Note

Saturation of convergence for $q$-Bernstein polynomials in the case $q \geq 1$

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Received 24 January 2007
Available online 18 April 2007
Submitted by M. Milman

Abstract

In the note, we discuss Voronovskaya type theorem and saturation of convergence for $q$-Bernstein polynomials for a function analytic in the disc $U_R := \{z : |z| < R\}$ ($R > q$) for arbitrary fixed $q \geq 1$. We give explicit formulas of Voronovskaya type for the $q$-Bernstein polynomials for $q > 1$. We show that the rate of convergence for the $q$-Bernstein polynomials is $o(q^{-n})$ ($q > 1$) for infinite number of points having an accumulation point on $U_R/q$ if and only if $f$ is linear.

Keywords: $q$-Bernstein polynomials; Voronovskaya type formulas; Saturation

1. Introduction

Let $q > 0$. For each nonnegative integer $k$, the $q$-integer $[k]$ and the $q$-factorial $[k]!$ are defined by

$$[k] := \begin{cases} 
(1 - q^k)/(1 - q), & q \neq 1, \\
1, & q = 1,
\end{cases}$$

and

$$[k]! := \begin{cases} 
[k][k-1] \cdots [1], & k \geq 1, \\
1, & k = 0,
\end{cases}$$

respectively. For the integers $n, k, n \geq k \geq 0$, the $q$-binomial coefficients are defined by (see [3, p. 12])

$$\binom{n}{k} := \frac{[n]!}{[k]![n-k]!}.$$

In 1997, Phillips proposed the $q$-Bernstein polynomials $B_{n,q}(f, x)$: for each positive integer $n$ and $f \in C[0, 1]$, the $q$-Bernstein polynomial of $f$ is (see [9])

\[\text{Supported by the Beijing Natural Science Foundation (Project No. 1062004) and by the National Natural Science Foundation of China.}\]
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\begin{equation}
B_{n,q}(f, x) := \sum_{k=0}^{n} f\left(\frac{[k]}{[n]}\right) \binom{n}{k} x^k \prod_{s=0}^{n-k-1} (1 - q^s x).
\tag{1.1}
\end{equation}

Note that for \( q = 1 \), \( B_{n,q}(f, x) \) is the classical Bernstein polynomial \( B_n(f, x) \),

\[ B_n(f, x) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1 - x)^{n-k}. \]

In recent years, the \( q \)-Bernstein polynomials have been investigated intensively by a number of authors (see [2, 4–10] and reference therein, [11–14]). From these researches we know that for \( q \neq 1 \), the convergence properties of the \( q \)-Bernstein polynomials differ essentially from those of the classical ones. In the case \( q > 1 \), the \( q \)-Bernstein polynomials are no longer positive operators, however, for a function analytic in a disc \( R/q \), and reference therein, \([11–14]\). From these researches we know that for \( q > 1 \), the convergence properties of the \( q \)-Bernstein polynomials differ essentially from those of the classical ones. In the case \( q > 1 \), the \( q \)-Bernstein polynomials are no longer positive operators, however, for a function analytic in a disc \( R/q \), the rate of convergence of \( \{B_{n,q}(f, z)\} \) to \( f(z) \) has the order \( q^{-n} \) (versus \( 1/n \) for the classical Bernstein polynomials). Note that the condition of analyticity of \( f \) is essential for convergence. In this note, we consider Voronovskaya type formulas and saturation of convergence of the \( q \)-Bernstein polynomials for such a function for arbitrary fixed \( q \geq 1 \). Let \( \Omega \) be a region in the complex plane \( \mathbb{C} \). Denote by \( H(\Omega) \) the space of all analytic functions on \( \Omega \). We say \( f_n \to f \) in \( H(\Omega) \) as \( n \to \infty \) if \( f_n \in H(\Omega) \) and the sequence \( \{f_n(z)\} \) converges to the limit function \( f(z) \) in \( \Omega \) as \( n \to \infty \), uniformly on every compact subset of \( \Omega \). The expression \( A(n) \propto B(n) \) means that there exists a positive constant \( c \) independent of \( n \) such that \( \frac{1}{c} B(n) \leq A(n) \leq c B(n) \); \( A(n) = o(B(n)) \) represents \( \lim_{n \to \infty} A(n)/B(n) = 0 \). For fixed \( q > 1 \), denote the \( q \)-derivative \( D_q f(z) \) of \( f \) by

\[ D_q f(z) = \begin{cases} \frac{f(qz) - f(z)}{(q-1)z}, & z \neq 0, \\ f'(0), & z = 0. \end{cases} \]

In the cases \( q = 1 \) and \( q \in (0, 1) \), the Voronovskaya type formulas and saturation of convergence for the \( q \)-Bernstein polynomials was obtained by Voronovskaya and the author, respectively, see [1, pp. 307–308], [14]. In the case \( q \geq 1 \), Ostrovska showed the following Voronovskaya type theorem for monomials (see [4]): for any \( m \in \mathbb{N}, z \in \mathbb{C} \),

\[ \lim_{n \to \infty} [n] B_{n,q}(m, z) - z^m = (1 + [2] + \cdots + [m - 1])(z^{m-1} - z^m). \tag{1.2} \]

In the note, we study Voronovskaya type formulas of the \( q \)-Bernstein polynomials of a function \( f \) analytic in the disc \( U_R \) \( (R > q) \) for fixed \( q \geq 1 \). Let \( R > q \geq 1 \) and let \( f \in H(U_R) \). For \( |z| < R/q \), we set

\[ L_q(f, z) := \frac{(1 - z)(D_q f(z) - f'(z))}{(q-1)} \quad \text{for } q > 1, \tag{1.3} \]

and for \( q = 1 \),

\[ L_1(f, z) = f''(z)(1 - z)/2. \tag{1.4} \]

Then we have the following Voronovskaya type theorem.

**Theorem 1.** Let \( R > q \geq 1 \). If a function \( f \) is analytic in the disc \( U_R \), then for any \( r, 0 < r < R/q \),

\[ \lim_{n \to \infty} [n](B_{n,q}(f, z) - f(z)) = L_q(f, z) \tag{1.5} \]

uniformly on the disc \( U_r \).

**Remark 1.** The above result is sharp in the following sense: the number \( R/q \) in Theorem 1 cannot be replaced by any other number strictly larger than \( R/q \), since for some points on \( \{z: |z| = R/q\} \), \( L_q(f, z) \) may be even undefined, let alone (1.5).

From Theorem 1 we conclude that for \( q \geq 1 \), \([n](B_{n,q}(f, z) - f(z)) \to L_q(f, z) \) in \( H(U_{R/q}) \) and therefore, \( L_q(f, z) \in H(U_{R/q}) \). Furthermore, we have the following saturation of convergence for the \( q \)-Bernstein polynomials for fixed \( q > 1 \).
Theorem 2. Let \( R > q > 1 \). If a function \( f \) is analytic in the disc \( U_R \), then \(|B_{n,q}(f,z) - f(z)| = o(q^{-n})\) for infinite number of points having an accumulation point on \( U_{R/q} \) if and only if \( f \) is linear.

It was proved in [4] that for \( R > q > 1 \) and \( f \in H(U_R) \), the rate of convergence of \( \{B_{n,q}(f,z)\} \) to \( f(z) \) on \( U_r \) (\( r < R/q \)) has the order \( q^{-n} \). The following corollary is the immediate consequence of Theorem 2.

Corollary 1. Let \( R > q > 1 \). If \( f \in H(U_R) \) is not a linear function, then for any \( r, 0 < r < R/q \),

\[
\sup_{|z| \leq r} |B_{n,q}(f,z) - f(z)| = q^{-n}; \\
\sup_{x \in [0,1]} |B_{n,q}(f,x) - f(x)| = q^{-n}.
\]

Remark 2. For \( q = 1 \), we have the following saturation: if \( f \in H(U_R) \), \( R > 1 \), then \(|B_{n,q}(f,z) - f(z)| = o(1/n)\) for infinite number of points having an accumulation point on the disc \( U_R \) if and only if \( f \) is linear. Even this result is possibly new.

The next theorem shows that \( L_q(f,x) \), \( q \geq 1 \), is continuous about the parameter \( q \) for \( f \in H(U_R) \), \( R > 1 \).

Theorem 3. Let \( R > 1 \) and let \( f \in H(U_R) \). Then for any \( r, 0 < r < R \),

\[
\lim_{q \to 1+} L_q(f,z) = L_1(f,z)
\]

uniformly on \( U_r \).

Corollary 2. Let \( q \geq 1 \). If \( f(z) \) is an entire function, then

\[
[n](B_{n,q}(f,z) - f(z)) \to L_q(f,z) \text{ in } H(\mathbb{C}) \text{ as } n \to \infty,
\]

and

\[
L_q(f,z) \to L_1(f,z) \text{ in } H(\mathbb{C}) \text{ as } q \to 1+.
\]

2. Proofs of Theorems 1–3

Lemma 1. (See [4].) Let \( q \geq 1 \) be fixed. Then for \( m \geq 2 \),

\[
B_{n,q}(t^m,z) = \alpha_1 z + \cdots + \alpha_j z^j, \quad j = \min(m,n),
\]

where \( \alpha_i \geq 0 \) (\( i = 1, \ldots, j \)) and \( \alpha_1 + \cdots + \alpha_j = 1 \). Besides, if \( n \geq m \), then

\[
\alpha_m = \prod_{i=1}^{m-1} \left( 1 - \frac{i}{n} \right), \quad \alpha_{m-1} = \frac{1 + [2] + \cdots + [m-1]}{n} \prod_{i=1}^{m-2} \left( 1 - \frac{i}{n} \right).
\]

Also, for any \( r \geq 1 \),

\[
[n]B_{n,q}(t^m,z) - z^m | \leq 2(m-1)[m-1]r^m \text{ for } |z| \leq r.
\]

Lemma 2. Let \( a_1, \ldots, a_k \in (0, 1) \). Then

\[
1 - \prod_{i=1}^{k} (1 - a_i) \leq \sum_{i=1}^{k} a_i
\]

and

\[
1 - \prod_{i=1}^{k} (1 - a_i) - \sum_{i=1}^{k} a_i \leq \sum_{1 \leq i < j \leq k} a_i a_j.
\]
Then by the assumption, we have

\[ 1 - \prod_{i=1}^{k+1} (1 - a_i) = 1 - \prod_{i=1}^{k} (1 - a_i) + a_{k+1} \prod_{i=1}^{k} (1 - a_i) \leq \sum_{i=1}^{k} a_i + a_{k+1} = \sum_{i=1}^{k+1} a_i, \]

which proves (2.4). Similarly, by the assumption and (2.4) we get

\[ |a_{k+1}| \leq \sum_{i=1}^{k} a_i a_j + a_{k+1} \sum_{i=1}^{k} a_i = \sum_{1 \leq i < j \leq k+1} a_i a_j, \]

which completes the proof of (2.5). Lemma 2 is proved.

**Lemma 3.** Let \( q \geq 1 \) be fixed. If \( n \geq m \geq 2 \) and \( r \geq 1 \), then for any \( z, |z| \leq r \),

\[
|[n](B_{n,q}(r^m, z) - z^m) - (1 + [2] + \cdots + [m - 1])(z^{m-1} - z^m)| \leq \frac{4(m - 1)^2[m - 1]^2}{n}r^m. \tag{2.6}
\]

**Proof.** It follows from (2.1) and (2.2) that for \( |z| \leq r \),

\[
I := |[n](B_{n,q}(r^m, z) - z^m) - (1 + [2] + \cdots + [m - 1])(z^{m-1} - z^m)| \\
\leq r^m [n] \sum_{i=1}^{m-1} a_i + r^m |[n] \alpha_{m-1} - \sum_{i=1}^{m-1} [i]| + r^m |[n](1 - \alpha_m) - \sum_{i=1}^{m-1} [i]| \\
\leq r^m [n](1 - \alpha_m - \alpha_{m-1}) + r^m |[n] \alpha_{m-1} - \sum_{i=1}^{m-1} [i]| + r^m |[n](1 - \alpha_m) - \sum_{i=1}^{m-1} [i]| \\
\leq 2r^m |[n] \alpha_{m-1} - \sum_{i=1}^{m-1} [i]| + 2r^m |[n](1 - \alpha_m) - \sum_{i=1}^{m-1} [i]| \\
= 2r^m \left( \sum_{i=1}^{m-1} [i] \right) \left( 1 - \prod_{i=1}^{m-2} \left( 1 - \frac{[i]}{[n]} \right) \right) + 2r^m [n] \sum_{1 \leq i < j \leq m-1} \frac{[i][j]}{[n][n]} \leq \frac{4(m - 1)^2[m - 1]^2}{n}r^m.
\]

Using (2.4) and (2.5) we get

\[
I \leq 2r^m \left( \sum_{i=1}^{m-1} [i] \right) \left( \sum_{i=1}^{m-2} [i] \right) + 2r^m [n] \sum_{1 \leq i < j \leq m-1} \frac{[i][j]}{[n][n]} \leq \frac{4(m - 1)^2[m - 1]^2}{n}r^m.
\]

Lemma 3 is proved.

**Proof of Theorem 1.** Let \( f(z) = \sum_{m=0}^{\infty} a_m z^m \) be a function analytic in the disc \( U_R, R > q \geq 1 \). We set

\[
V_q(f, z) := \sum_{m=2}^{\infty} a_m \left( \sum_{i=1}^{m-1} [i] \right) z^{m-1}(1 - z), \quad \text{for } |z| < R/q. \tag{2.7}
\]

It is easy to show that \( V_q(f, z) \in H(U_{R/q}) \). Let \( r \in [1, R/q) \) be fixed. First we show the sequence \( \{[n](B_{n,q}(f, z) - f(z))\} \) converges to \( V_q(f, z) \) uniformly on the disc \( U_r \), as \( n \to \infty \). By (2.1) we have

\[
B_{n,q}(f, z) = \sum_{m=0}^{\infty} a_m B_{n,q}(r^m, z) \quad \text{for } |z| < R.
\]
Since \( B_{n,q}(f,x) \) reproduce linear functions (see [9]), we get for \( |z| < R/q \),
\[
[n](B_{n,q}(f,z) - f(z)) - V_q(f,z) = \sum_{m=2}^{\infty} a_m \left[ n(B_{n,q}(t^m,z) - z^m) - \sum_{i=1}^{m-1} [i]z^{m-1}(1-z) \right].
\] (2.8)

Let \( \varepsilon > 0 \) be given. Choose \( t \in (0,1) \) such that \( q^{1+t} < R \). Since \( f \in H(U_R) \) and \( q^{1+t} < R \), we get \( \sum_{m=2}^{\infty} |a_m| m^4 q^{(1+t)m_r} < \infty \), so we can find \( N = N_{\varepsilon} \) such that \( \sum_{m=N}^{\infty} |a_m| m^2 q^m r_m < \varepsilon/8 \). Then for \( |z| \leq r \) and \( n > N \), by (2.8), (2.6) and (2.3) we have
\[
J := \left| n(B_{n,q}(f,z) - f(z)) - V_q(f,z) \right|
\leq \sum_{m=2}^{N-1} |a_m| \left[ n(B_{n,q}(t^m,z) - z^m) - \sum_{i=1}^{m-1} [i]z^{m-1}(1-z) \right]
\leq \frac{4}{|n|^2} \sum_{m=2}^{N-1} |a_m| m^2 |m-1|^{1+t} r_m + 4 \sum_{m=N}^{\infty} |a_m| m rm + \varepsilon/2.
\]

Since \( |n|^2 \to \infty \) as \( n \to \infty \) and \( \sum_{m=2}^{\infty} |a_m| m^4 q^{(1+t)m_r} < \infty \), we get \( J < \varepsilon \) for \( n \) sufficiently large. We conclude that
\[
\lim_{n \to \infty} [n](B_{n,q}(f,z) - f(z)) = V_q(f,z)
\]
uniformly on \( U_r \).

Now we show that \( L_q(f,z) = V_q(f,z) \). If \( q = 1 \), then for \( |z| < R \),
\[
V_q(f,z) = \sum_{m=2}^{\infty} a_m (1 + \cdots + m-1)z^{m-1}(1-z) = \frac{z(1-z)}{2} \sum_{m=2}^{\infty} a_m m(m-1)z^{m-2} = \frac{f''(z)}{2} z(1-z).
\]

For \( q > 1 \), it is easy to see that \( D_q(f,z) = \sum_{m=1}^{\infty} a_m[m]z^{m-1} \) and \( D_q(f,z) - f'(z) = \sum_{m=2}^{\infty} a_m [m]z^{m-1} \). Hence
\[
L_q(f,z) = \sum_{m=2}^{\infty} a_m \frac{[m] - m}{q-1} z^{m-1}(1-z), \quad q > 1.
\]

On the other hand, since
\[
1 + 2 + \cdots + [m] = \frac{[m] - m}{q-1},
\]
we obtain
\[
V_q(f,z) = \sum_{m=2}^{\infty} a_m \frac{[m] - m}{q-1} z^{m-1}(1-z) = L_q(f,z).
\]

The proof of Theorem 1 is complete. \( \square \)
Proof of Theorem 2. If \( f \) is linear, then \( B_{n,q}(f,z) - f(z) = 0 \) for any \( z \in \mathbb{C} \). Conversely, let \( f(z) = \sum_{m=0}^{\infty} a_m z^m \) be a function analytic in the disc \( UR \) and let \( |B_{n,q}(f,z) - f(z)| = o(q^{-m}) \) for infinite number of points having an accumulation point on \( UR/q \). Then by Theorem 1, we get \( L_q(f,z) = \lim_{n \to \infty} |n(B_{n,q}(f,z) - f(z))| = 0 \) for infinite number of points having an accumulation point on \( UR/q \). Since \( L_q(f,z) \in H(UR/q) \), by the Unicity Theorem for analytic functions we get \( L_q(f,z) = V_q(f,z) = 0 \), and therefore, by (2.7), \( a_m = 0, m = 2, 3, \ldots \). Thus, \( f \) is linear. Theorem 2 is proved. \( \square \)

Proof of Theorem 3. Let \( f(z) = \sum_{m=0}^{\infty} a_m z^m \) be a function analytic in the disc \( UR, R > 1 \). For any \( r \in [1, R) \), let \( q_0 \in (1, R/r) \) be fixed. Then for any \( q \in [1, q_0) \) and \( |z| \leq r \),

\[
L_q(f,z) = \sum_{m=2}^{\infty} a_m \left( \sum_{i=1}^{m-1} [i] \right) z^{m-1} (1 - z)
\]

and

\[
L_1(f,z) = \sum_{m=2}^{\infty} a_m \frac{(m - 1)m}{2} z^{m-1} (1 - z).
\]

Let \( \varepsilon > 0 \) be given. Since \( f \in H(UR) \), we get \( \sum_{m=2}^{\infty} |a_m|m^2 q_0^m r^m < \infty \), so we can find \( N = N_\varepsilon \) such that \( \sum_{m=N}^{\infty} |a_m|m^2 q_0^m r^m < \varepsilon/8 \). Using the inequality

\[
\left| \sum_{i=1}^{m-1} [i] - \frac{(m - 1)m}{2} \right| = \sum_{i=2}^{m-1} ((i) - i) = (q - 1) \sum_{i=2}^{m-1} [j] \leq (q - 1)m^2 \leq (q - 1)m^3 q_m,
\]

we get for \( |z| \leq r \) and \( q \in [1, q_0) \),

\[
K := \left| L_q(f,z) - L_1(f,z) \right| \\
\leq \sum_{m=2}^{N-1} |a_m| \left| \sum_{i=1}^{m-1} [i] - \frac{(m - 1)m}{2} \right| |z^{m-1} - z^m| + \sum_{m=N}^{\infty} |a_m| \left( \sum_{i=1}^{m-1} [i] + \frac{(m - 1)m}{2} \right) |z^{m-1} - z^m| \\
\leq 2(q - 1) \sum_{m=2}^{N-1} |a_m|m^3 q_0^m r^m + 4 \sum_{m=N}^{\infty} |a_m|(m - 1)[m - 1]r^m \\
\leq 2(q - 1) \sum_{m=2}^{N-1} |a_m|m^3 q_0^m r^m + 4 \sum_{m=N}^{\infty} |a_m|m^2 q_0^m r^m \\
\leq 2(q - 1) \sum_{m=2}^{N-1} |a_m|m^3 q_0^m r^m + \varepsilon/2.
\]

Since \( \sum_{m=2}^{\infty} |a_m|m^3 q_0^m r^m < \infty \), we get \( J < \varepsilon \) for \( q \) sufficiently close to 1 from the right. We conclude that

\[
\lim_{q \to 1^+} L_q(f,z) = L_1(f,z)
\]

uniformly on \( UR \). The proof of Theorem 3 is finished. \( \square \)

Acknowledgments

The authors are very grateful to the anonymous referees for many valuable comments and suggestions which helped to improve the draft.

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