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One-loop gluon scattering amplitudes in theories with $\mathcal{N} < 4$ supersymmetries

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Abstract

Generalised unitarity techniques are used to calculate the coefficients of box and triangle integral functions of one-loop gluon scattering amplitudes in gauge theories with $\mathcal{N} < 4$ supersymmetries. We show that the box coefficients in $\mathcal{N} = 1$ and $\mathcal{N} = 0$ theories inherit the same coplanar and collinear constraints as the corresponding $\mathcal{N} = 4$ coefficients. We use triple cuts to determine the coefficients of the triangle integral functions and present, as an example, the full expression for the one-loop amplitude $A^{\mathcal{N}=1}(1^-, 2^-, 3^-, 4^+, \ldots, n^+)$.

1. Introduction

The proposal of a “weak–weak” duality between $\mathcal{N} = 4$ super-Yang–Mills theory and a topological string theory [1] has led to significant progress in the computation of amplitudes in gauge theories.

At tree level, amplitudes display a structure which is inherited from the twistor string description. This has inspired several reformulations of tree level amplitudes. Specifically, Cachazo, Svrček and Witten [2] proposed a formulation for calculating tree amplitudes using “MHV-vertices” rather than using conventional three and four point Feynman vertices. A MHV vertex is an off-shell continuation of the Parke–Taylor formula [3,4] for physical on-shell tree amplitudes where two gluons have negative helicity and the remaining helicities are all positive (these are also known as “Maximally Helicity Violating” (MHV) amplitudes). This CSW formalism has proven very useful in obtaining compact expressions for tree amplitudes and has been extended to include external fermions and scalars [5] and even to theories with massive electroweak particles [6]. The MHV vertex approach extends to one-loop scattering amplitudes as demonstrated by the recomputation of the MHV one-loop amplitudes [7–9].

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At one loop level, over many years, various techniques have been developed to calculate loop-amplitudes more efficiently than conventional Feynman diagram approaches. A key ingredient is the careful organisation of the amplitude in terms of the physical properties and factorisation of the amplitudes. (In fact, an important feature of the CSW approach is that the MHV vertices are much closer to physical amplitudes than Feynman vertices.) Ideas such as the spinor helicity formalism [10] and colour-ordering [11], which organise amplitudes according to the physical outgoing states are very useful in determining tree amplitudes. Beyond tree level, the constraints demanded by unitarity have been used to compute one-loop gluon scattering amplitudes in various supersymmetric theories. In $\mathcal{N}=4$ super-Yang–Mills a one-loop amplitude is completely specified by the coefficients of scalar box functions [12,13]. The one-loop MHV amplitudes have been computed in both $\mathcal{N}=4$ super-Yang–Mills [12] and in $\mathcal{N}=1$ super-Yang–Mills [13]. The one-loop NMHV amplitudes with three negative helicities and the rest positive (known as next-to-MHV or NMHV amplitudes) have been calculated in $\mathcal{N}=4$ super-Yang–Mills, first at six points [13], then at seven points [14] and finally for all n [15]. These computations involve computing the two particle cuts [16] of an amplitude or more general cuts and factorisation properties [17,18].

These methods have been complemented by techniques derived or inspired by the twistor string approach. MHV and NMHV tree amplitudes have collinear and coplanar support in twistor space: these features correspond to annihilation of the amplitude by particular differential operators. By acting with these differential operators on the cuts of an amplitude one can obtain [19–23] algebraic equations which may be useful in computing the box-coefficients in one-loop amplitudes. The utility of this approach was demonstrated by the computation of one of the seven point $\mathcal{N}=4$ one-loop amplitudes [23]. More recently, Britto, Cachazo and Feng [24] demonstrated, by continuing three-point tree amplitudes to signature $(--++)$, how these box-coefficients could be computed directly as a quadruple product of tree amplitudes. (The continuation of the signature can best be seen as a Lorentzian signature with complex momenta. Although the unitarity properties are obscure in normal field theory, the signature $(--++)$ is more natural from a twistor space perspective [1].)

In this Letter we examine generalised unitarity techniques [15,17] for calculating amplitudes in theories with $\mathcal{N}<4$ supersymmetries. Firstly, we examine the box-coefficients for a variety of helicity configurations in $\mathcal{N}=1$ and $\mathcal{N}=0$ theories: determining these from the quadruple cuts [24]. These box coefficients satisfy collinearity and coplanarity constraints which have a geometric interpretation in twistor space. Interestingly, the box-coefficients obey these constraints independently of supersymmetry. Specifically the box-coefficients we compute are coplanar for NMHV amplitudes even in the $\mathcal{N}=0$ case.

Box coefficients are an important ingredient in these amplitudes but do not completely specify the amplitude. We demonstrate how triple cuts [15,17] can be used to determine the remaining triangle integrals and give the full result for the previously unknown amplitude,

$$A\left(1^-,2^-,3^-,4^+,5^+,\ldots,n^+\right), \tag{1.1}$$

in the $\mathcal{N}=1$ theory.

2. Generalised unitarity and relationships between box-coefficients

The idea that an amplitude might be reconstructed by its unitarity constraints was originally investigated within the context of S-matrix theories in the 1960s [25] with relatively limited success. However, these approaches assumed relatively little about the actual theories considered. If one restricts these investigations to theories which have a Quantum Field Theory description, e.g., gauge theories, then these techniques have proven extremely useful. In principle, a complete understanding of all cuts and factorisations in all channels should be sufficient to completely reconstruct all loop amplitudes. Part of the complete understanding is that cuts must in principle be evaluated with loop momentum in $4-2\epsilon$ dimensions. However, for supersymmetric theories, amplitudes are "cut-constructible" [12], meaning that it is sufficient to calculate the cuts using momenta restricted to four dimensions. This is an enormous simplification, allowing one to exploit the relatively simple expressions obtainable for on-shell tree amplitudes. While in special cases the two-particle cuts are enough to compute an amplitude exactly, in other cases we must use higher-point and more generalised cuts [13,17]. For example, at two-loops one must also consider three particle cuts and double–double cuts.

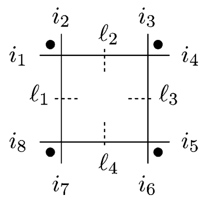
Within gauge-theories, amplitudes can be expanded in terms of various integral functions,

$$\mathcal{A} = \sum \hat{c}_i I_4 + \sum \hat{d}_i I_3 + \sum \hat{e}_i I_2 + \cdots, \tag{2.1}$$

where, in general, theories with more supersymmetry have a more restricted set of integral functions. For $\mathcal{N} = 4$ theories the series only contains the scalar box functions, I_4, and hence is entirely determined by the box-coefficients $\hat{c}_i$ [12]. For $\mathcal{N} = 1$ super-Yang–Mills we have to consider box functions together with scalar triangle and bubble functions, I_3 and I_2 [13]. For theories without supersymmetry the amplitude may also contain rational pieces which have only been calculated in a relatively small number of cases.

For $\mathcal{N} = 4$ amplitudes analysis of the two particle cuts has enabled a computation of the box-coefficients for arbitrary numbers of particles in the MHV [12] and NMHV cases [13–15], either by evaluating the cuts or by acting on the cut with differential operators [20,22,23].

Recently, Britto, Cachazo and Feng demonstrated, by analytically continuing tree amplitudes to a signature of $(- - + +)$, and using these to calculate quadruple cuts, that box coefficients can be determined algebraically from products of on-shell tree amplitudes [24]. Specifically, considering a generic amplitude containing the scalar box integral function,



its coefficient is given by the product of four tree amplitudes where the cut legs satisfy on-shell conditions

$$\hat{c} = \frac{1}{2} \sum_{\mathcal{S}} \left(A^{\text{tree}}(\ell_1, i_1, \ldots, i_2, \ell_2) A^{\text{tree}}(\ell_2, i_3, \ldots, i_4, \ell_3) A^{\text{tree}}(\ell_3, i_5, \ldots, i_6, \ell_4) A^{\text{tree}}(\ell_4, i_7, \ldots, i_8, \ell_1) \right), \tag{2.2}$$

where $\mathcal{S}$ indicates the set of helicity configurations and particle types of the legs ℓ_j giving a non-vanishing product of tree amplitudes. The analytic continuation allows this to be evaluated even when one or more of the tree amplitudes in Eq. (2.2) is a three point amplitude which would vanish in Minkowski signature.

In this section we restrict ourselves to a class of boxes where the amplitude at each corner is either a MHV amplitude with two negative helicity legs or a $\overline{\text{MHV}}$ amplitude with two positive helicity legs. This class of diagrams is quite large and includes all helicity cases up to six-point amplitudes and the MHV loop amplitudes themselves. For convenience, we describe such amplitudes as "MHV-deconstructible".

We will consider three possible matter contributions to the box-coefficients; the entire $\mathcal{N} = 4$ multiplet; the $\mathcal{N} = 1$ chiral multiplet consisting of a fermion and a scalar; and the contribution from a complex scalar circulating in the loop. We often, perhaps perversely, describe these last as the $\mathcal{N} = 0$ matter contribution. We can obtain the contribution of any matter content by summing over linear combinations of these three matter multiplets. Such decompositions arise very naturally in a string based approach [26].

For $\mathcal{N} = 1$ super-Yang–Mills with external gluons there are two possible supermultiplets contributing to the loop amplitude—the vector and the chiral matter multiplets. For simplicity we consider colour-ordered one-loop amplitudes. These can be decomposed into the contributions from single particle spins,

$$A_n^{\mathcal{N}=1\,\text{vector}} \equiv A_n^{[1]} + A_n^{[1/2]}, \qquad A_n^{\mathcal{N}=1\,\text{chiral}} \equiv A_n^{[1/2]} + A_n^{[0]}, \tag{2.3}$$

where $A_n^{[J]}$ is the one-loop amplitude with n external gluons and particles of spin-J circulating in the loop. (For spin 0 we mean a complex scalar.) For $\mathcal{N}=4$ super-Yang–Mills theory there is a single multiplet which is given by

$$A_n^{\mathcal{N}=4} \equiv A_n^{[1]} + 4A_n^{[1/2]} + 3A_n^{[0]}. \tag{2.4}$$

The contributions from these three multiplets are not independent but satisfy

$$A_n^{\mathcal{N}=1\,\text{vector}} \equiv A_n^{\mathcal{N}=4} - 3A_n^{\mathcal{N}=1\,\text{chiral}}. \tag{2.5}$$

Throughout we assume the use of a supersymmetry preserving regulator [26–28].

We first show that the box-coefficients for the three matter contributions are *not* independent for MHV-deconstructible boxes but that the $\mathcal{N}=0$ coefficient can be derived from the $\mathcal{N}=4$ and $\mathcal{N}=1$ coefficients. For MHV (and $\overline{\text{MHV}}$ by conjugation) tree amplitudes the contributions from the non-scalar particles can be related to that of the real scalar via supersymmetric Ward identities [4,29] and are simply,

$$A^{\text{tree}}\big((\ell_1)^{\mp}, i_1, \ldots, i_2, (\ell_2)^{\pm}\big) = (x)^{\pm 2h} A^{\text{tree}}\big((\ell_1)^{s}, i_1, \ldots, i_2, (\ell_2)^{s}\big), \tag{2.6}$$

where $h = 1/2$ for fermions and $h = 1$ for gluons and $x = \langle l_1\, i_a\rangle / \langle l_2\, i_a\rangle$ with i_a being the negative helicity gluon leg. The contribution to the box-coefficient will then be

$$(X)^{2h} \times \text{real scalar contribution}, \tag{2.7}$$

where $X = x_1x_2x_3x_4$, and x_j is the factor from the jth corner.

When we consider the contribution from a supersymmetric multiplet to the loop amplitude, we must sum over particle types. For the chiral multiplet the contribution, relative to the real scalar, has a factor

$$\rho^{\mathcal{N}=1} = -X + 2 - \frac{1}{X} = -\frac{(X-1)^2}{X}, \tag{2.8}$$

whilst for the $\mathcal{N}=4$ multiplet the factor is

$$\rho^{\mathcal{N}=4} = X^2 - 4X + 6 - 4\frac{1}{X} + \frac{1}{X^2} = \frac{(X-1)^4}{X^2} = \big(\rho^{\mathcal{N}=1}\big)^2. \tag{2.9}$$

For $\mathcal{N}=4$ boxes we also have solutions where the two cut legs attached to a corner have the same helicity. Such tree amplitudes are only non-zero if the cut legs are gluons. We refer to such configurations as “singlet” contributions. It is the remaining “non-singlet” contributions which can be obtained from the scalar by applying a factor of $\rho^{\mathcal{N}=4}$. We thus have

$$\hat{c}^{\mathcal{N}=4\,\text{non-singlet}} = \rho^{\mathcal{N}=4}\hat{c}^{\text{real scalar}}, \qquad \hat{c}^{\mathcal{N}=1\,\text{chiral}} = \rho^{\mathcal{N}=1}\hat{c}^{\text{real scalar}}, \tag{2.10}$$

which given that $\rho^{\mathcal{N}=4} = (\rho^{\mathcal{N}=1})^2$ yields

$$\hat{c}^{\mathcal{N}=0} = 2\frac{(\hat{c}^{\mathcal{N}=1\,\text{chiral}})^2}{\hat{c}^{\mathcal{N}=4\,\text{non-singlet}}}. \tag{2.11}$$

This formula applies to any box which is MHV-deconstructible. It can be used to determine the $\mathcal{N}=0$ (or scalar) coefficient from the two supersymmetric coefficients *provided* we have identified the non-singlet contribution in the $\mathcal{N}=4$ case.

Such a formula will have several analogs in gravity amplitudes. For graviton one-loop amplitudes explicit formulations [30,31] give

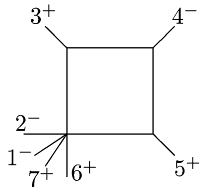
$$\hat{c}^{\mathcal{N}=0} = 2\frac{(\hat{c}^{\mathcal{N}=4\,\text{matter}})^2}{\hat{c}^{\mathcal{N}=8\,\text{non-singlet}}}, \tag{2.12}$$

where $\mathcal{N}=8$ denotes the full $\mathcal{N}=8$ multiplet [32], $\mathcal{N}=4$ *matter* denotes the $\mathcal{N}=4$ matter multiplet containing particles of spins 1, 1/2 and 0 and $\mathcal{N}=0$ denotes the scalar contribution.

Not all box-coefficients are MHV-deconstructible. For example, in the amplitude

$$A\left(1^-, 2^-, 3^+, 4^-, 5^+, 6^+, 7^+\right) \tag{2.13}$$

the box



will have a NMHV corner. The scalar tree amplitude at this corner is of the form

$$\frac{C_1}{K_{671}^2} + \frac{C_2}{K_{712}^2}, \tag{2.14}$$

where $K_{i\dots j} \equiv (k_i + \cdots + k_j)$ and the amplitudes for other particles types [5,33] are of the form

$$x_1^h \frac{C_1}{K_{671}^2} + x_2^h \frac{C_2}{K_{712}^2}, \tag{2.15}$$

which leads to box coefficients which are a sum of two terms

$$\hat{c} = \hat{c}_A + \hat{c}_B, \tag{2.16}$$

each of which satisfy Eq. (2.11) individually,

$$\hat{c}_A^{\mathcal{N}=0} = 2\frac{(\hat{c}_A^{\mathcal{N}=1\,\text{chiral}})^2}{\hat{c}_A^{\mathcal{N}=4\,\text{non-singlet}}} \quad \text{and} \quad \hat{c}_B^{\mathcal{N}=0} = 2\frac{(\hat{c}_B^{\mathcal{N}=1\,\text{chiral}})^2}{\hat{c}_B^{\mathcal{N}=4\,\text{non-singlet}}}. \tag{2.17}$$

This formula has obvious generalisations to higher point box coefficients.

3. Example box coefficients

In this section we present some specific examples of "MHV deconstructible" box-coefficients. We use colour-ordered amplitudes [11,34] throughout and only present the leading in colour expression.

There is a choice of representations for the box-integral functions. There are scalar box-integral functions and F-functions which have zero mass dimension and are related to the former by the removal of the momentum prefactors [12],

$$I_4 = \frac{1}{K} F. \tag{3.1}$$

We denote the coefficients of the scalar box functions as $\hat{c}_i$ and those of the F-functions as c_i. Both the $\hat{c}_i$ and c_i satisfy the relations (2.11).

In all cases we present the $\mathcal{N}=4$, $\mathcal{N}=1$ and $\mathcal{N}=0$ results. For the $\mathcal{N}=4$ case the results are generally already known [12–15] whilst the six point $\mathcal{N}=1$ box coefficients appear in [35].

3.1. MHV box-coefficients

Consider the case of MHV-amplitudes where all box coefficients are known and we may check the relationship (2.11). In general, the box functions are “two-mass-easy” boxes and single mass boxes. The $\mathcal{N}=4$ non-singlet terms occur where there is a single negative helicity leg in each massive corner. The $\mathcal{N}=4$ amplitude was calculated in Ref. [12] and the $\mathcal{N}=1$ in Ref. [13] (the five point amplitude appeared earlier in [36]) whilst the $\mathcal{N}=0$ coefficient was computed by Bedford et al. [9]. Denoting the two negative helicities as i and j and considering the box with two massless legs m_1 and m_2, the coefficients of the F-functions are

$$\begin{aligned} c^{\mathcal{N}=4} &= A^{\text{tree}} \times 1, \\ c^{\mathcal{N}=1} &= A^{\text{tree}} \times \frac{b^{ij}_{m_1m_2}}{2}, \\ c^{\mathcal{N}=0} &= A^{\text{tree}} \times \frac{(b^{ij}_{m_1m_2})^2}{2}, \end{aligned} \tag{3.2}$$

where

$$b^{ij}_{m_1m_2} = 2\frac{\langle i\, m_1\rangle\langle i\, m_2\rangle\langle j\, m_1\rangle\langle j\, m_2\rangle}{\langle i\ j\rangle^2\langle m_1\, m_2\rangle^2}, \tag{3.3}$$

and we use spinor inner-products, $\langle j\, l\rangle \equiv \langle j^-|l^+\rangle$, $[j\, l] \equiv \langle j^+|l^-\rangle$, where $|i^\pm\rangle$ is a massless Weyl spinor with momentum k_i and chirality $\pm$ [10,37].

Clearly these amplitudes satisfy the relation (2.11).

3.2. Six point NMHV box-coefficients

All boxes for the six point amplitudes are MHV-deconstructible and the box coefficients are known for both $\mathcal{N}=4$ and $\mathcal{N}=1$ [13,35], so we can apply (2.11) to generate the coefficients of the scalar boxes. The amplitudes with all-positive helicity legs and those with one-negative helicity leg are non-zero in non-supersymmetric theories, however these amplitudes are rational functions with no scalar box contributions. Thus, the two independent amplitudes with non-vanishing box-coefficients are the MHV case (or $\overline{\text{MHV}}$), which was covered in the previous section, and the NMHV case with three negative helicities.

There are three independent amplitudes with three negative helicity legs: $A(1^-,2^-,3^-,4^+,5^+,6^+)$, $A(1^-,2^-,3^+,4^-,5^+,6^+)$ and $A(1^-,2^+,3^-,4^+,5^-,6^+)$. Of these, the first has vanishing box-coefficients for $\mathcal{N}=1$ and $\mathcal{N}=0$,

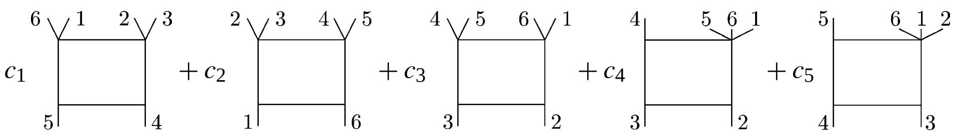
$$A^{\mathcal{N}=0,1}\big(1^-,2^-,3^-,4^+,5^+,6^+\big)\big|_{\text{box}} = 0. \tag{3.4}$$

The $\mathcal{N}=4$ amplitude only has singlet contributions in this case.

The second amplitude, $A(1^-,2^-,3^+,4^-,5^+,6^+)$, does have a non-trivial box structure,

$$A\big(1^-,2^-,3^+,4^-,5^+,6^+\big)\big|_{\text{box}} = c_1F^{2\text{m}\,h}_{4:4} + c_2F^{2\text{m}\,h}_{4:6} + c_3F^{2\text{m}\,h}_{4:2} + c_4F^{1\text{m}}_{4:2} + c_5F^{1\text{m}}_{4:3}, \tag{3.5}$$

which is depicted



Of these coefficients, only three are truly independent, since under flipping, conjugation and relabeling,

$$c_1 \leftrightarrow c_3, \qquad c_4 \leftrightarrow c_5. \tag{3.6}$$

Explicitly, the remaining box-coefficients are

$$c_1^{\mathcal{N}=4\,\text{non-singlet}} = i\,\frac{\langle 3^+|\not{K}|1^+\rangle^4}{[2\,3][3\,4]\langle 5\,6\rangle\langle 6\,1\rangle\langle 2^+|\not{K}|5^+\rangle\langle 4^+|\not{K}|1^+\rangle K^2},$$
$$c_1^{\mathcal{N}=1\,\text{chiral}} = i\,\frac{\langle 5\,1\rangle\langle 3^+|\not{K}|1^+\rangle^2\langle 3^+|\not{K}|5^+\rangle}{[2\,3]\langle 5\,6\rangle\langle 6\,1\rangle\langle 2^+|\not{K}|5^+\rangle\langle 4^+|\not{K}|5^+\rangle^2},$$
$$c_1^{\mathcal{N}=0} = 2i\,\frac{\langle 1\,5\rangle^2[3\,4]\langle 3^+|\not{K}|5^+\rangle^2\langle 4^+|\not{K}|1^+\rangle K^2}{[2\,3]\langle 5\,6\rangle\langle 6\,1\rangle\langle 2^+|\not{K}|5^+\rangle\langle 4^+|\not{K}|5^+\rangle^4}, \quad K = K_{234}, \tag{3.7}$$

$$c_2^{\mathcal{N}=4\,\text{non-singlet}} = i\,\frac{\langle 3^+|\not{K}|4^+\rangle^4}{[1\,2][2\,3]\langle 4\,5\rangle\langle 5\,6\rangle\langle 1^+|\not{K}|4^+\rangle\langle 3^+|\not{K}|6^+\rangle K^2},$$
$$c_2^{\mathcal{N}=1\,\text{chiral}} = i\,\frac{[3\,1]\langle 6\,4\rangle\langle 3^+|\not{K}|4^+\rangle^2}{[1\,2][2\,3]\langle 4\,5\rangle\langle 5\,6\rangle\langle 1^+|\not{K}|6^+\rangle^2},$$
$$c_2^{\mathcal{N}=0} = 2i\,\frac{[3\,1]^2\langle 6\,4\rangle^2\langle 1^+|\not{K}|4^+\rangle\langle 3^+|\not{K}|6^+\rangle K^2}{[1\,2][2\,3]\langle 4\,5\rangle\langle 5\,6\rangle\langle 1^+|\not{K}|6^+\rangle^4}, \quad K = K_{123}, \tag{3.8}$$

$$c_5^{\mathcal{N}=4\,\text{non-singlet}} = \frac{\langle 6^+|\not{K}|4^+\rangle^4}{[6\,1][1\,2]\langle 3\,4\rangle\langle 4\,5\rangle\langle 6^+|\not{K}|3^+\rangle\langle 2^+|\not{K}|5^+\rangle K^2},$$
$$c_5^{\mathcal{N}=1\,\text{chiral}} = i\,\frac{\langle 6^+|\not{K}|4^+\rangle^2\langle 6^+|\not{K}|5^+\rangle}{[6\,1][1\,2]\langle 3\,5\rangle^2\langle 2^+|\not{K}|5^+\rangle K^2},$$
$$c_5^{\mathcal{N}=0} = 2i\,\frac{\langle 3\,4\rangle\langle 4\,5\rangle\langle 6^+|\not{K}|5^+\rangle^2\langle 6^+|\not{K}|3^+\rangle^2}{\langle 3\,5\rangle^4[6\,1][1\,2]\langle 2^+|\not{K}|5^+\rangle K^2}, \quad K = K_{345}. \tag{3.9}$$

The remaining amplitude, $A(1^-, 2^+, 3^-, 4^+, 5^-, 6^+)$, contains all six one-mass and all six "two-mass-hard" boxes,

$$A\big(1^-, 2^+, 3^-, 4^+, 5^-, 6^+\big)_{\text{box}} = \sum_{i=1}^{6} a_i F^{1\text{m}}_{4:i} + \sum_{i=1}^{6} b_i F^{2\text{m},h}_{4:i}. \tag{3.10}$$

These are not all independent and symmetry demands relationships amongst the a_i's,

$$\begin{aligned}
&a_3(123456) = a_1(345612), &\quad& a_5(123456) = a_1(561234),\\
&a_4(123456) = a_2(345612), && a_6(123456) = a_2(561234),\\
&a_2(123456) = \bar{a}_1(234561), && a_1(123456) = a_1(321654),
\end{aligned} \tag{3.11}$$

where $\bar{a}_1$ denotes a_1 with $\langle i\,j\rangle \leftrightarrow [i\,j]$. Thus there is a single independent a_i. Similarly we can use symmetry to generate all the b_i's from b_2. The expressions for a_1 and b_2 are

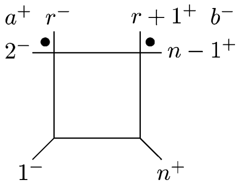
$$a_1^{\mathcal{N}=4\,\text{non-singlet}} = i\,\frac{\langle 2^+|\not{K}|5^+\rangle^4}{[1\,2][2\,3]\langle 4\,5\rangle\langle 5\,6\rangle\langle 1^+|\not{K}|4^+\rangle\langle 3^+|\not{K}|6^+\rangle K^2},$$
$$a_1^{\mathcal{N}=1\,\text{chiral}} = i\,\frac{\langle 2^+|\not{K}|5^+\rangle^2\langle 1^+|\not{K}|5^+\rangle\langle 3^+|\not{K}|5^+\rangle}{[1\,3]^2\langle 4\,5\rangle\langle 5\,6\rangle\langle 1^+|\not{K}|4^+\rangle\langle 3^+|\not{K}|6^+\rangle K^2},$$
$$a_1^{\mathcal{N}=0} = 2i\,\frac{[1\,2][2\,3]\langle 1^+|\not{K}|5^+\rangle^2\langle 3^+|\not{K}|5^+\rangle^2}{[1\,3]^4\langle 4\,5\rangle\langle 5\,6\rangle\langle 1^+|\not{K}|4^+\rangle\langle 3^+|\not{K}|6^+\rangle K^2}, \quad K = K_{123}, \tag{3.12}$$

$$b_2^{\mathcal{N}=4\,\text{non-singlet}} = i\,\frac{\langle 2^+|\not{K}|5^+\rangle^4}{[1\,2][2\,3]\langle 4\,5\rangle\langle 5\,6\rangle\langle 1^+|\not{K}|4^+\rangle\langle 3^+|\not{K}|6^+\rangle K^2},$$

$$b_2^{\mathcal{N}=1\,\text{chiral}} = i\frac{\langle 2^+|\not{K}|5^+\rangle^2\langle 3^+|\not{K}|5^+\rangle\langle 2^+|\not{K}|4^+\rangle}{[1\,2]\langle 5\,6\rangle\langle 3^+|\not{K}|6^+\rangle\langle 1^+|\not{K}|4^+\rangle\langle 3^+|\not{K}|4^+\rangle^2},$$
$$b_2^{\mathcal{N}=0} = 2i\frac{[2\,3]\langle 4\,5\rangle\langle 3^+|\not{K}|5^+\rangle^2\langle 2^+|\not{K}|4^+\rangle^2 K^2}{[1\,2]\langle 5\,6\rangle\langle 3^+|\not{K}|6^+\rangle\langle 1^+|\not{K}|4^+\rangle\langle 3^+|\not{K}|4^+\rangle^4}, \quad K = K_{123}. \tag{3.13}$$

3.3. Two-mass hard box

As an n-point example, we can consider the coefficient of the following box function



which has two massless corners, a corner with a single external positive helicity leg and a corner with a single external negative helicity leg. This box is thus MHV-deconstructible and can be computed using quadruple cuts and the technique of Britto, Cachazo and Feng [24] whereby the massless legs are analytically continued to signature $(--++)$ so that the massless corners do not vanish.

Solving for the box-coefficients we find

$$\rho_{\mathcal{N}=1} = -\frac{\langle 1^+|\not{K}|n^+\rangle^2\langle a^+|\not{K}|b^+\rangle^2}{K^2[a\,1]\langle n\,b\rangle(K^2[a\,1]\langle n\,b\rangle - \langle 1^+|\not{K}|n^+\rangle\langle a^+|\not{K}|b^+\rangle)}, \tag{3.14}$$

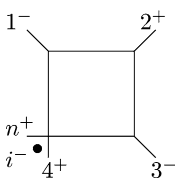
where $K = K_{1\ldots r}$ and the box coefficients

$$c^{\mathcal{N}=4\,\text{non-singlet}} = i\frac{s_{n1}\langle a^+|\not{K}|b^+\rangle^4}{[1\,2]\cdots[r-1\,r]\langle r+1\,r+2\rangle\cdots\langle n-1\,n\rangle\langle 1^+|\not{K}|r+1^+\rangle\langle r^+|\not{K}|n^+\rangle},$$
$$c^{\mathcal{N}=1\,\text{chiral}} = i\frac{[a\,1]\langle b\,n\rangle(K^2[a\,1]\langle n\,b\rangle - \langle 1^+|\not{K}|n^+\rangle\langle a^+|\not{K}|b^+\rangle)s_{n1}(K^2)\langle a^+|\not{K}|b^+\rangle^2}{[1\,2]\cdots[r-1\,r]\langle r+1\,r+2\rangle\cdots\langle n-1\,n\rangle\langle 1^+|\not{K}|n^+\rangle^2\langle 1^+|\not{K}|r+1^+\rangle\langle r^+|\not{K}|n^+\rangle},$$
$$c^{\mathcal{N}=0} = 2i\frac{[a\,1]^2\langle b\,n\rangle^2(K^2[a\,1]\langle n\,b\rangle - \langle 1^+|\not{K}|n^+\rangle\langle a^+|\not{K}|b^+\rangle)^2 s_{n1}(K^2)^2}{[1\,2]\cdots[r-1\,r]\langle r+1\,r+2\rangle\cdots\langle n-1\,n\rangle\langle 1^+|\not{K}|n^+\rangle^4\langle 1^+|\not{K}|r+1^+\rangle\langle r^+|\not{K}|n^+\rangle}. \tag{3.15}$$

3.4. The one-mass boxes

For a one-mass box, adjacent massless legs must have opposite helicity [24] to yield a non-vanishing result upon analytic continuation. Using parity we need only consider the case where the massive corner is mostly positive. The case where exactly two of the massless legs have positive helicity is just the MHV case considered previously.

The remaining case where exactly two of the massless legs have negative helicity is a contribution to the NMHV amplitudes. Specifically we have the one-mass scalar box:

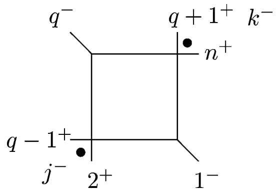


Using the quadruple cuts we can easily determine the coefficients in the three cases:

$$
\begin{aligned}
c^{\mathcal{N}=4\,\text{non-singlet}} &= i\frac{\langle 2^+|\not{K}|i^+\rangle^4}{[1\,2][2\,3]\langle 4\,5\rangle\cdots\langle i\,i+1\rangle\cdots\langle n-1\,n\rangle\langle 1^+|\not{K}|4^+\rangle\langle 3^+|\not{K}|n^+\rangle K^2},\\
c^{\mathcal{N}=1\,\text{chiral}} &= i\frac{\langle 2^+|\not{K}|i^+\rangle^2\langle 1^+|\not{K}|i^+\rangle\langle 3^+|\not{K}|i^+\rangle}{[1\,3]^2\langle 4\,5\rangle\cdots\langle i\,i+1\rangle\cdots\langle n-1\,n\rangle\langle 1^+|\not{K}|4^+\rangle\langle 3^+|\not{K}|n^+\rangle K^2},\\
c^{\mathcal{N}=0} &= 2i\frac{[1\,2][2\,3]\langle 1^+|\not{K}|i^+\rangle^2\langle 3^+|\not{K}|i^+\rangle^2}{[1\,3]^4\langle 4\,5\rangle\cdots\langle i\,i+1\rangle\cdots\langle n-1\,n\rangle\langle 1^+|\not{K}|4^+\rangle^2\langle 3^+|\not{K}|n^+\rangle^2 K^2},\quad K=K_{123}.
\end{aligned}
\tag{3.16}
$$

3.5. The two-mass-easy boxes

In the case of two mass easy boxes, there are no solutions to the kinematic constraints if the massless legs have opposite parity, so $c^{\mathcal{N}=0}$, $c^{\mathcal{N}=1\,\text{chiral}}$ and $c^{\mathcal{N}=4\,\text{non-singlet}}$ vanish for such configurations. As an example of a non-vanishing two mass easy box we consider the box below, which has a single negative helicity leg at each corner.



Setting, $K_2 = k_2 + k_3 + \cdots + k_j + \cdots + k_{q-1}$ and $K_4 = k_{q+1} + \cdots + k_k + \cdots + k_n$, we find

$$
\begin{aligned}
c^{\mathcal{N}=4\,\text{non-singlet}} &= \frac{i}{\mathcal{D}}\langle j^-|K_2K_4|k^-\rangle^4,\\
c^{\mathcal{N}=1\,\text{chiral}} &= -\frac{i}{\mathcal{D}}\frac{\langle q^+|\not{K}_2|j^+\rangle\langle 1^+|\not{K}_2|j^+\rangle\langle 1^+|\not{K}_4|k^+\rangle\langle q^+|\not{K}_4|k^+\rangle\langle j^-|K_2K_4|k^+\rangle^2}{[1\,q]^2},\\
c^{\mathcal{N}=0} &= 2\frac{i}{\mathcal{D}}\frac{\langle q^+|\not{K}_2|j^+\rangle^2\langle 1^+|\not{K}_2|j^+\rangle^2\langle 1^+|\not{K}_4|k^+\rangle^2\langle q^+|\not{K}_4|k^+\rangle^2}{[1\,q]^4},
\end{aligned}
\tag{3.17}
$$

where

$$
\begin{aligned}
\mathcal{D} = {}& K_2^2K_4^2\langle q^+|\not{K}_2|2^+\rangle\langle 1^+|\not{K}_2|q-1^+\rangle\langle 1^+|\not{K}_4|q+1^+\rangle\langle q^+|\not{K}_4|n^+\rangle\\
&\times\langle 2\,3\rangle\langle 3\,4\rangle\cdots\langle q-2\,q-1\rangle\langle q+1\,q+2\rangle\langle q+2\,q+3\rangle\cdots\langle n-1\,n\rangle.
\end{aligned}
\tag{3.18}
$$

4. Twistor-related properties of box coefficients

The results for the twistor structure of the box-coefficients are relatively simple. We find that the box-coefficients within the MHV amplitudes have collinear support in twistor space

$$F_{ijk}c^{\mathcal{N}=4\,\text{MHV}} = F_{ijk}c^{\mathcal{N}=1\,\text{MHV}} = F_{ijk}c^{\mathcal{N}=0\,\text{MHV}} = 0, \tag{4.1}$$

while box-coefficients within NMHV amplitudes have coplanar support

$$K_{ijkl}c^{\mathcal{N}=4\,\text{NMHV}} = K_{ijkl}c^{\mathcal{N}=1\,\text{NMHV}} = K_{ijkl}c^{\mathcal{N}=0\,\text{NMHV}} = 0, \tag{4.2}$$

in twistor space. The coplanarity of the box-coefficients for the $\mathcal{N}=4$ amplitudes was shown in Refs. [14,38]. It was verified for the $\mathcal{N}=1$ box coefficients in [35].

In the generic NMHV case, where we have a three mass box, the legs will have support upon three intersecting lines in twistor space, with the legs at each massive corner being collinear. The geometric picture of this is identical to that of $\mathcal{N}=4$ [15] and indeed this pattern is also inherited by gravity amplitudes [31]. Since the three contributions in a supersymmetric decomposition obey the same twistor space conditions, it follows that these conditions will apply to gluon scattering in many massless gauge theories.

5. Triangles from triple cuts

To obtain the coefficients of triangle integral functions we consider triple cuts [17]. This corresponds to inserting three $\delta(\ell_i^2)$ functions into the four-dimensional integrals. Specifically, we consider

$$\begin{aligned}&\int d^4\ell_1\, d^4\ell_2\, d^4\ell_3\, \delta^4(\ell_1-\ell_2-K_1)\delta^4(\ell_2-\ell_3-K_2)\delta\big(\ell_1^2\big)\delta\big(\ell_2^2\big)\delta\big(\ell_3^2\big)\\ &\quad\times A^{\text{tree}}(\ell_1,k_1,\ldots,k_r,\ell_2)A^{\text{tree}}(-\ell_2,k_{r+1},\ldots,k_{r'},\ell_3)A^{\text{tree}}(-\ell_3,k_{r'+1},\ldots,k_n,-\ell_1).\end{aligned} \tag{5.1}$$

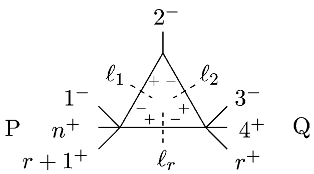
Both triangle functions and box functions contribute to this triple cut. As a strategy, one can first determine the box-coefficients from quadruple cuts and then subtract these from the triple cut to obtain the triangle coefficients. Unlike the quadruple cuts case, the three $\delta(\ell_i^2)$ functions do not freeze the integral, so we must carry out manipulations within the cut integral to recognise the coefficient.

As an example application of triple cuts, consider the amplitude

$$A^{\mathcal{N}=1}\big(1^-,2^-,3^-,4^+,5^+,\ldots,n^+\big). \tag{5.2}$$

This amplitude is particularly amenable in that it contains no box integral functions. This can be seen by examining the integrals in a two-particle cut [39] or, fairly obviously, by observing that there are no solutions to the quadruple cuts.

Consider the following triple cut:



with the momenta on the two massive legs being $P \equiv k_{r+1}+\cdots+k_n+k_1$ and $Q \equiv k_3+k_4+\cdots+k_r$. Within the cut integral, where the cut legs are scalars, the product of the three tree amplitudes is

$$\frac{\langle 1\,\ell_1\rangle^2\langle 1\,\ell_r\rangle^2}{\langle r+1\,r+2\rangle\cdots\langle n\,1\rangle\langle 1\,\ell_1\rangle\langle \ell_1\,\ell_r\rangle\langle \ell_r\,r+1\rangle}\,\frac{\langle 3\,\ell_2\rangle^2\langle 3\,\ell_r\rangle^2}{\langle 3\,4\rangle\cdots\langle r-1\,r\rangle\langle r\,\ell_r\rangle\langle \ell_r\,\ell_2\rangle\langle \ell_2\,3\rangle}\,\frac{\langle 2\,\ell_1\rangle\langle 2\,\ell_2\rangle}{\langle \ell_1\,\ell_2\rangle}. \tag{5.3}$$

To obtain the contribution from the $\mathcal{N}=1$ multiplet we must multiply this by $\rho^{\mathcal{N}=1}$ within the integral. Using

$$\frac{1}{\langle \ell_1\,\ell_r\rangle}=\frac{[\ell_1\,\ell_r]}{P^2},\qquad \frac{1}{\langle \ell_2\,\ell_r\rangle}=\frac{[\ell_2\,\ell_r]}{Q^2},\qquad \frac{1}{\langle r\,\ell_r\rangle}=\frac{[\ell_r\,2]}{\langle r\,\ell_r\rangle[\ell_r\,2]}=\frac{[\ell_r\,2]}{\langle 2^+|\not{P}|r^+\rangle}, \tag{5.4}$$

this product can be rearranged to give

$$\frac{F[\ell_i]\rho^{\mathcal{N}=1}}{\langle 2^+|P|r^+\rangle\langle 2^+|P|r+1^+\rangle\langle 3\,4\rangle\cdots\langle r-1\,r\rangle\langle r+1\,r+2\rangle\cdots\langle n\,1\rangle P^2Q^2\langle \ell_1\,\ell_2\rangle}, \tag{5.5}$$

where much of the denominator can now be taken outside the cut integral and

$$F[\ell_i]=\langle 1\,\ell_1\rangle\langle 1\,\ell_r\rangle^2\langle 3\,\ell_2\rangle\langle 3\,\ell_r\rangle^2\langle 2\,\ell_1\rangle\langle 2\,\ell_2\rangle[\ell_2\,\ell_r][\ell_1\,\ell_r][2\,\ell_r]^2. \tag{5.6}$$

When combining the different particles' contributions we have

$$X=\frac{\langle 1\,\ell_1\rangle}{\langle 1\,\ell_r\rangle}\frac{\langle 2\,\ell_2\rangle}{\langle 2\,\ell_1\rangle}\frac{\langle 3\,\ell_r\rangle}{\langle 3\,\ell_2\rangle},\quad \text{so that } \rho^{\mathcal{N}=1}=\frac{(\langle 1\,\ell_1\rangle\langle 2\,\ell_2\rangle\langle 3\,\ell_r\rangle-\langle 1\,\ell_r\rangle\langle 2\,\ell_1\rangle\langle 3\,\ell_2\rangle)^2}{\langle 1\,\ell_1\rangle\langle 1\,\ell_r\rangle\langle 2\,\ell_2\rangle\langle 2\,\ell_1\rangle\langle 3\,\ell_r\rangle\langle 3\,\ell_2\rangle}. \tag{5.7}$$

Thus the loop momentum dependent part of the integrand is

$$\frac{F[\ell_i]\rho^{\mathcal{N}=1}}{\langle\ell_1\,\ell_2\rangle}=\frac{\langle 1\,\ell_r\rangle\langle 3\,\ell_r\rangle[\ell_2\,\ell_r][\ell_1\,\ell_r][2\,\ell_r]^2(\langle 1\,\ell_1\rangle\langle 2\,\ell_2\rangle\langle 3\,\ell_r\rangle-\langle 1\,\ell_r\rangle\langle 2\,\ell_1\rangle\langle 3\,\ell_2\rangle)^2}{\langle\ell_1\,\ell_2\rangle}. \tag{5.8}$$

To evaluate this we use the identity

$$\langle 1\,\ell_1\rangle\langle 2\,\ell_2\rangle\langle 3\,\ell_r\rangle-\langle 1\,\ell_r\rangle\langle 2\,\ell_1\rangle\langle 3\,\ell_2\rangle=\langle 3^-|QP|1^+\rangle\frac{\langle\ell_1\,\ell_2\rangle}{[2\,\ell_r]}, \tag{5.9}$$

which is valid due to the momentum constraints. The part of the integrand which still depends on the loop momentum can be rearranged

$$\begin{aligned}\langle 1\,\ell_r\rangle\langle 3\,\ell_r\rangle[\ell_2\,\ell_r][\ell_1\,\ell_r]\langle\ell_1\,\ell_2\rangle&=\langle 1\,\ell_r\rangle[\ell_r\,\ell_1]\langle\ell_1\,\ell_2\rangle[\ell_2\,\ell_r]\langle\ell_r\,3\rangle\\&=\langle 1^-|\not\ell_r\not\ell_1\not\ell_2\not\ell_r|3^+\rangle=\langle 1^-|\not P\not\ell_1\not\ell_2\not Q|3^+\rangle,\end{aligned} \tag{5.10}$$

using $\not\ell_r=\not\ell_1+\not P$, $\not\ell_r=\not\ell_2-\not Q$. Finally we can reduce this to a linear function by using $\not\ell_1=\not\ell_2+\not k_2$,

$$\frac{1}{2}\langle 1^-|\not P(\not k_2\not\ell_2-\not\ell_1\not k_2)\not Q|3^+\rangle, \tag{5.11}$$

where we chose to perform the algebra in such a way as to reflect the symmetry of the diagram: this facilitates the identification of the triangle coefficients. To solve this triangle we first Feynman parameterise and make a shift of momenta

$$\ell_1^\mu\longrightarrow\ell_1^{\mu\,\prime}-k_2^\mu a_3-(k_2+Q)^\mu a_{r+1},\qquad \ell_2^\mu\longrightarrow\ell_1^{\mu\,\prime}-k_2^\mu a_3-(k_2+Q)^\mu a_{r+1}-k_2^\mu, \tag{5.12}$$

leading to

$$\frac{1}{2}\langle 1^-|\not P(\not k_2\not Q-\not Q\not k_2)\not Q|3^+\rangle a_{r+1}. \tag{5.13}$$

Finally, the Feynman parameter integral $I[a_{r+1}]$ can be expressed in terms of the L_0 functions

$$I[a_{r+1}]=\frac{\mathrm{L}_0[P^2/Q^2]}{Q^2}, \tag{5.14}$$

where we use the integral functions

$$\mathrm{L}_0[r]=\frac{\ln(r)}{1-r}+\mathcal{O}(\epsilon)\quad\text{and}\quad \mathrm{K}_0(s)=\left(-\ln(-s)+2+\frac{1}{\epsilon}\right)+\mathcal{O}(\epsilon). \tag{5.15}$$

From the triple cut we can now identify the coefficient of the L_0 triangle function:

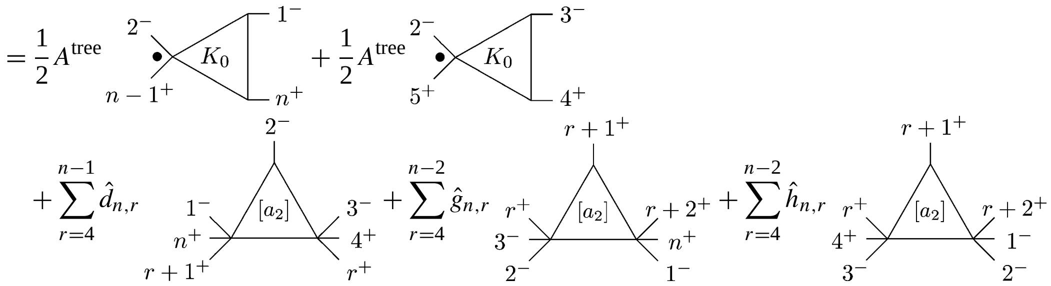
$$\frac{(\langle 3^-|QP|1^+\rangle)^2\langle 3^-|(Q(2P-P2)P)|1^+\rangle}{\langle 2^+|P|r^+\rangle\langle 2^+|P|r+1^+\rangle\langle 3\,4\rangle\cdots\langle r-1\,r\rangle\langle r+1\,r+2\rangle\cdots\langle n\,1\rangle P^2Q^2}. \tag{5.16}$$

Similarly, we can determine all the triangle functions present in the amplitude using triplet cuts, obtaining the expression for the full amplitude

$$A^{\mathcal{N}=1}\big(1^-,2^-,3^-,4^+,5^+,\ldots,n^+\big)=\frac{A^{\text{tree}}}{2}\big(\mathrm{K}_0(s_{n1})+\mathrm{K}_0(s_{34})\big)-\frac{i}{2}\sum_{r=4}^{n-1}\hat{d}_{n,r}\frac{\mathrm{L}_0[t_3^{[r-2]}/t_2^{[r-1]}]}{t_2^{[r-1]}}$$
$$-\frac{i}{2}\sum_{r=4}^{n-2}\hat{g}_{n,r}\frac{\mathrm{L}_0[t_2^{[r-1]}/t_2^{[r]}]}{t_2^{[r]}}-\frac{i}{2}\sum_{r=4}^{n-2}\hat{h}_{n,r}\frac{\mathrm{L}_0[t_3^{[r-2]}/t_3^{[r-1]}]}{t_3^{[r-1]}},\qquad(5.17)$$

which can be depicted in the following way:

$$A^{\mathcal{N}=1}\big(1^-,2^-,3^-,4^+,5^+,\ldots,n^+\big)$$



where

$$\hat{d}_{n,r}=\frac{(\langle 3^-|K_{r-3}\bar{K}_{r-3}|1^+\rangle)^2\langle 3^-|K_{r-3}(k_2\bar{K}_{r-3}-\bar{K}_{r-3}k_2)\bar{K}_{r-3})|1^+\rangle}{\langle 2^+|\bar{K}_{r-3}|r^+\rangle\langle 2^+|\bar{K}_{r-3}|r+1^+\rangle\langle 3\,4\rangle\cdots\langle r-1\,r\rangle\langle r+1\,r+2\rangle\cdots\langle n\,1\rangle\bar{K}_{r-3}^2K_{r-3}^2},$$
$$\hat{g}_{n,r}=\sum_{i=1}^{r-3}\frac{\langle 3^-|K_i\bar{K}_i|1^+\rangle^2\langle 3^-|K_i\bar{K}_i(k_{r+1}\bar{K}_{r-3}-\bar{K}_{r-3}k_{r+1})|1^+\rangle\langle i+3\,i+4\rangle}{\langle 2^+|\not{K}_i|i+3^+\rangle\langle 2^+|\not{K}_i|i+4^+\rangle\langle 3\,4\rangle\langle 4\,5\rangle\cdots\langle n\,1\rangle K_i^2\bar{K}_i^2},$$
$$\hat{h}_{n,r}=\hat{g}_{n,n-r+2}|_{(123\ldots n)\to(321n\ldots 4)},\qquad(5.18)$$

with $K_i=k_3+k_4+\cdots+k_{i+3}$ and $\bar{K}_i=k_2+k_3+\cdots+k_{i+3}$. We have checked that this expression satisfies the correct collinear and soft limits thus confirming the normalisation.

6. Conclusion

Perturbative amplitudes in quantum field theories are complex objects which contain a great deal of information, some of which is rather well understood and some less so. The recently proposed relationships between perturbative gauge theories and twistor strings provide a fascinating insight into gauge theories and may be very useful in perturbative calculations. It also remains an open question as to whether a string theory can be completely reconstructed from its states and its on-shell tree amplitudes using unitarity and other techniques.

Although relations with twistor string theories have been observed for $\mathcal{N}=4$ super-Yang–Mills, it is an open question as to what degree theories with less or no supersymmetry are related to a twistor string theory [40]. Until a direct connection is uncovered it is reasonable to gather evidence by studying the properties of amplitudes. The box-coefficients are a physically meaningful subset of an amplitude being the coefficients of distinct functions of the class $\ln(s)\ln(s')$. By computing some special examples, we have observed that even for non-supersymmetric theories (but still massless) box-coefficients satisfy the same collinearity and coplanarity constraints as in $\mathcal{N}=4$ theories. These constraints can be seen as a consequence of the construction of box coefficients using unitarity but may be a hint of the underlying string structure.

For $\mathcal{N} = 4$ theories the amplitudes are completely determined from the box-coefficients. For theories with less supersymmetry the amplitudes contain additional, and important, functional information. As an example of using unitarity constraints, we have presented the full structure of the simplest NMHV configuration for n-gluons in $\mathcal{N} = 1$ super-Yang–Mills. This amplitude is entirely expressed in terms of (specific) triangle functions. The coefficients of these functions were determined by carrying out triple cuts of the amplitude. These coefficients do not have an obvious twistor property such as coplanarity.

Theories without supersymmetry are the most interesting phenomenologically and, arguably, formally. Unitarity techniques, generalised sufficiently, may in principle determine such perturbative amplitudes [41,42] but practical computations are extremely sparse at this point. It remains a challenge to develop techniques and perform calculations for theories without any supersymmetry.

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