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TORSIONFREE MODULES OVER ONE-DIMENSIONAL NOETHERIAN RINGS

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Introduction

Let R be a reduced one-dimensional Noetherian ring with finite integral closure $\bar{R}$, and conductor C and let $\Phi(A)$ denote the set of isomorphism classes of finitely generated torsionfree R-modules. In a paper of Wiegand [7], an operation of the group of units $(\bar{R}/C)^*$ on the set $\Phi(A)$ was defined. This was applied to problems related to the cancellation of direct summands from torsionfree modules, especially when R is a Bass ring, that is a ring with finite integral closure such that each ideal of R can be generated by two elements. The object of this note is to obtain a similar group action in the absence of the finite integral closure assumption and to derive some consequences of this action. We define such a group action in Section two, and look at some of the applications in Sections three and four. It turns out that trying to remove the finite integral closure assumption leads naturally to a division of the cancellation problem into two cases (that of cancelling projectives and of cancelling nonprojectives), which is enlightening even in the case that R has finite integral closure. This allows extensions of several of the results of [3] and [7] from Bass rings to more general reduced one-dimensional Noetherian rings. Many of our results reduce to those in [3] and [7] whenever R has the property that stably isomorphic finitely generated torsionfree R-modules are isomorphic.

In Section three we generalize some of the results in [7] on when $A \oplus C \cong B \oplus C \Rightarrow A \cong B$ for finitely generated R-modules A, B and C with A torsionfree. Some of the history of this problem and the related problems which are considered in Section four is discussed in [3] and [6]. In Section four we extend some of the results of [3] on power cancellation over Bass rings to more general one-dimensional Noetherian rings. To be more specific let $A^{(q)}$ denote the direct sum of q copies of A, and consider the following statements:

(a) $A^{(q)} \cong B^{(q)}$ for some $q \geq 1$;

(b) $A \oplus X \cong B \oplus X$ for some finitely generated R-module X;

(c) $A_m \cong B_m$ for each maximal ideal m of R. In [3], it was shown that if R is a Bass ring, then (b)$\Rightarrow$(a) for all $A, B \in \Phi(R)$ if and only if $D(\bar{R}) = \ker(\mathrm{Pic}(R) \to \mathrm{Pic}(\bar{R}))$ is a torsion group, and (c)$\Rightarrow$(a) for all $A, B \in \Phi(R)$ if and only if Pic(R) is a torsion group. Information was also obtained on the size of the exponent q required in (a) in terms of the degree of torsion in $D(\bar{R})$ and Pic(R) respectively. We obtain similar results for a wider class of Noetherian one-dimensional rings. In [3] Levy and Wiegand associated to a finitely generated R-module A an ideal cl(A), called the *ideal class* of A, such that cl(A) equals the determinant of A if A is projective. We conclude this note by giving a result on the behavior of the group action defined in Section two on the ideal class of an element $A \in \Phi(R)$.

1. Preliminary results

First we recall the definitions. In this paper R always denotes a reduced one-dimensional commutative Noetherian ring with total quotient ring K. Recall that if M is a finitely generated torsionfree R-module, then M is said to be *torsionfree* if the canonical map $M \to M \otimes_R K$ is injective. If S is an overring of R and M is a finitely generated torsionfree R-module, we denote by SM the S-submodule of $M \otimes_R K$ generated by the canonical image of M in $M \otimes_R K$. Since R is reduced and Noetherian we can write $K = Ke_1 \oplus \cdots \oplus Ke_n$ where the e_i are idempotents and the Ke_i are fields. We call these e_i the *fundamental idempotents* of R.

Lemma 1.1. *Let R be a reduced noetherian ring and let A and B be finitely generated torsionfree R-modules. If $\bar{R}A \cong \bar{R}B$, then there exists a finite overring S of R such that $SA \cong SB$.*

Proof. Let $f: \bar{R}A \to \bar{R}B$ be an isomorphism and let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ generate A and B respectively over R. For $i = 1, \ldots, n$ let $t_{i1}, t_{i2}, \ldots, t_{in} \in \bar{R}$ be such that $f(a_i) = \sum_j t_{ij} b_j$. Then if $S_0 = R[t_{ij} \mid i, j = 1, \ldots, n]$, then $f(SA) \subseteq SB$ for every overring S such that $S_0 \subseteq S \subseteq \bar{R}$. Let $S_1 = S_0[r_{ij} \mid i, j = 1, \ldots, n]$ where the $r_{ij} \in \bar{R}$ are such that $b_i = \sum_j r_{ij} f(a_j)$ for $i = 1, \ldots, n$. Then f induces an isomorphism from S_1A to S_1B. □

Corollary 1.2. *Let R be a reduced one-dimensional Noetherian ring and let A be a finitely generated torsionfree R-module. Then there exists a finite overring S of R such that SA is S-projective.*

Proof. Let $R_1 = R[e_1, \ldots, e_n]$ where the e_i are the fundamental idempotents of R. By the above lemma it suffices to find a projective R_1-module B such that $\bar{R}A \cong \bar{R}B$. Since R_1 is a direct product of integral domains it suffices to consider the case that R is a domain. We can then write $\bar{R}A = I_1 \oplus \cdots \oplus I_t$ where each I_j is an invertible ideal of $\bar{R}$. If S is a finite overring of R_1 with conductor C, then from the Mayer–Vietoris sequence of the following square we see that $\mathrm{Pic}(R_1) \to \mathrm{Pic}(S)$ is onto.

$$\begin{array}{ccc} R_1 & \longrightarrow & S \\ \downarrow & & \downarrow \\ R_1/C & \longrightarrow & S/C \end{array}$$

It follows that $\mathrm{Pic}(R_1) \to \mathrm{Pic}(\bar{R})$ is also onto and thus there are projective ideals J_i of R_1 such that $I_i \cong J_i\bar{R}$ for each i. Let $B = J_1 \oplus \cdots \oplus J_t$. □

2. A group operation

Let R be a one-dimensional reduced Noetherian ring with finite integral closure $\bar{R}$ and conductor C. In [7] Wiegand defined an action of the group of units $(\bar{R}/C)^*$ or $\bar{R}/C$ on the set $\Phi(R)$ of isomorphism classes of finitely generated torsionfree R-modules, and some interesting properties and applications of this action were obtained. In this section we extend this construction. First we remove the requirement that R have finite integral closure by replacing $\bar{R}$ by a finite overring S which depends on the module A under discussion. Then we consider the problem of putting the constructions obtained from the various finite overrings of R together to get a group operation on $\Phi(R)$. Much of the presentation in this section and the next parallels the presentation in [7], to which we refer the reader for many of the details.

Definition 2.1. Let A be a finitely generated torsionfree R-module and S a finite overring of R with conductor C. The following pullback diagram is called the *standard pullback diagram* for A with respect to S:

$$\begin{array}{ccc} A & \longrightarrow & SA \\ \downarrow & & \downarrow \\ A/CA & \longrightarrow & SA/CA \end{array}$$

Lemma 2.2. *Let R be a one-dimensional reduced Noetherian ring, S a finite overring of R and let $C = (R:_R S)$. Let P be a torsionfree S-module, M an R/C-module, and $j: M \to P/CP$ an R-monomorphism whose image generates P/CP as an S-module. Denote by $[M, P, j]$ the R-module A defined by the following pullback*:

$$\begin{array}{ccc} A & \longrightarrow & P \\ \downarrow & & \downarrow \\ M & \longrightarrow & P/CP \end{array}$$

Then $[M, P, j]$ is a torsionfree R-module and the above pullback diagram is isomorphic to the standard pullback diagram for A. Thus each torsionfree R-module is of the form $[M, P, j]$ for some M, P, and j as above.

Proof. The same argument as in [7] works with the exception of the proof that β is a monomorphism. That β is a monomorphism follows from the fact that $h: A \to SA$ is essential. □

Corollary 2.3. *If R is as above and $f: A \to B$ is a homomorphism of finitely generated R-modules, then f is an isomorphism if and only if for some (and thus for every) finite overring S of R such that SA, SB are S-projective the induced maps $A/CA \to B/CB$ and $SA \to SB$ are isomorphisms, where C is the conductor of S into R.* □

The following construction, with $S = \bar{R}$, was used by Wiegand [7] in the case that $\bar{R}$ is a finite R-module with conductor C to define an operation of the group of units $(\bar{R}/C)^*$ on the set of isomorphism classes of finitely generated torsionfree R-modules.

Let A be a finitely generated torsionfree R-module and let S be a finite overring of R containing the fundamental idempotents of K such that SA is a projective S-module. Let $C = R:_R S$ and let e be the idempotent of S that generates $(0:_S SA)$. Then the decomposition $Se \oplus S(1-e)$ gives decompositions of the groups S^*, and $(S/C)^*$ via $x \in S^* \to x = (ex + (1-e))(e + (1-e)x)$ etc. For $x \in (S/C)^*$ define A^x as follows. Choose an automorphism φ of SA/CA whose determinant is $x^A = e + (1-e)x$. This is possible since SA/CA is a projective S/C-module. Define A^x by the pullback

$$\begin{array}{ccccc} A^x & & \longrightarrow & & SA \\ \downarrow & & & & \downarrow \\ A/CA & \longrightarrow & SA/CA & \xrightarrow{\varphi} & SA/CA \end{array}$$

To simplify our presentation we will restrict our attention to faithful modules. By choosing our finite overrings S to contain the fundamental idempotents of R the results extend without difficulty to nonfaithful torsionfree modules. The proofs of the following three results are similar to the arguments given in [7] and [8]. In these results we let A be a finitely generated torsionfree R-module, S a finite overring of R such that SA is S-projective, $C = (R:_R S)$, and $x \in (S/C)^*$.

Lemma 2.4. *The isomorphism class of the module A^x does not depend on the choice of the automorphism φ with determinant x.* □

Lemma 2.5. *The following statements are equivalent:*
(i) $A^x \cong A$;
(ii) $x = \bar{u}v$ *where* $\bar{u} \in (S/C)^*$ *lifts to* $u \in S^*$ *and* v *is the determinant of an automorphism of* SA/CA *which carries* A/CA *into itself;*
(iii) *There exists* $\vartheta \in \mathrm{End}_R(A)$ *such that the induced map* $\bar{\vartheta} \in \mathrm{End}_R(SA/CA)$ *is an isomorphism with* $\det(\bar{\vartheta})\bar{u} = x$ *for some* $u \in S^*$. □

Proposition 2.6. *The following statements hold:*
(i) $\bar{R} \otimes_R A^x \cong \bar{R} \otimes_R A$;
(ii) $(A^x)_M \cong A_M$ *for every maximal ideal* M *of* R;
(iii) *If* SB *is* S*-projective, then* $A^x \oplus B \cong (A \oplus B)^{x^A}$;
(iv) $A^{xy} \cong (A^x)^y$. □

Let $\Phi(R)$ denote the set of isomorphism classes of finitely generated faithful torsionfree R-modules. The above procedure would be sufficient to allow extensions, to the case that the integral closure $\bar{R}$ of R is not a finite R-module, of most of the applications in [3, 6–8] of Wiegand's action of $(\bar{R}/C)^*$ on $\Phi(R)$. Our next objective however, is to turn the constructions for the various finite overrings of R into a group action.

Lemma 2.7. *Let $S \subseteq S_1$ be finite overrings of R such that SA is S-projective and let C, C_1 be the conductors of S and S_1 respectively to R. Let $u \in (S/C_1)^*$ have images $\bar{u}$ and u_1 in S/C and S_1/C_1 respectively. Then $A^{\bar{u}} \simeq A^{u_1}$.*

Proof. Let $SA = I \oplus P$ where I is an invertible ideal of S. Let A^u be defined to make the left square below a pullback. The right square is also a pullback and thus the large square is a pullback.

$$\begin{array}{ccccc}
A^u & \longrightarrow & I \oplus P & \longrightarrow & S_1 I \oplus S_1 P \\
\downarrow & & \downarrow & & \downarrow \\
A/C_1A \longrightarrow I/C_1I \oplus P/C_1P & \longrightarrow & I/C_1I \oplus P/C_1P & \longrightarrow & S_1I/C_1I \oplus S_1P/C_1P
\end{array}$$

But the bottom and right side of the large square are the same as in the pullback diagram which defines A^{u_1}. Thus $A^u \cong A^{u_1}$. Similarly, the two smaller squares in the following diagram are pullbacks; so the large one is also.

$$\begin{array}{ccccc}
A^u & \longrightarrow & & & I\oplus P \\
\downarrow & & & & \downarrow \\
A/C_1A & \longrightarrow & I/C_1I\oplus P/C_1P & \longrightarrow & I/C_1I\oplus P/C_1P \\
\downarrow & & & & \downarrow \\
A/CA & \longrightarrow & I/CI\oplus P/CP & \longrightarrow & I/CI\oplus P/CP
\end{array}$$

Thus $A^u \cong A^{\bar{u}}$. □

It follows that the 'partial operations' of $(R/C)^*$ on $\Phi(R)$, where C ranges over the conductors of the finite overrings of R, induce an operation of $\varprojlim(R/C)^*$ on $\Phi(R)$. The following result, which is a straightforward generalization of the main result of [8], shows that the orbits of this operation are the stable isomorphism classes of elements of $\Phi(R)$.

Theorem 2.8. *Let A and B be finitely generated torsionfree R-modules. The following statements are equivalent*:

(i) *There exists a finite overring S of R with conductor C such that SA, SB are S-projective and an $x\in(R/C)^*$ such that $A^x\cong B$*;

(ii) $A\oplus R\cong B\oplus R$;

(iii) $A\oplus R^n\cong B\oplus R^n$ *for some* $n\geq 0$. □

If S is an overring of R let $D(S)=\ker(\mathrm{Pic}(R)\to\mathrm{Pic}(S))$. Since the groups $(S/C)^*$, S a finite overring of R with conductor C, are not conveniently connected together by homomorphisms, to try to obtain an operation on $\Phi(R)$ we switch to the groups $D(S)=\ker(\mathrm{Pic}(R)\to\mathrm{Pic}(S))$. For certain rings this yields an action of $D(\bar{R})=\bigcup D(S)$ on $\Phi(R)$, where the union is over all finite overrings S of R. To get a well-defined action for all one-dimensional reduced Noetherian rings however we must allow $D(\bar{R})$ to operate on a set of equivalence classes of $\Phi(R)$, namely the set of orbits of the operation of $\varprojlim(R/C)^*$. To be more precise let $\langle A\rangle$ denote the isomorphism class of a finitely generated torsionfree R-module A. We write $\langle A\rangle\sim\langle B\rangle$ if $A\oplus R^n\cong B\oplus R^n$ for some n. This is an equivalence relation on $\Phi(R)$. Let $\bar{\Phi}(R)$ denote the set of equivalence classes, and let $[A]$ denote the equivalence class of A in $\bar{\Phi}(R)$. The operation of $D(\bar{R})$ on $\bar{\Phi}(R)$ is now defined as follows. If S is an integral overring of R and I is an invertible ideal of R such that $IS\cong S$, let $[I]$ denote the image of I in $D(S)=\ker(\mathrm{Pic}(R)\to\mathrm{Pic}(S))$.

If $[A]\in\bar{\Phi}(R)$ and $[I]\in D(\bar{R})$, then $[I]\in D(S)$ for some finite overring S of R such that SA is S-projective. Let $C=R:_RS$. From the exactness of the Mayer–Vietoris sequence

$$R^*\to S^*\times(R/C)^*\to(S/C)^*\to\mathrm{Pic}(R)$$

$$\to\mathrm{Pic}(S)\times\mathrm{Pic}(R/C)\to\mathrm{Pic}(S/C)$$

we get that $I=[R/C, S, u]$ for some $u \in (S/C)^*$, and that I represents zero in $\mathrm{Pic}(R)$ if and only if $u=\bar{x}y$ for some $x \in S^*$ and $y \in (R/C)^*$. It follows that $[A]^{[I]}=[A/CA, SA, u]$ is a well-defined member of $\bar{\Phi}(R)$.

It is of interest to find subclasses of $\Phi(R)$ on which $\varprojlim(R/C)^*$ acts trivially. One such class is the set $F(R)$ of isomorphism classes of finitely generated torsionfree R-modules of the form $A=I \oplus X$ where I is an ideal with $(0:_R I)=(0:_R A)$. For later reference we record this as

Lemma 2.9. *The group* $\varprojlim(R/C)^*$ *operates trivially on* $F(R)$.

Proof. In Lemma 2.5(iii), take $\vartheta: I \oplus X \to I \oplus X$ to be multiplication by x on I and the identity on X. □

3. Applications to cancellation of summands

The following result is essentially the same as [7, Theorem 2.3]. Removing the finite integral closure assumption required adjusting (i) and (iv) slightly, and (v) has been added.

Theorem 3.1. *Let A and B be finitely generated torsionfree R-modules. The following are equivalent*:

(i) *There exists a module-finite overring S of R with conductor C such that SA, SB are S-projective and an $x \in (S/C)^*$ such that $A^x \cong B$*;

(ii) $\bar{R}A \cong \bar{R}B$ *and* $A_M \cong B_M$ *for each maximal ideal M of R*;

(iii) $A \oplus C \cong B \oplus C$ *for some finitely generated R-module C*;

(iv) $A \oplus S \cong B \oplus S$ *for some finite overring S of R*;

(v) $[A]^{[I]}=[B]$ *for some* $[I] \in D(\bar{R})$.

Proof. (iv)$\Rightarrow$(iii). Trivial.

(iii)$\Rightarrow$(ii). This follows since one can cancel modules over local rings [5] and Dedekind domains.

(ii)$\Rightarrow$(i). Since A and B are torsionfree, $\bar{R}A$ and $\bar{R}B$ are $\bar{R}$-projective. It follows that there exists a module-finite overring S of R such that $SA \cong SB$ is S-projective and S contains the fundamental idempotents of K. Now use the argument of [7].

(i)$\Rightarrow$(v). Let $I=[R/C, S, x]$. Then $[B]=[A^x]=[A]^{[I]}$.

(v)$\Rightarrow$(iv). Let S be a finite overring of R with conductor C such that SA, SB are projective and let $I \in D(S)$, say $I=[R/C, S, u]$, $u \in (S/C)^*$. Then $[A]^{[I]}=[B]$ implies $[A^u]=[B]$. Thus $[B \oplus S]=[A^u \oplus S]=[(A \oplus S)^u]=[A \oplus S^u]=[A \oplus S]$. Therefore $B \oplus S \cong A \oplus S$ by Lemma 2.9. □

Definition 3.2. We will say that R has *torsionfree cancellation* if $A \oplus C \cong$

$B \oplus C \Rightarrow A \cong B$ whenever A, B and C are finitely generated R-modules with A torsionfree.

It follows from local cancellation [5, Corollary to Proposition 1.3] that if $A \oplus G \cong B \oplus G$ with A, B, G finitely generated and A torsionfree, then B is torsionfree. In [7, Corollary 2.4] torsionfree cancellation is compared to the property $D(\bar{R}) = 0$, but no characterization of either of these properties was obtained without further hypotheses on R. The following two results give the relationship between these two properties for general reduced one-dimensional Noetherian ring. Recall that R-modules A and B are called stably isomorphic if $A \oplus R^n \cong B \oplus R^n$ for some n.

Corollary 3.3. *The ring R has torsionfree cancellation if and only if $D(\bar{R}) = 0$ and stably isomorphic finitely generated torsionfree R-modules A and B are isomorphic.*

Proof. '$\Leftarrow$'. If $A \oplus M \cong B \oplus M$ for some finitely generated R-module M, then by the above theorem there exists an $I \in D(\bar{R})$ such that $[A]^{[I]} = [B]$. But $D(\bar{R}) = 0$ implies that $[A] = [B]$. Thus A and B are stably isomorphic by Theorem 2.8, and hence $A \cong B$ by hypothesis.

'$\Rightarrow$'. Let $A \in D(\bar{R})$. Then $A_M \cong R_M$ for every maximal ideal M of R and $\bar{R}A \cong \bar{R}R = \bar{R}$. Thus by Theorem 3.1, $A \oplus S \cong R \oplus S$ for some finite overring S of R. Then by hypothesis $A \cong R$. Therefore $D(\bar{R}) = 0$. □

Corollary 3.4. *The following properties are equivalent*:

(i) $D(\bar{R}) = 0$;

(ii) *If $A \in \Phi(R)$ and $A \oplus M \cong B \oplus M$ with M finitely generated, then A and B are in the same orbit under* $\varprojlim(R/C)^*$;

(iii) *If $A \in \Phi(R)$ and $A \oplus M \cong B \oplus M$ with M finitely generated, then A and B are stably isomorphic.* □

Recall that $\varprojlim (R/C)^*$ acts trivially on the set $F(R)$ of isomorphism classes of finitely generated torsionfree R-modules of the form $A = I \oplus X$ where I is an ideal with $(0:_R I) = (0:_R A)$. It follows that the operation of $D(\bar{R})$ on $\bar{\Phi}(R)$ induces an operation of $D(\bar{R})$ on $F(R)$. Thus as in [7, Corollary 2.5] we have

Corollary 3.5. *The following properties are equivalent*:

(i) *If $A \in F(R)$ and $A \oplus C \cong B \oplus C$ with C finitely generated, then $A \cong B$*;

(ii) *If A is projective of positive rank and $A \oplus C \cong B \oplus C$ with C finitely generated, then $A \cong B$*;

(iii) *If $A \oplus S \cong R \oplus S$ for some finite overring S of R, then $A \cong R$*;

(iv) $D(\bar{R}) = 0$. □

It follows that torsionfree cancellation for R is equivalent to $D(\bar{R})=0$ if R has the property that $\varprojlim(R/C)^*$ operates trivially on $\Phi(R)$. This holds if R is a Bass ring, since in this case $F(R)=\Phi(R)$ [7, Proposition 2.6]. Thus we get the result [7, Theorem 2.7] that if R is a Bass ring, then $D(\bar{R})=0$ if and only if R has torsionfree cancellation.

Remark 3.6. In [7] it was stated that it was probably known that if R is the ring of an affine curve over an algebraically closed field, then $D(\bar{R})=0 \Leftrightarrow R=\bar{R}$, and a nice argument was given for this result. This result can also be found in [4, Theorem 3.2].

The argument given in [7, Theorem 2.10] easily extends to give the following:

Theorem 3.7. *Let A and B be finitely generated torsionfree R-modules such that $A_M = B_M$ for each maximal ideal M of R, and let F be a faithful torsionfree R-module. Then there exists an R-module G such that $A \oplus F = B \oplus G$.* □

4. Power cancellation

The objective in this section is to give some relationships among the following statements:

(a) $A^{(q)} \cong B^{(q)}$ for some $q \geq 1$;

(b) $A \oplus X \cong B \oplus X$ for some finitely generated R-module X;

(c) $A_m \cong B_m$ for each maximal ideal m or R.

In order to include the case that R does not have the property that stably isomorphic elements of $\Phi(R)$ are isomorphic, we are also interested in the property

(a′) $A^{(q)}$ and $B^{(q)}$ are stably isomorphic for some $q \geq 1$.

It turns out that (a) and (a′) are equivalent (with different exponents) if A has constant rank. Indeed by [8, Proposition 2.9], the following holds:

Proposition 4.1. *If A and B are stably isomorphic torsionfree R-modules of constant rank r, then $A^{(r)} \cong B^{(r)}$.* □

(Although this result was given only for the case that R has finite integral closure, the given argument is easily adapted to the general case.)

It is clear that the direct sum induces a well-defined addition on the set $\bar{\Phi}(R)$ of stable isomorphism classes of $\Phi(R)$, and it follows from Lemma 1.3(iii) that $([A]+[B])^{[I]} = [A]+[B]^{[I]}$ for $[I] \in D(\bar{R})$. This observation will simplify the proof of the following result:

Proposition 4.2 (Power cancellation). (i) *If $D(\bar{R})$ is a torsion group with finite* *exponent e, then e is the least exponent q such that* (b) *implies* (a′) *for all* $A, B \in \Phi(R)$;

(ii) *If* $D(\bar{R})$ *is a torsion group with infinite exponent, then* (b) *implies* (a′) *but no single value q works for all* $A, B \in \Phi(R)$;

(iii) *If* $D(\bar{R})$ *is not a torsion group, then there exist* $A, B \in \Phi(R)$ *satisfying* (b) *but not* (a′) *for each q*.

Proof. (i) From Theorem 3.1 we get that (b) implies that $[A]^{[I]} = [B]$ for some $[I] \in D(\bar{R})$. Taking e-fold sums of both sides, the above observation together with the fact that $I^e \cong R$, yields the result $[A^{(e)}] = [B^{(e)}]$.

For the other half of (i) let $[I] \in D(\bar{R})$ have exponent e. By Theorem 3.1, $I \oplus X \cong R \oplus X$ for some finitely generated R-module X. By [3, Lemma 4.1], $[I^{(q)}] = [I^q \oplus R^{(q-1)}]$, and this is $[R^{(q)}]$ if and only if $I^q = R$.

Parts (ii) and (iii) are similar. □

Before comparing (a′) and (c) we compare (c) and the following property:

(d) $A^{(q)} \oplus X \cong B^{(q)} \oplus X$ for some finitely generated R-module X and some $q \geq 1$.

Proposition 4.3. (i) *If* $\mathrm{Pic}(\bar{R})$ *is a torsion group with finite exponent e, then e is the least exponent q such that* (c) *implies* (d) *for all* $A, B, \in \Phi(R)$;

(ii) *If* $\mathrm{Pic}(\bar{R})$ *is a torsion group with infinite exponent, then* (c) *implies* (d) *but no single value q works for all* $A, B, \in \Phi(R)$;

(iii) *If* $\mathrm{Pic}(\bar{R})$ *is not a torsion group, then there exists* $A, B, \in \Phi(R)$ *satisfying* (c) *but not* (d) *for each q*.

Proof. (i) Let $A, B \in \Phi(R)$ with $A_m \cong B_m$ for each maximal ideal m of R. The hypothesis implies that $\det_{\bar{R}}((\bar{R}A)^{(e)}) = \det_{\bar{R}}((\bar{R}B)^{(e)})$. Thus $\bar{R}A^{(e)} \cong \bar{R}B^{(e)}$, and hence (d) holds for $q = e$ by Theorem 3.1.

For the other half of (i) let $[I] \in \mathrm{Pic}(R)$ be such that $[I\bar{R}] \in \mathrm{Pic}(\bar{R})$ has exponent e. Then $I^{(q)} \oplus X \cong R^{(q)} \oplus X$ implies $I^q \oplus R^{(q-1)} \oplus X \cong R^{(q)} \oplus X$ and this implies $I^q\bar{R} = \bar{R}$. Thus e divides q.

The other parts are similar. □

Combining the above two propositions we get

Corollary 4.4. $\mathrm{Pic}(R)$ *is a torsion group if and only if* (c) *implies* (a′) *for all* $A, B \in \Phi(R)$. □

5. Ideal classes

We continue to let R be a reduced commutative Noetherian ring with total quotient ring K and M a finitely generated torsionfree R-module. Following [3] we let $\bar{\Lambda}^n_R M = (\Lambda^n_R M)/t(\Lambda^n_R M)$, where $t(A)$ denotes the torsion submodule of an R-module A. As observed in [3] the R-module $\bar{\Lambda}^n_R M$ is canonically isomorphic to the image of $\Lambda^n_R M$ in $\Lambda^n_K(KM)$.

Write $K = K_1 \oplus \cdots \oplus K_m$ where the K_i are fields, and let $\varepsilon_j(M)$ denote the sum of the units $e_i \in K_i$ such that $\dim_{K_i}(K_i M) = j$. The ideal class $\mathrm{cl}(M)$ of a finitely generated torsionfree R-module M is then defined as the isomorphism class of

$$\bigoplus_{n \geq 0} (\bar{\Lambda}^n_R M)(\varepsilon_n(M)) .$$

The main properties of $\mathrm{cl}(M)$ are [3]:

Proposition 5.1. (a) $\mathrm{cl}(M \oplus N) = \mathrm{cl}(M)\mathrm{cl}(N)$;
(b) $\mathrm{cl}(M)$ *is the isomorphism class of a faithful ideal of* R;
(c) M *is projective if and only if* $\mathrm{cl}(M)$ *is projective.* □

In the case that R is a one-dimensional reduced Noetherian ring with finite integral closure $\bar{R}$ and conductor C, the following theorem can be considered as an analogue for finitely generated torsionfree modules of part of the natural transformation from the Mayer–Vietoris sequence involving $K_0(R)$ and $K_1(R)$ to the Mayer–Vietoris sequence involving $\mathrm{Pic}(R)$ and $(R)^*$. We do most of the work in the following lemma:

Lemma 5.2. *Let A be a finitely generated torsionfree R-module and S a finite overring of R with conductor C such that SA is S-projective. If $\varphi : SA/CA \to SA/CA$ is an isomorphism, then the following pullback diagram*:

$$\begin{array}{ccccc} B & & \xrightarrow{\quad i \quad} & & SA \\ {\scriptstyle f}\downarrow & & & & \downarrow{\scriptstyle g} \\ A/CA & \xrightarrow{j} & SA/CA & \xrightarrow{\varphi} & SA/CA \end{array}$$

induces for each n a pullback diagram as follows:

$$\begin{array}{ccccc} \bar{\Lambda}^n_R B & & \xrightarrow{\quad \bar{i} \quad} & & \Lambda^n_S(SA) \\ {\scriptstyle \bar{f}}\downarrow & & & & \downarrow{\scriptstyle \bar{g}} \\ \bar{\Lambda}^n_R A/C[\bar{\Lambda}^n_R A] & \xrightarrow{\bar{j}} & \Lambda^n_S(SA)/C[\Lambda^n_S(SA)] & \xrightarrow{\bar{\varphi}} & \Lambda^n_S(SA)/C[\Lambda^N_S(SA)] \end{array}$$

Proof. The original pullback is a composition of pullback diagrams

$$\begin{array}{ccccc} B & & \xrightarrow{\quad i \quad} & & SA \\ {\scriptstyle f_n}\downarrow & & & & \downarrow{\scriptstyle g_n} \\ A/C^nA & \xrightarrow{j_n} & SA/C^nA & \xrightarrow{\varphi_n} & SA/C^nA \end{array}$$

and

$$\begin{array}{ccccc} A/C^nA & \xrightarrow{j_n} & SA/C^nA & \xrightarrow{\varphi_n} & SA/C^nA \\ \downarrow & & & & \downarrow \\ A/CA & \xrightarrow{j} & SA/CA & \xrightarrow{\varphi} & SA/CA \end{array}$$

Recall that taking exterior powers commutes with change of rings. Let f_0 denote the composition of the canonical maps $\Lambda^n_R B \to \Lambda^n_R(A/C^nA) \to \Lambda^n_{R/C^n}(A/C^nA) \to (\Lambda^n_R A)/C^n(\Lambda^n_R A) \to (\bar{\Lambda}^n_R A)/C^n(\bar{\Lambda}^n_R A)$, and let i_0 denote the composition $\Lambda^n_R B \to \Lambda^n_R SA \to \Lambda^n_S SA$. We have a commutative diagram

$$\begin{array}{ccccc} \Lambda^n_R B & & \xrightarrow{\quad\quad i_0 \quad\quad} & & \Lambda^n_S(SA) \\ {\scriptstyle f_0}\downarrow & & & & \downarrow{\scriptstyle \bar{g}} \\ (\bar{\Lambda}^n_R A)/C^n(\bar{\Lambda}^n_R A) & \xrightarrow{\bar{j}_n} & (\Lambda^n_S SA)/C^n(\Lambda^n_S SA) & \xrightarrow{\bar{\varphi}_n} & (\Lambda^n_S SA)/C^n(\Lambda^n_S SA) \end{array}$$

and since the image of i_0 is torsionfree, i_0 factors through the canonical map $\pi : \Lambda^n_R B \to \bar{\Lambda}^n_R B$. Further, since the maps in the bottom line are injective, it follows that f_0 also factors through $\pi : \Lambda^n_R B \to \bar{\Lambda}^n_R B$. Let $\bar{i} : \bar{\Lambda}^n_R B \to \Lambda^n_S SA$ and $\bar{f} : \bar{\Lambda}^n_R B \to \bar{\Lambda}^n_R A / C^n(\bar{\Lambda}^n_R A)$ be the induced maps. To see that the resulting diagram is a pullback let $J \subseteq \Lambda^n_S(SA)$ be the submodule of $\Lambda^n_S(SA)$ making the diagram a pullback. Then $\bar{g}(\bar{i}(\bar{\Lambda}^n_R B)) = \bar{g}(i_0(\Lambda^n_R B))$ and $J \subseteq i_0(\Lambda^n_R B) + C^n[\Lambda^n_S SA] = i_0(\Lambda^n_R B) + \Lambda^n_S CA \subseteq i_0(\Lambda^n_R B)$. It follows that the induced diagram is a pullback. Composing with the following pullback we get the desired result.

$$\begin{array}{ccccc} (\bar{\Lambda}^n_R A)/C^n(\bar{\Lambda}^n_R A) & \xrightarrow{\bar{j}_n} & (\Lambda^n_S SA)/C^n(\Lambda^n_S SA) & \xrightarrow{\bar{\varphi}_n} & (\Lambda^n_S SA)/C^n(\Lambda^n_S SA) \\ \downarrow & & & & \downarrow \\ (\bar{\Lambda}^n_R A)/C(\bar{\Lambda}^n_R A) & \xrightarrow{\bar{j}} & (\Lambda^n_S SA)/C(\Lambda^n_S SA) & \xrightarrow{\bar{\varphi}} & (\Lambda^n_S SA)/C(\Lambda^n_S SA) \end{array} \qquad \square$$

Theorem 5.3. *Let R be a one-dimensional reduced Noetherian ring, let A and B be finitely generated torsionfree R-modules, and let S be a finite overring of R such that SA and SB are S-projective. If $B = A^x$ for $x \in (S/C)^*$, then* $\mathrm{cl}(B) = \mathrm{cl}(A)^x$.

Proof. This is immediate from the above lemma and the observation that $S\bar{\Lambda}^n_R A = \Lambda^n_S SA$. $\square$

Remark 5.4. As an illustration of how the above result may be applied, let R be a one-dimensional reduced Noetherian ring and let M and N be finitely generated torsionfree R-modules such that $M_m \cong N_m$ for each maximal ideal m of R and $\mathrm{cl}(M) = \mathrm{cl}(N)$. Let S be a finite overring of R such that SM and SN are

S-projective. Since $\mathrm{cl}_S(SM) = S\mathrm{cl}_R(M)$ it follows that $SM \cong SN$, and since $M_m \cong N_m$ for each maximal ideal m of R, we get $M^x \cong N$ for some $x \in (S/C)^*$ by Theorem 3.1 (or [7, Theorem 2.3] if R has finite integral closure). Thus by Theorem 5.3 we have $\mathrm{cl}(M)^x = \mathrm{cl}(N)$. But by hypothesis we have $\mathrm{cl}(M) = \mathrm{cl}(N)$. Thus by Lemma 2.5, $x = \bar{u}v$ where $\bar{u} \in (S/C)^*$ lifts to $u \in S^*$ and v is the determinant of an automorphism of $S\mathrm{cl}_R(M)/C[\mathrm{cl}_R(M)]$ which carries $\mathrm{cl}_R(M)/C[\mathrm{cl}_R(A)]$ into itself. For any R-module J such that $v(J/CJ) \subseteq J/CJ$ we get $N \oplus J \cong M^x \oplus J \cong M^v \oplus J \cong M \oplus J^v \cong M \oplus J$. For instance, letting $I = \mathrm{cl}(M)$, if $I/CI \cong R/C$ or $I:_K I = R$, then it follows that M and N are stably isomorphic. It follows from Proposition 5.1(a) and local cancellation that conversely, if M and N are stably isomorphic, then $M_m \cong N_m$ for each maximal ideal m of R and $\mathrm{cl}(M) = \mathrm{cl}(N)$.

The following gives some other simple consequences of Theorem 5.3. Recall that an ideal I is called *regular* if it contains a non-zerodivisor.

Remark 5.5. (a) If J is a regular ideal of R and $[I] \in D(\bar{R})$, then $[J]^{[I]} = [IJ]$.
(b) If J and L are regular ideals of R, and S is an overring of R such that SJ and SL are projective over S, then $J(L^x) = (JL)^x$ for each $x \in (S/C)^*$.

Proof. First consider (b). We have $(JL)^x = \mathrm{cl}(J \oplus L)^x = \mathrm{cl}(J \oplus L^x) = J(L^x)$. Now to prove (a) let S be an overring of S with conductor C such that SJ is S-projective, and write $I = [R/C, S, x]$ with $x \in (S/C)^*$. Then $J^x = (JR)^x = J(R^x) = JI$. □

Remark 5.6. An alternate approach to Theorem 5.3 is to prove Remark 5.5(b) directly and then Theorem 5.3 follows easily from this.

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