A small note on symmetric geodesic curvature on $D^2$ ✩

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Abstract
Some new results are obtained for the problem of prescribing geodesic curvature $k$ on $D$ when $k$ possesses some kinds of symmetries.
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1. Introduction and main results

Let $(D, g_0)$ be the unit disk with the Euclidean metric $g_0$. Given a continuous function $k(x)$ on $S^1 = \partial D$, we want to find a condition on $k(x)$ so that there is a flat metric $g$, which is pointwise conformal to the standard metric $g_0$, i.e., $g = e^{2u} g_0$, for some function $u$ defined on $D$ such that the geodesic curvature $k_g = k$.

This problem is equivalent to the existence of a solution to the following equation:

$$\begin{cases}
-\Delta u = 0 & \text{in } D, \\
\frac{\partial u}{\partial n} + 1 = ke^u & \text{on } \partial D,
\end{cases}$$

(1)

where $\frac{\partial u}{\partial n}$ is the outer normal derivative of $u$.

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This is an analogue of the well-known Nirenberg’s problem, i.e., what kinds of functions $K$ can be the Gaussian curvature of a metric $g$ on $S^2$, which is pointwise conformal to the standard metric $g_0$? The latter has been studied extensively, see [2,4–7,9–11,13].

Our Eq. (1) is quite similar to that of Nirenberg’s problem. There are Kazdan–Warner type conditions on Eq. (1) as follows: if $u$ is a solution of Eq. (1), then

$$\int_{S^1} k' e^u = 0,$$

where $k'$ denotes the tangential derivative of $k$. And hence our Eq. (1) may be insolvable for general $k$.

In this present work we are interested in the case when $k$ possesses some kinds of symmetries. Let $G$ be a subgroup of the orthogonal transformation group in $\bar{D}$, and let $f_G := \{x \in S^1 \mid g \cdot x = x, \forall g \in G\}$, the set of fixed points on $\partial D = S^1$ under the action of $G$.

The following results are known [3,8]: one can solve Eq. (1) if

1. $f_G = \emptyset$, so in particular if $k(x) = k(-x)$ and $k(x) > 0$ somewhere (Moser type theorem);
2. if $G$ is the unit group and if $k > 0 \in C^2$ and $\mu_0 \neq \mu_1 + 1$, where $\mu_0$ and $\mu_1$ are the numbers of local maxima and local minima of $k$ in the region $\Omega = \{z \in \partial D \mid k(z) > 0, \hat{k}'(z) > 0\}$ and $\hat{k}$ denotes the conjugate function of $k$.

Closely related to the above problem is the following Moser–Trudinger type inequality (Lebedev–Milin inequality [12]):

$$\int_{S^1} e^u ds \leq 2\pi \exp \left\{ \frac{1}{4\pi} \int_{S^1} u \frac{\partial u}{\partial n} ds + \int_{S^1} u ds \right\}, \forall u \in H^{1/2}(S^1),$$

where $\int_{S^1} u$ denotes the mean value of $u$.

Note that the case (1) $f_G = \emptyset$ occurs when the group $G$ is generated by the rotation $\theta = 2\pi/m$, where $m \in \mathbb{N}$ is a natural number, and the Lebedev–Milin inequality can be improved for this kind of functions, which plays a key role to the proof of the above result.

The more interesting points lie in the case that $f_G \neq \emptyset$, in which the usual Moser–Trudinger type inequality cannot be improved: for instance, we consider a family of functions $u_\lambda(x) = \ln \frac{1}{1 + 2\lambda \cos \theta + \lambda^2}$, where $x = e^{i\theta} \in S^1$ and $\lambda \in (-1, 1)$ is some real parameter. Note that $u_\lambda$ satisfies the condition $u(e^{-i\theta}) = u(e^{i\theta})$ and it is easy to check that only the usual Lebedev–Milin inequality holds for this family of functions $u_\lambda$.

Let $x = e^{i\theta} \in S^1$, where $\theta \in [-\pi, \pi]$. Note that any given function $u : \partial D \to \mathbb{R}$ has the harmonic extension (still denoted by $u$) which is uniquely determined by its boundary value.

In order to study Eq. (1), we use variational method. Let $H^l(S^1)$ be the Sobolev space, $l \geq 0$. We set

$$C_\theta^{\infty}(S^1) = \{u \in C^{\infty}(S^1) \mid u(e^{-i\theta}) = u(e^{i\theta})\},$$

$$H^{1/2}_\theta(S^1) = \text{the closure of } C_\theta^{\infty} \text{ in } H^{1/2}(S^1).$$

Consider the functional

$$I(u) = \frac{1}{2} \int_{S^1} \frac{\partial u}{\partial n} u ds + \int_{S^1} u ds, \forall u \in H^{1/2}(S^1),$$
and set $\mu = \inf I(u)$ for all $u \in C_\theta^\infty(S^1)$ satisfying $\int_{S^1} ke^u = 2\pi$.

**Theorem 1.1.** If $k \in C_\theta^\infty(S^1)$ and $\max(k(1), k(-1)) > 0$, where $x = 1$ ($x = -1$) corresponds $\theta = 0$ ($\theta = \pi$), respectively, then

$$\mu \leq 2\pi \ln \frac{1}{\max(k(1), k(-1))}.$$  

Moreover, if

$$\mu < 2\pi \ln \frac{1}{\max(k(1), k(-1))},$$

then Eq. (1) has a solution $u \in C_\theta^\infty(S^1)$.

**Remark 1.** This result resembles that of Aubin [1] for the Yamabe problem and the one of C. Hong [7].

**Theorem 1.2.** Suppose that $k \in C_\theta^\infty(S^1)$, $\max(k(1), k(-1)) \leq 0$ and $k(x) > 0$ somewhere. Then Eq. (1) has a solution $u \in C_\theta^\infty(S^1)$.

**Corollary 1.3.** Suppose that $k \in C_\theta^\infty(S^1)$, $k(x) > 0$ somewhere and

$$\bar{k} := \frac{1}{2\pi} \int_{S^1} k \geq \max(k(1), k(-1)).$$

Then Eq. (1) has a solution $u \in C_\theta^\infty(S^1)$.

2. Proofs of existence results

Given $k \in C_\theta^\infty(S^1)$, let $\{u_n\}$ be a minimizing sequence in $C_\theta^\infty(S^1)$, i.e.,

$$I(u_n) \rightarrow \mu \quad \text{and} \quad \int_{S^1} ke^{u_n} = 2\pi, \quad \forall n \in \mathbb{N}. \quad (2)$$

**Lemma 2.1.** If there exists a constant $C > 0$ such that

$$\|\nabla u_n\|^2 := \int_{S^1} \frac{\partial u_n}{\partial n} u_n \leq C, \quad \forall n \in \mathbb{N}, \quad (3)$$

then Eq. (1) has a solution $u \in C_\theta^\infty(S^1)$.

**Proof.** A similar arguments proceed as in Aubin [1, Section 5]: indeed, the functional $I$ satisfies (PS) condition in $I(u) \leq C$. From the conditions $I(u_j) \leq C$ and $\int_{S^1} ke^{u_j} = 2\pi$, it follows that $\|u_j\|_{1/2} \leq 3C$, i.e., a (PS) sequence $u_j$ is bounded in $H^{1/2}(S^1)$. On the other hand, from

$$\frac{\partial u_j}{\partial n} + 1 = \lambda_j e^{u_j} + o(1)$$

we have $\lambda_j \rightarrow 1$. By using the Sobolev trace inequality, which says $\{e^{u_j}\}$ is compact in $L^2(S^1)$, we obtain a subsequence of $u_j$ converging in $H^{1/2}$. This proves the lemma. \qed
Thus we are led to find a condition to ensure (3), so we can prove that Eq. (1) has a solution.

**Lemma 2.2.** If \( u \in C_0^\infty(S^1) \) and \( \exists 0 < \delta \leq \pi/2 \) and \( c_1, c_2 \in \mathbb{R}^+ \), \( \delta \leq |\theta_0| \leq \pi \) such that

\[
I(u) \leq c_1 \quad \text{and} \quad u(e^{i\theta_0}) \geq c_2,
\]

then

\[
\|\nabla u\|_2 \leq C(\delta, c_1, c_2), \tag{4}
\]

where \( C = C(\delta, c_1, c_2) \) depends only on \( \delta, c_1, c_2 \).

**Proof.** By a modified version of the Poincaré inequality, we have, for all \( v \in C_0^\infty(S^1) \) with \( v(e^{i\theta_0}) = 0 \) for some \( \theta_0: |\theta_0| \geq \delta \), that the following holds:

\[
\int_{S^1} |v|^2 \leq C(\delta) \int_{S^1} \frac{\partial v}{\partial n} v \, ds.
\]

Hence we have

\[
c_1 \geq \frac{1}{2} \int_{S^1} \frac{\partial u}{\partial n} u \, ds + \int_{S^1} u \, ds \\
\geq \frac{1}{2} \int_{S^1} \frac{\partial u}{\partial n} u \, ds + 2\pi c_2 + \int_{S^1} (u - u(e^{i\theta_0})) \, ds
\]

\[
\geq \frac{1}{2} \int_{S^1} \frac{\partial u}{\partial n} u \, ds + 2\pi c_2 - C(\delta) \left( \frac{1}{2} \int_{S^1} \frac{\partial u}{\partial n} u \, ds \right)^{1/2},
\]

which implies the desired inequality (4). \( \square \)

**Proof of Theorem 1.1.** (1) Without loss of generality, assume that \( k(1) \geq k(-1) \) and \( k(1) > 0 \). We set

\[
u_\lambda = \ln \frac{(1 - \lambda^2)}{k(1)(1 - 2\lambda \cos \theta + \lambda^2)}, \quad \lambda \to 1^-.
\]

We have

\[
\int_{S^1} ke^{u_\lambda} = \frac{1 - \lambda^2}{k(1)} \int_{-\pi}^{\pi} \frac{k(e^{i\theta}) \, d\theta}{(1 - 2\lambda \cos \theta + \lambda^2)}
\]

\[
= (1 - \lambda^2) \int_{-\pi}^{\pi} \frac{d\theta}{(1 - 2\lambda \cos \theta + \lambda^2)} + \frac{(1 - \lambda^2)}{k(1)} \int_{-\pi}^{\pi} \frac{k(e^{i\theta}) - k(1) \, d\theta}{(1 - 2\lambda \cos \theta + \lambda^2)}
\]

\[
= 2\pi + \frac{(1 - \lambda^2)}{k(1)} \left[ \int_{[-\pi, \pi] \setminus \{\theta \leq \delta(\epsilon)\}} \frac{k(e^{i\theta}) - k(1) \, d\theta}{(1 - 2\lambda \cos \theta + \lambda^2)} \\
+ \int_{|\theta| \leq \delta(\epsilon)} \frac{k(e^{i\theta}) - k(1) \, d\theta}{(1 - 2\lambda \cos \theta + \lambda^2)} \right]
\]

\[
= 2\pi + \epsilon(\lambda), \quad \text{where} \ \epsilon(\lambda) \to 0 \ \text{as} \ \lambda \to 1^-,
\]
and

\[
\frac{1}{2} \int_{S^1} \frac{\partial u_\lambda}{\partial n} u_\lambda \, ds + \int_{S^1} u_\lambda \, ds = 2\pi \ln \frac{1}{k(1)},
\]

which imply the first conclusion of Theorem 1.1.

(2) If \( \mu < 2\pi \ln \frac{1}{\max(k(1), k(-1))} \), consider the minimizing sequence \( \{u_n\} \subset C^{\infty}_\theta(S^1) \) satisfying (2). Note that \( \forall \epsilon > 0, \exists \delta > 0 \) such that \( k(e^{i\theta}) \leq k(1) + \epsilon \) if \( |\theta| \leq \delta \). Suppose that \( \int_{S^1} \frac{\partial u_n}{\partial n} u_n \, ds \to +\infty \) as \( n \to +\infty \). Then by Lemma 2.2 we have \( u_n(e^{i\theta}) \to -\infty \) uniformly in \( |\theta| \geq \delta \) as \( n \to \infty \). Thus by the Lebedev–Milin inequality we have

\[
2\pi = \int_{S^1} ke^{u_n} \leq \eta_n + (k(1) + \epsilon) \int_{S^1} e^{u_n} \leq \eta_n + (k(1) + \epsilon) 2\pi \exp \left\{ \frac{1}{4\pi} \int_{S^1} \frac{\partial u_n}{\partial n} u_n \, ds + \frac{1}{2\pi} \int_{S^1} u_n \right\},
\]

where

\[
\eta_n = \int_{\delta \leq |\theta| \leq \pi} ke^{u_n} \to 0.
\]

Hence

\[
I(u_n) \geq 2\pi \ln \frac{2\pi - \eta_n}{2\pi (k(1) + \epsilon)}.
\]

Since \( \eta_n \) and \( \epsilon > 0 \) can be arbitrarily small, we get \( \mu \geq 2\pi \ln(1/k(1)) \), a contradiction. Therefore there exists a subsequence, still denoted by \( \{u_n\} \), such that \( \int_{S^1} \frac{\partial u_n}{\partial n} u_n \, ds \leq C \). Then by Lemma 2.1, (1) has a solution \( u \in C^{\infty}_\theta(S^1) \).

**Proof of Theorem 1.2.** Case 1. \( \max(k(1), k(-1)) < 0 \). By the continuity of \( k \), \( \exists \delta > 0 \) such that \( k(e^{i\theta}) \leq 0 \) if \( |\theta| \leq \delta \). Considering the minimizing sequence \( \{u_n\} \) as above, we have

\[
2\pi = \int_{S^1} ke^{u_n} \leq \int_{|\theta| \geq \delta} ke^{u_n} \leq 2\pi \max k(x) \cdot \exp \left\{ \max_{|\theta| \geq \delta} u_n \right\},
\]

that means

\[
\max_{|\theta| \geq \delta} u_n(e^{i\theta}) \geq \ln \frac{1}{\max k}.
\]

Thus, by Lemmas 2.2 and 2.1 we obtain a solution \( u \in C^{\infty}_\theta(S^1) \) of Eq. (1).

Case 2. \( \max(k(1), k(-1)) = 0 \). Again, consider the minimizing sequence \( \{u_n\} \subset C^{\infty}_\theta(S^1) \). Assuming that \( \int_{S^1} \frac{\partial u_n}{\partial n} u_n \, ds \to +\infty \) as \( n \to \infty \), we proceed as in the proof of Theorem 1.1, (2) and get \( I(u_n) \to \infty \), a contradiction. Then by Lemma 2.1, Eq. (1) has a solution \( u \in C^{\infty}_\theta(S^1) \).

**Proof of Corollary 1.3.** Case 1. \( \max(k(1), k(-1)) \leq 0 \). The conclusion follows directly from Theorem 1.2.
Case 2. \( \bar{k} > \max(k(1), k(-1)) > 0 \). We set \( w = \ln(1/\bar{k}) \). Then \( \int_{S^1} ke^w = 2\pi \) and
\[
\mu \leq \frac{1}{2} \int_{S^1} \frac{\partial w}{\partial n} w \, ds + 2 \int_{S^1} w = 2\pi \ln \frac{1}{\bar{k}} < 2\pi \ln \frac{1}{\max(k(1), k(-1))}.
\]
Thus Corollary 1.3 follows from Theorem 1.1.

Case 3. \( \bar{k} = \max(k(1), k(-1)) > 0 \). If Eq. (1) has no solution, then by Theorem 1.1,
\[
\mu = 2\pi \ln \frac{1}{\max(k(1), k(-1))} = 2\pi \ln \frac{1}{\bar{k}}.
\]
But \( w = \ln 1/\bar{k} \) achieves the infimum \( \mu \), so we obtain a contradiction.

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References

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