation shows that $\rho(d)$ satisfies the equation

$$d - \rho(d) = \frac{2d-2}{\rho^2(d) + 3\rho(d) + 2} > \frac{2d-2}{d^2 + 3d + 2}, \tag{7}$$

where the last inequality follows since $\rho(d) < d$. Using (7), we deduce that

$$d - \frac{2}{d+4} < \rho(d) < d - \frac{2}{d+5} \tag{8}$$

for $d \geqslant 5$. We are now ready to prove Theorem 1.4.

Proof of Theorem 1.4. We will prove the contrapositive, namely we will show that if G is a d-regular graph with edge-connectivity 1 (containing a bridge), then $\lambda_2(G) \geqslant \lambda_2(X_d) = \rho(d)$. We also show that equality happens if and only if $G = X_d$.

Consider a connected d-regular graph G that contains a bridge x_1x_2. Deleting the edge x_1x_2 partitions $V(G)$ into two connected components G_1 and G_2 such that $x_i \in V(G_i)$ for $i = 1, 2$. Let $n_i = |V(G_i)|$ for $i = 1, 2$. Without loss of generality, we assume from now on that $n_1 \leqslant n_2$. The graph G_1 contains $n_1 - 1$ vertices of degree d and one vertex of degree $d - 1$. Because d is odd, this implies that n_1 is odd. By symmetry, n_2 is also odd. Thus, $n_2 \geqslant n_1 \geqslant d + 2$. Note that $G = X_d$ if and only if $n_1 = n_2 = d + 2$.

Let A_2 be the quotient matrix of the partition of $V(G)$ into $V(G_1)$ and $V(G_2)$. Then

$$A_2 = \begin{bmatrix} d - \frac{1}{n_1} & \frac{1}{n_1} \\ \frac{1}{n_2} & d - \frac{1}{n_2} \end{bmatrix}. \tag{9}$$

The eigenvalues of A_2 are: d and $\lambda_2(A_2) = d - \frac{1}{n_1} - \frac{1}{n_2}$.

Let A_3 be the quotient matrix of the partition of $V(G)$ into $V(G_1), \{x_2\}, V(G_2) \setminus \{x_2\}$. Then

$$A_3 = \begin{bmatrix} d-a & a & 0 \\ 1 & 0 & d-1 \\ 0 & b & d-b \end{bmatrix}, \tag{10}$$

where $a = \frac{1}{n_1}$ and $b = \frac{d-1}{n_2-1}$. The eigenvalues of A_3 are $d, \lambda_2(A_3) = \frac{d-a-b+\sqrt{(d-a+b)^2+4(a-b)}}{2}$, $\lambda_3(A_3) = \frac{d-a-b-\sqrt{(d-a+b)^2+4(a-b)}}{2}$. Taking partial derivatives with respect to a and to b respectively, we find that when $d > 1$, the eigenvalue $\lambda_2(A_3)$ is strictly monotone decreasing with respect to both a and b and so is strictly monotone increasing with respect to both n_1 and n_2.

We consider now a partition of $V(G)$ into the following six parts: $V(G_1) \setminus (x_1 \cup N(x_1)), N(x_1) \setminus \{x_2\}, \{x_1\}, \{x_2\}, N(x_2) \setminus \{x_1\}$ and $V(G_2) \setminus (x_2 \cup N(x_2))$. Here $N(u)$ denotes the neighbourhood of vertex u in G. Let e_1 denote the number of edges between $N(x_1) \setminus \{x_2\}$ and $V(G_1) \setminus (x_1 \cup N(x_1))$. Let e_2 denote the number of edges between $N(x_2) \setminus \{x_1\}$ and $V(G_2) \setminus (x_2 \cup N(x_2))$. Note that $e_i \leqslant (d-1)(n_i - d)$ for $i = 1, 2$.

Let A_6 be the quotient matrix of the previous partition of $V(G)$ into six parts. Then

$$A_6 = \begin{bmatrix} d - \frac{e_1}{n_1-d} & \frac{e_1}{n_1-d} & 0 & 0 & 0 & 0 \\ \frac{e_1}{d-1} & d-1-\frac{e_1}{d-1} & 1 & 0 & 0 & 0 \\ 0 & d-1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & d-1 & 0 \\ 0 & 0 & 0 & 1 & d-1-\frac{e_2}{d-1} & \frac{e_2}{d-1} \\ 0 & 0 & 0 & 0 & \frac{e_2}{n_2-d} & d-\frac{e_2}{n_2-d} \end{bmatrix}. \tag{11}$$

Eigenvalue interlacing (see [8,9,10]) implies that

$$\lambda_2(G) \geqslant \max(\lambda_2(A_2), \lambda_2(A_3), \lambda_2(A_6)). \tag{12}$$

We will use this as well as the inequalities (7) and (8) to show that $\lambda_2(G) \geqslant \lambda_2(X_d) = \rho(d)$ and that equality happens if and only if $G = X_d$.

Recall that $n_2 \geqslant n_1 \geqslant d+2$ and that d, n_1 and n_2 are all odd.

If $n_1 \geqslant d+6$ and $d \geqslant 5$, then we use the partition of $V(G)$ into two parts whose quotient matrix A_2 is given in (9). Using inequality (8), we have that

$$\lambda_2(G) \geqslant \lambda_2(A_2) = d - \frac{1}{n_1} - \frac{1}{n_2} \geqslant d - \frac{2}{d+6} > \rho(d).$$

If $n_1 \geqslant d+6$ and $d = 3$, then we use the partition of $V(G)$ into three parts whose quotient matrix A_3 is given in (10). We have that

$$\begin{aligned}\lambda_2(G) \geqslant \lambda_2(A_3) &\geqslant \frac{3 - \frac{1}{9} - \frac{2}{8} + \sqrt{\left(3 - \frac{1}{9} - \frac{2}{8}\right)^2 + 4\left(\frac{1}{9} - \frac{1}{4}\right)}}{2} \\ &= \frac{95 + \sqrt{12049}}{72} > 2.84 > \rho(3).\end{aligned}$$

If $n_2 \geqslant n_1 = d+4$, then we use the partition of $V(G)$ into three parts whose quotient matrix A_3 is shown in (10). We have that

$$\begin{aligned}\lambda_2(G) \geqslant \lambda_2(A_3) &\geqslant \frac{d - \frac{1}{d+4} - \frac{d-1}{d+3} + \sqrt{\left(d - \frac{1}{d+4} + \frac{d-1}{d+3}\right)^2 + 4\left(\frac{1}{d+4} - \frac{d-1}{d+3}\right)}}{2} \\ &= \frac{d^3 + 6d^2 + 8d + 1 + \sqrt{d^6 + 16d^5 + 88d^4 + 174d^3 + 8d^2 - 96d + 385}}{2(d^2 + 7d + 12)} \\ &= \frac{d^3 + 6d^2 + 8d + 1 + \sqrt{(d^3 + 8d^2 + 12d - 9)^2 + 8d^2 + 120d + 304}}{2(d^2 + 7d + 12)}.\end{aligned}$$

If $d = 3$, the right hand side of the previous inequality equals $\frac{106+\sqrt{16612}}{84} > 2.7962 > \rho(3)$. When $d \geqslant 5$, from the previous inequality and (8) we obtain that

$$\begin{aligned}\lambda_2(G) &\geqslant \frac{d^3+6d^2+8d+1+\sqrt{(d^3+8d^2+12d-9)^2+8d^2+120d+304}}{2(d^2+7d+12)}\\ &> \frac{d^3+6d^2+8d+1+(d^3+8d^2+12d-9)}{2(d^2+7d+12)} = \frac{d^3+7d^2+10d-4}{d^2+7d+12}\\ &= d-\frac{2d+4}{d^2+7d+12} > d-\frac{2}{d+5} > \rho(d).\end{aligned}$$

If $n_1 = d+2$ and $n_2 \geqslant d+6$, then we use the partition of $V(G)$ into three parts whose quotient matrix is is given in (10). We obtain that

$$\begin{aligned}\lambda_2(G) \geqslant \lambda_2(A_3) &\geqslant \frac{d-\frac{1}{d+2}-\frac{d-1}{d+5}+\sqrt{\left(d-\frac{1}{d+2}+\frac{d-1}{d+5}\right)^2+4\left(\frac{1}{d+2}-\frac{d-1}{d+5}\right)}}{2}\\ &= \frac{d^3+6d^2+8d-3+\sqrt{d^6+16d^5+80d^4+118d^3-24d^2+56d+329}}{2(d^2+7d+10)}.\end{aligned}$$

When $d=3$, the right hand side equals $\frac{102+\sqrt{14564}}{80} > 2.7835 > \rho(3)$. If $5 \leqslant d \leqslant 17$, then from the previous identity, we deduce that

$$\begin{aligned}\lambda_2(G) &\geqslant \frac{d^3+6d^2+8d-3+\sqrt{(d^3+8d^2+8d-5)^2-8d^2+136d+304}}{2(d^2+7d+10)}\\ &> \frac{d^3+6d^2+8d-3+(d^3+8d^2+8d-5)}{2(d^2+7d+10)}\\ &= \frac{d^3+7d^2+8d-4}{d^2+7d+10} = d-\frac{2}{d+5} > \rho(d).\end{aligned}$$

When $d \geqslant 19$, from the previous inequality we obtain that

$$\begin{aligned}\lambda_2(G) &\geqslant \frac{d^3+6d^2+8d-3+\sqrt{(d^3+8d^2+8d-6)^2+2d^3+8d^2+152d+293}}{2(d^2+7d+10)}\\ &> \frac{d^3+6d^2+8d-3+(d^3+8d^2+8d-6)}{2(d^2+7d+10)} = \frac{d^3+7d^2+8d-4.5}{d^2+7d+10}\\ &= d-\frac{2d+4.5}{d^2+7d+10} > d-\frac{2d-2}{d^2+3d+2} > \rho(d).\end{aligned}$$

The only case which remains to consider is $n_1 = d+2$ and $n_2 = d+4$. In this case, $e_1 = 2d-4$, $n_2 - d = 4$ and e_2 is an even integer with $4d-12 \leqslant e_2 \leqslant 4d-4$. Thus,

$$A_6 = \begin{bmatrix} 1 & d-1 & 0 & 0 & 0 & 0\\ 2 & d-3 & 1 & 0 & 0 & 0\\ 0 & d-1 & 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0 & d-1 & 0\\ 0 & 0 & 0 & 1 & d-1-\frac{e_2}{d-1} & \frac{e_2}{d-1}\\ 0 & 0 & 0 & 0 & \frac{e_2}{4} & d-\frac{e_2}{4} \end{bmatrix}.$$

Let $P_{A_6}(x)$ denote the characteristic polynomial of A_6. Since each row sum of A_6 is d, it follows that d is a root of $P_{A_6}(x)$. Thus, $P_{A_6}(x) = (x-d)P_5(x)$. The second largest root of $P_{A_6}(x)$ is the largest root of $P_5(x)$. Using the results of [3, p. 130], we observe that $P_5(x)$ equals the characteristic polynomial of the following matrix:

$$\begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 \\ 0 & d-1 & d-2 & d-1 & 0 \\ 0 & 0 & 1 & 0 & \frac{e_2}{d-1} \\ 0 & 0 & 0 & 1 & d-\frac{e_2}{d-1}-\frac{e_2}{n_2-d} \end{bmatrix}.$$

Using Maple to divide $P_5(x)$ by the polynomial $x^3+(3-d)x^2+(2-3d)x-2$, we obtain the quotient $Q(x)=x^2-\frac{4d^2-4d-de_2-3e_2}{4(d-1)}x-\frac{e_2}{d-1}$ and the remainder $R(x)=(2d-e_2-2)x+\frac{de_2+e_2+4d-4d^2}{2}$. Thus,

$$P_5(x)=(x^3+(3-d)x^2+(2-3d)x-2)Q(x)+(2d-e_2-2)x + \frac{de_2+4d+e_2-4d^2}{2}.$$

Because $\rho(d)$ satisfies the equation $x^3+(3-d)x^2+(2-3d)x-2=0$, it follows that

$$\begin{aligned} P_5(\rho(d)) &= (2d-e_2-2)\rho(d)+\frac{de_2+4d+e_2-4d^2}{2} \\ &= e_2\left(-\rho(d)+\frac{d+1}{2}\right)+(2d-2)(\rho(d)-d). \end{aligned}$$

Because $d>\rho(d)>d-1\geqslant\frac{d+1}{2}$, we deduce that

$$P_5(\rho(d))<0. \tag{13}$$

Assume that $\rho(d)\geqslant\lambda_2(A_6)$. Then $\rho(d)$ is larger than any root of $P_5(x)$. This implies that $P_5(\rho(d))=\prod_{\theta \text{ root of } P_5}(\rho(d)-\theta)\geqslant 0$ which is a contradiction with (13). Thus, $\lambda_2(G)\geqslant\lambda_2(A_6)>\rho(d)$.

Hence, $\lambda_2(G)>\rho(d)$ whenever $n_2\geqslant d+4$. The last case to be considered is $n_1=n_2=d+2$. This means that $G=X_d$ and $\lambda_2(G)=\rho(d)$ which finishes the proof.

4. Proof of Theorem 1.5

In this section we present the proof of Theorem 1.5. We describe first the d-regular graph Y_d having $\kappa'(Y_d)=2$ and $\lambda_2(Y_d)=\frac{d-3+\sqrt{(d+3)^2-16}}{2}$.

Consider the graph $H_d=K_{d-1}\vee\overline{K_2}$. It has $d-1$ vertices of degree d and two vertices of degree $d-1$. We construct Y_d by taking two disjoint copies of $K_{d-1}\vee\overline{K_2}$ and adding two disjoint edges between the vertices of degree $d-1$ in different copies of $K_{d-1}\vee\overline{K_2}$. Fig. 4 describes Y_3 and Fig. 5 shows Y_4.

The graph Y_d is d-regular, has $2d+2$ vertices and its edge-connectivity is 2. It has an equitable partition into four parts of sizes $d-1,2,2,d-1$ with the following quotient matrix:

$$\widetilde{Y_d}=\begin{bmatrix} d-2 & 2 & 0 & 0 \\ d-1 & 0 & 1 & 0 \\ 0 & 1 & 0 & d-1 \\ 0 & 0 & 2 & d-2 \end{bmatrix}. \tag{14}$$

The characteristic polynomial of this matrix equals

$$P_{\widetilde{Y_d}}(x)=(x-d)(x+1)(x^2+(3-d)x+(4-3d)).$$

Thus, the eigenvalues of $\widetilde{Y_d}$ are $d,1$ and $\frac{d-3\pm\sqrt{(d+3)^2-16}}{2}$. To simplify our notation, let $\theta(d)=\frac{d-3+\sqrt{(d+3)^2-16}}{2}$. Note that $\theta(d)$ is the largest root of

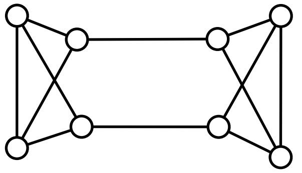

Fig. 4. Y_3 is 3-regular with $\lambda_2(Y_3) = \theta(3) = \sqrt{5}$.

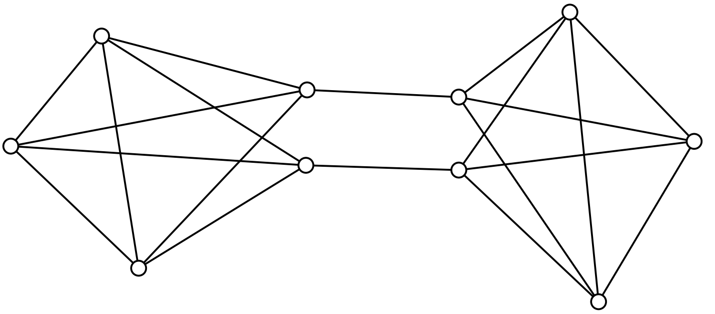

Fig. 5. Y_4 is 4-regular with $\lambda_2(Y_4) = \theta(4) = \frac{1+\sqrt{33}}{2}$.

$$T(x) = x^2 + (3-d)x + (4-3d), \tag{15}$$

and

$$d - \frac{4}{d+2} < \theta(d) < d - \frac{4}{d+3}. \tag{16}$$

Lemma 4.1. *The second largest eigenvalue of Y_d equals $\theta(d)$.*

Proof. Since the previous partition of $V(Y_d)$ into four parts is equitable, it follows that the four eigenvalues of $\widetilde{Y_d}$ are also eigenvalues of Y_d. Obviously, the second largest of the eigenvalues of $\widetilde{Y_d}$ is $\theta(d)$. By an argument similar to the one of Lemma 3.2, one can show that the other eigenvalues of Y_d are $(-1)^{(2d-2)}$ which implies the desired result. $\square$

We are ready now to prove Theorem 1.5.

Proof of Theorem 1.5. We will prove the contrapositive, namely we will show that among all d-regular graphs with edge-connectivity less than or equal to 2, the graph Y_d has the smallest λ_2.

Let G be a d-regular graph with $\kappa'(G) \leqslant 2$. We will prove that $\lambda_2(G) \geqslant \theta(d)$ with equality if and only if $G = Y_d$.

If $\kappa'(G) = 1$, then Theorem 1.4, (8) and (16) imply that $\lambda_2(G) \geqslant \rho(d) \geqslant d - \frac{2(d-1)}{(d+1)(d+2)} \geqslant d - \frac{4}{d+3} > \theta(d)$.

If $\kappa'(G) = 2$, then there exists a partition of $V(G)$ into two parts V_1 and V_2 such that $e(V_1, V_2) = 2$. Let $S_1 \subset V_1$ and $S_2 \subset V_2$ denote the endpoints of the two edges between V_1 and V_2. We have that $(|S_1|, |S_2|) \in \{(1,2),(2,2),(2,1)\}$. Let $n_i = |V_i|$ for $i \in \{1,2\}$. It is easy to see that $n_i \geqslant d+1$ for each $i \in \{1,2\}$. Note that $n_1 = n_2 = d+1$ is equivalent to $G = Y_d$.

Without loss of generality assume that $n_2 \geqslant n_1 \geqslant d+1$. If $n_1 \geqslant d+3$, consider the partition of $V(G)$ into V_1 and V_2. The quotient matrix of this partition is

$$\begin{bmatrix} d - \frac{2}{n_1} & \frac{2}{n_1} \\ \frac{2}{n_2} & d - \frac{2}{n_2} \end{bmatrix},$$

and its eigenvalues are d and $d-\frac{2}{n_1}-\frac{2}{n_2}$. Eigenvalue interlacing and $n_2 \geqslant n_1 \geqslant d+3$ imply that $\lambda_2(G) \geqslant d-\frac{2}{n_1}-\frac{2}{n_2} \geqslant d-\frac{4}{d+3}$. Using inequality (16), we obtain that $\lambda_2(G) \geqslant d-\frac{4}{d+3} > \theta(d)$ which finishes the proof of this case.

If $n_1 = d+2$, then we have a few cases to consider.

If $|S_1| = 2$, then $e(S_1) = 0$ or $e(S_1) = 1$. If $e(S_1) = 0$, consider the partition of $V(G)$ into three parts: $V_1 \setminus S_1, S_1, V_2$. The quotient matrix of this partition is

$$\begin{bmatrix} d-\frac{2d-2}{d} & \frac{2d-2}{d} & 0 \\ d-1 & 0 & 1 \\ 0 & \frac{2}{n_2} & d-\frac{2}{n_2} \end{bmatrix}.$$

Its characteristic polynomial equals

$$P_3(x) = (x-d)\left(x^2 - \frac{d^2n_2+2n_2-2dn_2-2d}{dn_2}x - \frac{2(d^2n_2+n_2+2-2dn_2-d)}{dn_2}\right).$$

If $P_2(x) = x^2 - \frac{d^2n_2+2n_2-2dn_2-2d}{dn_2}x - \frac{2(d^2n_2+n_2+2-2dn_2-d)}{dn_2}$, then

$$P_2(x) = T(x) + R(x),$$

where $R(x) = \frac{dn_2+2n_2-2d}{dn_2}\left(\frac{d^2n_2+2d-2n_2-4}{dn_2+2n_2-2d} - x\right)$. The expression $\frac{d^2n_2+2d-2n_2-4}{dn_2+2n_2-2d}$ is decreasing with n_2 and thus it attains its maximum when $n_2 = d+2$. This maximum equals $\frac{d^3+2d^2-8}{d^2+2d+4} = d - \frac{4(d+2)}{d^2+2d+4} < d - \frac{4}{d+2}$. Thus, $P_2(\theta(d)) = R(\theta(d)) < \frac{dn_2+2n_2-2d}{dn_2}\left(d-\frac{4}{d+2}-\theta(d)\right) < 0$ where the last inequality follows from (16). This fact and eigenvalue interlacing imply that $\lambda_2(G) > \theta(d)$.

If $e(S_1) = 1$, then we consider the same partition into three parts: $V_1 \setminus S_1, S_1, V_2$. The quotient matrix is the following

$$\begin{bmatrix} d-\frac{2d-4}{d} & \frac{2d-4}{d} & 0 \\ d-2 & 1 & 1 \\ 0 & \frac{2}{n_2} & d-\frac{2}{n_2} \end{bmatrix}.$$

Its characteristic polynomial equals

$$P_3(x) = (x-d)\left(x^2 - \frac{d^2n_2+4n_2-dn_2-2d}{dn_2}x - \frac{d^2n_2+4n_2+8-6dn_2}{dn_2}\right).$$

If $P_2(x) = x^2 - \frac{d^2n_2+4n_2-dn_2-2d}{dn_2}x - \frac{d^2n_2+4n_2+8-6dn_2}{dn_2}$ then

$$P_2(x) = T(x) + R(x),$$

where $R(x) = \frac{2(dn_2+2n_2-d)}{dn_2}\left(\frac{d^2n_2+dn_2-2n_2-4}{dn_2+2n_2-d} - x\right)$. The expression $\frac{d^2n_2+dn_2-2n_2-4}{dn_2+2n_2-d}$ decreases with n_2 and thus, its maximum is attained at $n_2 = d+2$. This maximum equals $\frac{d^3+3d^2-8}{d^2+3d+4} = d - \frac{4(d+2)}{d^2+3d+4} < d - \frac{4}{d+2}$. As before, the previous inequality and eigenvalue interlacing imply that $\lambda_2(G) > \theta(d)$.

If $|S_1| = 1$, then d must be even. Indeed, the subgraph induced by V_1 contains $d+1$ vertices of degree d and one vertex of degree $d-2$. This cannot happen when d is odd. Thus, d is even and $d \geqslant 4$. Consider the partition of G into three parts: $V_1 \setminus S_1, S_1, V_2$. The quotient matrix of this partition is

$$\begin{bmatrix} d-\frac{d-2}{d+1} & \frac{d-2}{d+1} & 0 \\ d-2 & 0 & 2 \\ 0 & \frac{2}{n_2} & d-\frac{2}{n_2} \end{bmatrix}.$$

Its characteristic polynomial is

$$P_3(x) = (x-d)\left(x^2 - \frac{d^2n_2+2n_2-2d-2}{(d+1)n_2}x - \frac{d^2n_2+2d+4n_2+8-4dn_2}{(d+1)n_2}\right).$$

If $P_2(x) = x^2 - \frac{d^2n_2+2n_2-2d-2}{(d+1)n_2}x - \frac{d^2n_2+2d+4n_2+8-4dn_2}{(d+1)n_2}$, then

$$P_2(x) = T(x) + R(x),$$

where $R(x) = \frac{2dn_2+5n_2-2d-2}{(d+1)n_2}\left(\frac{2d^2n_2+3dn_2-8n_2-2d-8}{2dn_2+5n_2-2d-2} - x\right)$. Because $d \geqslant 4$, the expression $\frac{2d^2n_2+3dn_2-8n_2-2d-8}{2dn_2+5n_2-2d-2}$ is decreasing with n_2 and thus, its maximum is attained when $n_2 = d+2$. This maximum equals $\frac{2d^3+7d^2-4d-24}{2d^2+7d+8} = d - \frac{12d+24}{2d^2+7d+8} < d - \frac{4}{d+2}$. This fact and eigenvalue interlacing imply $\lambda_2(G) > \theta(d)$.

Assume now that $n_1 = d+1$. This implies that V_1 induces a subgraph isomorphic to $K_{d-1} \vee \overline{K_2}$ and consequently, $|S_1| = 2$ and $e(S_1) = 0$.

If $n_2 \geqslant d+3$, then consider the partition of G into three parts: $V_1 \setminus S_1, S_1, V_2$. The quotient matrix of this partition is

$$\begin{bmatrix} d-2 & 2 & 0 \\ d-1 & 0 & 1 \\ 0 & \frac{2}{n_2} & d - \frac{2}{n_2} \end{bmatrix}.$$

Its characteristic polynomial is

$$P_3(x) = (x-d)\left(x^2 - \left(d - 2 - \frac{2}{n_2}\right)x + 2 + \frac{2}{n_2} - 2d\right).$$

If $P_2(x) = x^2 - \left(d - 2 - \frac{2}{n_2}\right)x + 2 + \frac{2}{n_2} - 2d$, then

$$P_2(x) = T(x) + \frac{dn_2 + 2 - 2n_2 - (n_2-2)x}{n_2},$$

which implies that

$$P_2(\theta(d)) = \frac{dn_2 + 2 - 2n_2 - (n_2-2)\theta(d)}{n_2}.$$

The expression $\frac{dn_2+2-2n_2-(n_2-2)\theta(d)}{n_2}$ is decreasing with n_2 and therefore, its maximum is attained when $n_2 = d+3$. Thus,

$$\begin{aligned} P_2(\theta(d)) &\leqslant \frac{d^2 + d - 4 - (d+1)\theta(d)}{d+3} \\ &= \frac{(d+1)\left(d - \frac{4}{d+1} - \theta(d)\right)}{d+3} < 0. \end{aligned}$$

This inequality and eigenvalue interlacing imply that $\lambda_2(G) > \theta(d)$.

Thus, the remaining case is $n_1 = d+1$ and $n_2 \leqslant d+2$. If d is odd, then $n_2 \neq d+2$. Otherwise, the sum of the degrees of the graph induced by V_2 would equal $d|V_2| - 2 = d(d+2) - 2$ which is an odd number. This is impossible and thus, $n_2 = d+1$. This implies $G = Y_d$.

If $n_2 = d+2$ and d is even, then we have a few cases to consider. If $|S_2| = 1$, then both vertices in S_1 are adjacent to the vertex a of S_2. Thus, a has exactly $d-2$ neighbours in V_2 which means there are 3 vertices of V_2 which are not adjacent to a. Each of these three vertices has degree d and the only way this can happen is they form a clique and each of them is adjacent to the $d-2$ neighbours of a in V_2. Using a degree argument, it also follows that the $d-2$ neighbours of a in V_2 induce a subgraph isomorphic to the complement of a perfect matching on $d-2$ vertices $\overline{M_{d-2}}$. Using the notation from the previous section, it follows that $G = K_{d-1} \vee \overline{K_2} \vee K_1 \vee \overline{M_{d-2}} \vee K_3$. The graph G has an obvious equitable partition into five parts whose quotient matrix is

$$\begin{bmatrix} d-2 & 2 & 0 & 0 & 0 \\ d-1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & d-2 & 0 \\ 0 & 0 & 1 & d-4 & 3 \\ 0 & 0 & 0 & d-2 & 2 \end{bmatrix}.$$

Its characteristic polynomial equals $P_5(x) = (x-d)P_4(x)$ where $P_4(x) = x^4 + (-d+4)x^3 + (-4d+4)x^2 - 8x + 6d - 12$. Dividing $P_4(x)$ by $T(x)$ we get that $P_4(x) = T(x)Q_4(x) + R_4(x)$ where $Q_4(x) = x^2 + x - 3$ and $R_4(x) = -3x - 3d$. It follows that $P_4(\theta(d)) = R_4(\theta(d)) = -3\theta(d) - 3d < 0$. This fact and eigenvalue interlacing imply that $\lambda_2(G) > \theta(d)$.

The case remaining is $|S_2| = 2$. If $e(S_2) = 0$, then each vertex of S_2 has exactly $d-1$ neighbours in V_2. Let $S_2 = \{x_2, y_2\}$. If x_2 and y_2 are adjacent to the same $d-1$ vertices of V_2, then because $|V_2| = n_2 = d+2$, there is exactly one vertex of V_2 outside the vertices of S_2 and their $d-1$ common neighbours. This vertex cannot have degree d. Thus, this case cannot happen and therefore, both x_2 and y_2 have exactly $d-2$ common neighbours in V_2. We call this set U. Let $\{u_2, w_2\} = V_2 \setminus (\{x_2, y_2\} \cup U)$. Because x_2 and y_2 have $d-1$ neighbours in V_2, we may assume that x_2 is adjacent to u_2 and y_2 is adjacent to w_2. A degree argument implies that u_2 and w_2 must be adjacent and both of them are adjacent to each vertex of U. Finally, the subgraph induced by U must be $(d-4)$-regular.

The following is a five-part equitable partition of G: $V_1 \setminus S_1, S_1, S_2, U, \{u_2, w_2\}$. The quotient matrix of this partition is the following:

$$\begin{bmatrix} d-2 & 2 & 0 & 0 & 0 \\ d-1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & d-2 & 1 \\ 0 & 0 & 2 & d-4 & 2 \\ 0 & 0 & 1 & d-2 & 1 \end{bmatrix}.$$

Its characteristic polynomial equals $P_5(x) = (x-d)P_4(x)$ where $P_4(x) = x^4 + (5-d)x^3 + (10-5d)x^2 + (-6d+7)x - d$. Dividing $P_4(x)$ by $T(x)$ we get that $P_4(x) = T(x)Q_4(x) + R_4(x)$ where $Q_4(x) = x^2 + 2x$ and $R_4(x) = -x - d$. It follows that $P_4(\theta(d)) = R_4(\theta(d)) = -\theta(d) - d < 0$. This and eigenvalue interlacing show that $\lambda_2(G) > \theta(d)$.

The final case of our proof is $e(S_2) = 1$. Because $|V_2 \setminus S_2| = d$ and both x_2 and y_2 have $d-2$ neighbours in $V_2 \setminus S_2$, it follows that x_2 and y_2 have at least $d-4$ common neighbours in $V_2 \setminus S_2$. If x_2 and y_2 have at least $d-3$ common neighbours in $V_2 \setminus S_2$, we deduce that there exists at least one vertex z of $V_2 \setminus S_2$ that is not adjacent to x_2 nor y_2. The vertex z cannot have degree d since its only possible neighbours are in $V_2 \setminus (S_2 \cup \{z\})$ which has size $d-1$. We conclude that x_2 and y_2 must have precisely $d-4$ common neighbours in $V_2 \setminus S_2$. There are 4 remaining vertices a_2, b_2, c_2, d_2 in $V_2 \setminus S_2$ and without loss of generality, assume that x_2 is adjacent to both a_2 and b_2 and y_2 is adjacent to both c_2 and d_2. Each of the vertices a_2, b_2, c_2, d_2 will be adjacent to every other vertex of $V_2 \setminus S_2$. The degree constraint implies that the remaining $d-4$ vertices of $V_2 \setminus S_2$ induce a $(d-6)$-regular subgraph of G. Obviously, this argument shows this case is only possible when $d \geqslant 6$.

The following is a five-part equitable partition of G: $V_1 \setminus S_1, S_1, S_2, \{a_2, b_2, c_2, d_2\}, V_2 \setminus (S_2 \cup \{a_2, b_2, c_2, d_\}\})$. Its quotient matrix is the following:

$$\begin{bmatrix} d-2 & 2 & 0 & 0 & 0 \\ d-1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & d-4 \\ 0 & 0 & 1 & 3 & d-4 \\ 0 & 0 & 2 & 4 & d-6 \end{bmatrix}.$$

Its characteristic polynomial equals $P_5(x) = (x-d)P_4(x)$ where $P_4(x) = x^4 + (-d+4)x^3 + (-4d+6)x^2 + (-2d-1)x + 3d - 6$. Dividing $P_4(x)$ by $T(x)$ we get that $P_4(x) = T(x)Q_4(x) + R_4(x)$ where $Q_4(x) = x^2 + x - 1$ and $R_4(x) = -2x - 2$. It follows that $P_4(\theta(d)) = R_4(\theta(d)) = -2\theta(d) - 2 < 0$. This and eigenvalue interlacing imply that $\lambda_2(G) > \theta(d)$ which finishes our proof. $\square$

5. Some remarks

Any strongly regular graph of degree $d \geqslant 3$ satisfies the condition $\lambda_2 \leqslant d - 2$ and thus, is d-edge-connected. The fact that the edge-connectivity of a strongly regular graph equals its degree, was observed by Plesńik in 1975 (cf. [2]). As mentioned in the introduction, much more is true, namely the vertex-connectivity of any distance-regular graph equals its degree (see [5]). It is known that any vertex transitive d-regular graph whose second largest eigenvalue is simple has $\lambda_2(G) \leqslant d - 2$ and consequently, is d-edge-connected. In fact, any vertex transitive d-regular graph is d-edge-connected as shown by Mader in 1971 (see [14] or Chapter 3 of [8]).

I expect that Theorems 1.4 and 1.5 can be extended to other values of edge-connectivity and vertex-connectivity. For example, it seems that $\lambda_2(G) \leqslant d - \frac{1}{2}$ implies $\kappa(G) \geqslant 2$. Note however that in many cases, Fiedler's bound $\kappa(G) \geqslant d - \lambda_2$ cannot be improved. When $2k - 2 \geqslant d$ and dk is even, consider the graph $2K_{d-k+1} \vee H$ where H is a $(2k - 2 - d)$-regular graph on k vertices. This graph is d-regular, has vertex connectivity k and its second largest eigenvalue equals $d - k$.

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