Spaces of valuations as quasimetric domains

Philipp Sünderhauf [1]

Department of Computing
Imperial College
London SW7 2BZ, England
P.Sunderhauf@doc.ic.ac.uk

E. N. T. C. S.

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Abstract

We define a natural quasimetric on the set of continuous valuations of a topological space and investigate it in the spirit of quasimetric domain theory. It turns out that the space of valuations of an (ordinary) algebraic domain D is an algebraic quasimetric domain. Moreover, it is precisely the lower powerdomain of D, where D is regarded as a quasimetric domain. The essential tool for proving these results is a generalization of the Splitting Lemma which characterizes the quasimetric for simple valuations and holds for valuations on arbitrary topological spaces.

1 Introduction

A *continuous valuation* [8,11] on a topological space $(X, \mathcal{T})$, where $\mathcal{T}$ denotes the collection of open subsets of X, is a function $\mu: \mathcal{T} \to [0, \infty]$ satisfying

- $\mu(\emptyset) = 0$ (strictness)
- $\mu(O \cup O') + \mu(O \cap O') = \mu(O) + \mu(O')$ (modularity)
- $O \subseteq O' \Longrightarrow \mu(O) \leq \mu(O')$ (monotonicity)
- $\mu(\bigcup_{i \in I} O_i) = \sup_{i \in I} \mu(O_i)$ for all directed (wrt $\subseteq$) families $(O_i)_{i \in I}$ of open sets (continuity).

Continuous valuations were introduced in [8] and, for a continuous lattice in place of the topology $\mathcal{T}$, in [11].

Continuous valuations play an important role in denotational semantics, where the set of all continuous valuations μ with $\mu(X) = 1$, the *probabilistic*

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powerdomain of X, is used to model probabilistic nondeterminism [8,9]. They are also the main ingredient in the domain-theoretic approach to measure theory which was developed in [5] and lead to the definition of the generalized Riemann integral [4].

The present paper investigates the set of continuous valuations as a quasimetric space. We define a natural quasimetric, the *sup-distance* for continuous valuations. Although defined in terms of the masses of all open sets, the distance can be characterized for simple valuations purely in terms of the weights and points. This result, proved in Section 2.2, generalizes the well-known *Splitting Lemma* [8], a combinatorial characterization of the order on simple valuations which is an invaluable tool when working with valuations.

Then the space is investigated in the setting of quasimetric domain theory [14,6,3], where notions from (ordinary) domain theory are generalized to quasimetric spaces. In the present paper, we provide the necessary definitions and results from quasimetric domain theory where needed, namely at the beginning of Sections 3 and 4.

The space of valuations on an (ordinary) algebraic domain D turns out to be an algebraic domain in the setting of quantitative domain theory. Moreover, it is exactly the quasimetric powerdomain of D as defined in [15] when D is regarded as a quasimetric domain.

1.1 Preliminaries and notations

The set of all continuous valuations on X is denoted by $V(X)$. It is ordered pointwise, i.e.

$$\mu \sqsubseteq \nu \iff \mu(O) \leq \nu(O) \text{ for all } O \in \mathcal{T}.$$

We write $V_{\leq 1}(X) = \{\mu \in V(X) \mid \mu(X) \leq 1\}$ for the probabilistic powerdomain of X and $V_{=1}(X) = \{\mu \in V(X) \mid \mu(X) = 1\}$ for the set of continuous valuations with total mass exactly 1.

Continuous valuations may be added and multiplied by scalars from $[0, \infty]$. Both operations are defined pointwise. The *point valuations* δ_a for $a \in X$ are defined by

$$\delta_a(O) = \begin{cases} 1 & a \in O \\ 0 & a \notin O. \end{cases}$$

Finite linear combinations of point valuations, i.e. valuations of the form $\sum_{a \in A} r_a \delta_a$ are referred to as *simple valuations.* Imposing the additional condition that all weights r_a are non-zero ensures uniqueness of the points and weights specifying a simple valuation [10].

Addition and multiplication on the set of extended nonnegative real numbers $[0, \infty]$ are defined by $\infty + x = x + \infty = \infty$ for all $x \in [0, \infty]$ and by $\infty \cdot x = x \cdot \infty = \infty$ for $x \neq 0$ and $0 \cdot \infty = \infty \cdot 0 = 0$. Truncated subtraction is given by $x \dot{-} y = (x - y) \vee 0$ for $x, y \in [0, \infty)$ and $\infty \dot{-} y = \infty$ for all $y < \infty$

and $x \dot{-} \infty = 0$ for all $x \in [0, \infty]$.

A *quasimetric* on a set X is a function $d: X \times X \to [0, \infty]$ such that (1) $d(x, x) = 0$, (2) if $d(x, y) = 0 = d(y, x)$ then $x = y$, and (3) such that the triangle inequality $d(x, y) + d(y, z) \geq d(x, z)$ holds for all $x, y, z \in X$. We frequently refer to d as *distance function*. The *order derived from* d is defined by $x \sqsubseteq_d y$ iff $d(x, y) = 0$. A *quasimetric space* is a pair (X, d), where X is a set and d a quasimetric. A function $f: X \to Y$ between quasimetric spaces is *nonexpansive* if $d(x, y) \geq d(f(x), f(y))$ holds for all $x, y \in X$. We denote with $B_\varepsilon(x) = \{y \in X \mid \varepsilon > d(x, y)\}$ the open ε-ball around x. These are the basic neighbourhoods for the *ε-ball topology* derived from d. The *opposite* quasimetric space of (X, d) has distance $d^{-1}(x, y) = d(y, x)$ and is denoted by X^{-1}.

2 A distance for valuations

2.1 The sup-distance

We want to supply $V(X)$ with a quasimetric. A natural choice is the *sup-distance*

$$d(\mu, \nu) = \sup_{O \in \mathcal{T}} \Big(\mu(O) \dot{-} \nu(O) \Big),$$

which is the distance of the infinite product $[0, \infty]^{\mathcal{T}}$ restricted to the set of continuous valuations. It is immediate that this quasimetric induces the pointwise order on $V(X)$: we have $\mu \leq \nu$ iff $\forall O \in \mathcal{T}.\ \mu(O) \leq \nu(O)$ iff $\forall O \in \mathcal{T}.\ \mu(O) \dot{-} \nu(O) = 0$ iff $d(\mu, \nu) = 0$.

The probabilistic powerdomain is the closed 1-ball with respect to the inverse quasimetric d^{-1} around the constant-0 valuation:

$$V_{\leq 1}(X) = \{\mu \in V(X) \mid d(\mu, 0) \leq 1\}.$$

The boundary of this ball is exactly the set $V_{=1}(X)$.

This distance function has very natural properties. Recall that the *information order* on the topological space $(X, \mathcal{T})$ is defined by $x \sqsubseteq_{\mathcal{T}} y$ iff $(x \in O \Longrightarrow y \in O)$ for all $O \in \mathcal{T}$.

Proposition 2.1 *The distance on $V(X)$ satisfies*

(1) $d(r\delta_x, r\delta_y) = \begin{cases} 0 & x \sqsubseteq_{\mathcal{T}} y \\ r & x \not\sqsubseteq_{\mathcal{T}} y \end{cases}$

(2) $d(r\delta_x, s\delta_x) = r \dot{-} s$

(3) $d(\mu, \nu) \geq d(\mu + \eta, \nu + \eta)$

(4) $m d(\mu, \nu) \geq d(m\mu, m\nu)$

for all $x, y \in X$, $r, s \in [0, \infty]$ and $\mu, \nu, \eta \in V(X)$.

Proof. Note that $r\delta_x(O) \dot{-} r\delta_y(O)$ equals r if $x \in O$ and $y \notin O$ and is zero otherwise. Hence (1) follows as $x \not\sqsubseteq_{\mathcal{T}} y$ iff there is $O \in \mathcal{T}$ with the above property. Equation (2) is trivial and (3) follows from the fact that $(a+x) \dot{-} (b+x) \leq a \dot{-} b$ holds for all $a, b, x \in [0, \infty]$. Finally, we have $m(a \dot{-} b) \geq ma \dot{-} mb$ for all $m, a, b \in [0, \infty]$ which implies (4). □

Choosing for η the constant-∞ valuation and $m = \infty$ reveals that equality holds neither in (3) nor in (4).

2.2 The distance for simple valuations

For simple valuations, it is possible to define the distance only in terms of the points and weights, without referring to open sets. In the sequel we will define a distance $\hat{d}$ on the set of simple valuations and then prove it to coincide with d. First, we need to define an auxiliary function d_0. Proposition 2.1(1,2), tells us how to start the definition.

$$d_0(r\delta_x, r\delta_y) := \begin{cases} 0 & x \sqsubseteq_{\mathcal{T}} y \\ r & x \not\sqsubseteq_{\mathcal{T}} y \end{cases}$$

$$d_0(r\delta_x, s\delta_x) := r \dot{-} s$$

for all $x, y \in X$ and $r, s \in [0, \infty]$. Inspired by (3) and (4) in Proposition 2.1, we extend this function by setting

(5) $d_0(r\delta_x + \eta, r\delta_y + \eta) := d_0(r\delta_x, r\delta_y)$

(6) $d_0(r\delta_x + \eta, s\delta_x + \eta) := d_0(r\delta_x, s\delta_x)$

for all simple valuations η and x, y, r, s as before.

Let $\sim$ be the domain of d_0, i.e. write $\mu \sim \nu$ whenever $d_0(\mu, \nu)$ is defined by one of the above cases. The distance function $\hat{d}$ on the set of simple valuations is now defined as follows. For simple valuations μ and ν, we write $[\mu, \nu]$ for the set of all sequences $\alpha = (\eta_0, \eta_1, \ldots, \eta_n)$ with $\eta_0 = \mu$, $\eta_n = \nu$ and $\eta_{i-1} \sim \eta_i$ for all $i = 1, \ldots, n$. For such a sequence α we set

$$w(\alpha) = \sum_{i=1}^{n} d_0(\eta_{i-1}, \eta_i).$$

Now we define

$$\hat{d}(\mu, \nu) = \inf_{\alpha \in [\mu, \nu]} w(\alpha).$$

Note that there is a path from the zero valuation to every simple valuation and vice versa. Moreover, paths can be concatenated: if $\alpha = (\eta_0, \ldots, \eta_n) \in [\mu, \nu]$ and $\beta = (\eta'_0, \ldots, \eta'_m) \in [\nu, \eta]$ then $\alpha\beta := (\eta_0, \ldots, \eta_n, \eta'_1, \ldots, \eta'_m) \in [\mu, \eta]$ and $w(\alpha\beta) = w(\alpha) + w(\beta)$. As a consequence, we conclude that $\hat{d}$ is indeed defined for all simple valuations and satisfies the triangle inequality.

We will see that $\hat{d} = d$. For the case $d = \hat{d} = 0$ this is a reformulation of the Splitting Lemma [8,9]. The essential tool in the proof is the Max-Flow Min-Cut Theorem from graph theory, see e.g. [2].

Definition 2.2 Suppose $G = (V, E)$ is a directed graph with capacity $c\colon E \to [0, \infty]$ and two distinct vertices $\bot$ and $\top$. A *flow from* $\bot$ *to* $\top$ is a function $f\colon E \to [0, \infty]$ such that (1) $f \le c$ (pointwise) and (2) for all $x \in V \setminus \{\bot, \top\}$ the inflow at vertex x equals the outflow: $\sum_{w \in V,\ \langle w,x\rangle \in E} f(w, x) = \sum_{y \in V,\ \langle x,y\rangle \in E} f(x, y)$. The *value* of the flow is given by $\sum_{x \in V,\ \langle \bot,x\rangle \in E} f(\bot, x)$. A *cut* is a set $C \subseteq V$ such that $\bot \in C$ and $\top \notin C$. The value of the cut C is $\sum \{f(a, b) \mid a \in C;\ b \in V \setminus C\}$.

Theorem 2.3 (Max-Flow Min-Cut Theorem) *There exists a flow with maximum value. This value equals the minimum value of a cut.*

Theorem 2.4 (Generalized Splitting Lemma) *The sup-distance d coincides for simple valuations with $\hat{d}$.*

Proof. If $\alpha \in [\mu, \nu]$ then $d(\mu, \nu) \le w(\alpha)$ by the triangle inequality in conjunction with the definition of d_0 and Proposition 2.1. Thus it is clear that $d \le \hat{d}$ holds whenever the latter is defined. To prove the reverse inequality, suppose

$$\eta = \sum_{a \in A} r_a \delta_a \quad \text{and} \quad \eta' = \sum_{b \in B} s_b \delta_b$$

for certain finite sets $A, B \subseteq X$. Write $r = \sum_{a \in A} r_a$ and let $d^* = d(\eta, \eta') = \sup_{O \in \mathcal{T}} (\eta(O) \dot{-} \eta'(O))$. If $K \subseteq A$ then

$$\sum_{a \in K} r_a \le \sum_{b \in B \cap \uparrow K} s_b + d^* \tag{7}$$

as we can pick an open set $O \in \mathcal{T}$ such that for all $x \in A \cup B$ it is true that $x \in O$ holds iff $x \in \uparrow K = \{y \in X \mid \exists k \in K.\ k \sqsubseteq_{\mathcal{T}} y\}$. Now we construct a directed graph as follows. The first part is the graph from Jones' proof of the Splitting Lemma [8, Theorem 4.10]: we have nodes $\bot$ and $\top$ and one node for each $a \in A$ and $b \in B$ as well as edges from $\bot$ to each a, from each b to $\top$ and an edge $\langle a, b\rangle$ whenever $a \sqsubseteq b$. Capacities are as before: the edges $\langle \bot, a\rangle$ have capacity r_a, the edges $\langle a, b\rangle$ have capacity r and finally s_b for the edges $\langle b, \top\rangle$. Now we add one further node, $*$, with edges $\langle *, \top\rangle$ and $\langle a, *\rangle$, for all $a \in A$. All these edges have capacity d^*. We claim that the minimal value of a cut is r. If this is true then, by the Max-Flow Min-Cut Theorem, there is a flow with value r. This flow may be described by numbers $t_{a,b}$ for $a \sqsubseteq b$ (denoting the flow from a to b) and u_a for $a \in A$ (denoting the flow from a to $*$) such that inflow=outflow at all nodes. As the value of the flow is r and $\sum_{a \in A} r_a = r$, it must be true that the flow from $\bot$ to a equals r_a for all $a \in A$. Hence, the inflow=outflow condition for these nodes yields

$$r_a = u_a + \sum_{b \in B \cap \uparrow a} t_{a,b} \tag{8}$$

for all $a \in A$. Evaluating the condition at the vertices $b \in B$ gives

$$\sum_{a \in A \cap \downarrow b} t_{a,b} \le s_b \tag{9}$$

for all $b \in B$. Finally node $*$. We get

$$\sum_{a \in A} u_a \leq d^*. \tag{10}$$

We now define a path from η to η' as follows, writing $\sim_5$ and $\sim_6$ to denote a finite number of $\sim$-steps in the path involving only $\sim$-relations due to equation (5) or (6), respectively.

$$\begin{aligned}
(11)\ \eta = \sum_{a \in A} r_a \delta_a &= \sum_{a \in A} (u_a + \sum_{b \in \uparrow a \cap B} t_{a,b}) \delta_a \\
(12) \qquad &\sim_6 \sum_{a \in A} (0 + \sum_{b \in \uparrow a \cap B} t_{a,b}) \delta_a \\
(13) \qquad &= \sum_{b \in B} \sum_{a \in A \cap \downarrow b} t_{a,b} \delta_a \\
(14) \qquad &\sim_5 \sum_{b \in B} \sum_{a \in A \cap \downarrow b} t_{a,b} \delta_b \\
(15) \qquad &= \sum_{b \in B} \Big(\sum_{a \in A \cap \downarrow b} t_{a,b} \Big) \delta_b \\
(16) \qquad &\sim_6 \sum_{b \in B} s_b \delta_b = \eta'
\end{aligned}$$

Here the equality in (11) is valid by (8) whereas (13) and (15) are merely rearrangements. The weight of this path is the sum of the d_0-distances in the $\sim$-steps. All the steps in (14) are of the form $(\mu + q\delta_a) \sim (\mu + q\delta_b)$ with $a \sqsubseteq b$, so the distance is zero. Also, the steps in (16) yield $d_0 = 0$ as they are of the form $(\mu + q_1\delta_b) \sim (\mu + q_2\delta_b)$, where $q_1 \leq q_2$ by (9). Let us finally consider the steps in (12). They are of the form $(\mu + u_a\delta_a) \sim (\mu + 0\delta_a)$ for $a \in A$. Such a step costs $d_0(u_a\delta_a, 0) = u_a$. Hence the weight of the path is $\sum_{a \in A} u_a$ which, by (10), is at most d^*. Thus $\hat{d}(\eta, \eta') \leq d^* = d(\eta, \eta')$.

It remains to prove the claim, i.e. that the minimum value of a cut is r. Observe that the cut $\{\bot\}$ has value $\sum_{a \in A} r_a = r$, so we have to show that there is no cut with smaller value. Let C be a cut. If there are $a \in A$ and $b \in B$ such that $a \sqsubseteq b$ and $a \in C$ but $b \notin C$, then the value of the cut is certainly at least r since this is the capacity of the edge $\langle a, b \rangle$. We have already seen that the value is at least r if there is no $a \in A \cap C$. We are left with the case that $A' = A \cap C$ is not empty and that $\uparrow A' \cap B \subseteq C$. Observe that there is at least one edge with weight d^* leading out of the cut, either $\langle *, \top \rangle$ or one of the $\langle a, * \rangle$. Using (7), we see that the value of such a cut is at least

$$\sum_{a \in A \setminus A'} r_a + \sum_{b \in \uparrow A' \cap B} s_b + d^* \geq \sum_{a \in A \setminus A'} r_a + \sum_{a \in A'} r_a = r.$$

So the cut's value is at least r. This verifies the claim and finishes the proof of the theorem. □

Corollary 2.5 *If μ and ν are simple valuations then there is a path $\alpha \in [\mu, \nu]$ with $d(\mu, \nu) = \hat{d}(\mu, \nu) = w(\alpha)$.*

As a consequence, we derive the usual Splitting Lemma in a formulation using *elementary steps*, see [16, Lemma 2.10].

Corollary 2.6 *If μ and ν are simple valuations then $\mu \sqsubseteq \nu$ if and only if there is a chain $\mu = \eta_0 \sqsubseteq \eta_1 \sqsubseteq \cdots \sqsubseteq \eta_n = \nu$ where for each $i \in \{1, \ldots, n\}$ the simple valuation η_i is obtained from η_{i-1} by an elementary step of the form $\eta_{i-1} = \eta + r\delta_a \sqsubseteq \eta + r\delta_b = \eta_i$ with $a \sqsubseteq b$ or $\eta_{i-1} = \eta + r\delta_a \sqsubseteq \eta + s\delta_a = \eta_i$ with $r \leq s$ for some η, a, b, r, s.*

3 V(X) as quasimetric domain

We are now going to investigate the quasimetric space $(V(X), d)$ in the spirit of quasimetric domain theory.

3.1 Basic notions of quasimetric domain theory

In this section, we briefly introduce the basic notions of quasimetric domain theory as a special case of quantitative domain theory developed in [6]. This theory goes back to ideas of Smyth [14] and Lawvere [12]. Rather than using the approach via ideals as in [6], however, we take the approach using nets. It was initiated in [14] and developed (with sequences in place of nets) in [13,3]. In Section 8 of [6] the two approaches are shown to be equivalent.

Quasimetric spaces may be interpreted as generalized partially ordered sets. To do so, think of $[0, \infty]$ as the set of truth-values, of 0 as `true`, of $+$ as logical conjunction &, and as the relation $p \geq q$ as entailment $p \vdash q$. The values of the distance function d may be thought of as the "truth value" of the assertion "$x \sqsubseteq y$". In this setting, the triangle inequality corresponds to the law of transitivity, $d(x, x) = 0$ reveals reflexivity and the second assumption is antisymmetry. This logical interpretation is studied in greater detail in [6]. A net $(x_i)_{i \in I}$ on X is *forward Cauchy* if, for all $\varepsilon > 0$, there is $i \in I$ such that whenever $i \leq j \leq k$ we have $\varepsilon > d(x_j, x_k)$. A point $x \in X$ is the *directed limit* of the net, denoted by $x = \lim^{\uparrow}_{i \in I} x_i$, if $d(x, y) = \inf_{i \in I} \sup_{j \geq i} d(x_j, y)$ holds for all $y \in X$. A quasimetric space is *directed complete*, or a *qmdcpo,* if all forward Cauchy nets have a directed limit.

A set $O \subseteq X$ is Scott-open iff it is open with respect to the ε-ball topology and whenever $\lim^{\uparrow}_i x_i \in O$ for a forward Cauchy net $(x_i)_{i \in I}$, then there is $i \in I$ such that $B_\varepsilon(x_j) \subseteq O$ for all $j \geq i$; the collection of all these subsets is the *Scott topology.* A function $f \colon X \to Y$ is *Scott-continuous,* if it is continuous with respect to the Scott topologies on X and Y. A nonexpansive function between qmdcpos is Scott-continuous iff it preserves directed limits of forward Cauchy nets.

An element k in a qmdcpo X is *compact* if $d(k, \lim^{\uparrow}_i x_i) = \lim_i d(k, x_i)$ holds for all forward Cauchy nets $(x_i)_{i \in I}$. A subset $Y \subseteq X$ *generates* X if every element of X is the directed limit of a forward Cauchy net in Y. A qmdcpo is *algebraic,* or an *algebraic qm-domain,* if it is generated by its set of compact elements.

3.2 Directed completeness

The first subject is directed completeness and we first look at infinite products in general. The product in the category of quasimetric spaces and non-expansive maps may be constructed by taking the cartesian product with the *sup-distance* $d((x_i)_{i\in I},(y_i)_{i\in I}) = \sup_{i\in I} d(x_i,y_i)$. This product is well-behaved with respect to directed completeness (cf. Theorem 6.5 of [13]):

Proposition 3.1 *If (X,d) is a qmdcpo, and Y is any set, then the infinite product X^Y, endowed with the sup-distance, is a qmdcpo. Limits of forward Cauchy nets are calculated pointwise.*

Proof. Let $(f_i)_{i\in I}$ be a Cauchy net in X^Y. It is clear that the coordinate nets $(f_i(y))_{i\in I}$ are Cauchy for all $y \in Y$. We have to check that the pointwise limit f, defined by $f(y) = \lim_{i\in I} f_i(y)$ is the limit of the original Cauchy net, i.e. that for all $g \in X^Y$ we have

$$d(f,g) = \inf_{i\in I}\sup_{j\geq i} d(f_j,g). \tag{17}$$

The function f is the pointwise limit, hence

$$d(f(y),g(y)) = \inf_{i\in I}\sup_{j\geq i} d(f_j(y),g(y)) \tag{18}$$

for all $y \in Y$. But certainly $d(f_j(y),g(y)) \leq d(f_j,g)$, so (18) implies that $d(f(y),g(y))$ is smaller than or equal to the RHS in (17) for all $y \in Y$. Thus $\leq$ in (17) holds. For $\geq$ observe that $\inf_{i\in I}\sup_{j\geq i} d(f_j,g) \leq \inf_{i\in I}\sup_{j\geq i}(d(f_j,f)+d(f,g)) = d(f,g) + \inf_{i\in I}\sup_{j\geq i} d(f_j,f)$, so it remains to show

$$\inf_{i\in I}\sup_{j\geq i} d(f_j,f) = 0. \tag{19}$$

Suppose $\varepsilon > 0$. As the net $(f_i)_{i\in I}$ is Cauchy, there is $i \in I$ such that $i \leq j \leq k$ implies $\varepsilon > d(f_j,f_k)$ for all $j,k \in I$. For each $y \in Y$, equation (19) holds with $d(f_j(y),f(y))$ in place of $d(f_j,f_y)$ by pointwise convergence of the net. So there is an index $i_y \in I$ such that $\varepsilon \geq \sup_{j\geq i_y} d(f_j(y),f(y))$. Fix $j \geq i$. Then, for each $y \in Y$, there is $k \in I$ larger than both j and i_y. Hence $\varepsilon+\varepsilon \geq d(f_j,f_k) + d(f_k(y),f(y)) \geq d(f_j(y),f(y))$ by construction. Thus $\varepsilon+\varepsilon \geq \sup_{j\geq i} d(f_j,f)$ and equation (19) follows. □

This result enables us to see that the spaces of valuations are qmdcpos.

Theorem 3.2 *The spaces $(V(X),d)$, $(V_{\leq 1}(X),d)$ and $(V_{=1}(X),d)$ are qmdcpos.*

Proof. We consider $V(X)$ as a subset of the infinite product $[0,\infty]^{\mathcal{T}}$ which is directed complete by Proposition 3.1. By the same proposition, limits of forward Cauchy nets are calculated pointwise, so it is a standard observation that $V(X)$ is closed under their formation: Suppose $(\mu_i)_{i\in I}$ is such a net with pointwise limit μ and $O, O' \in \mathcal{T}$. Then $\mu(O \cup O') + \mu(O \cap O') = \lim_{i \in I}(\mu_i(O \cup O') + \mu_i(O \cap O')) = \lim_{i \in I}(\mu_i(O) + \mu_i(O')) = \mu(O) + \mu(O')$ and similarly for the other equations and for continuity.

Directed completeness of the subspaces $(V_{\leq 1}(X), d)$ and $(V_{=1}(X), d)$ is shown in a similar fashion: if the equation $\mu_i(X) \leq 1$ or $\mu_i(X) = 1$, respectively, holds for all $i \in I$, where $(\mu_i)_{i \in I}$ is a forward Cauchy net in $V(X)$, then it also holds for the limit since this is calculated pointwise. □

3.3 Algebraic domains

Now let us turn our attention to the case when the topological space $(X, \mathcal{T})$ is an (ordinary) algebraic domain $(D, \sqsubseteq)$ in its Scott-topology $\sigma_{\sqsubseteq}$. See [1] for domain-theoretic definitions and notations. Then the set of continuous valuations $V(D)$ is a continuous domain with the set of all simple valuations as basis [8]. (In fact, the same is true for the case of the domain D being continuous.) Of our special interest are *finite simple valuations based on compact elements,* by which we understand simple valuations $\sum_{a \in a} r_a \delta_a$ such that each $a \in A$ is a compact element of $(D, \sqsubseteq)$ and such that each r_a is finite. We denote the set of all finite simple valuations based on compact elements by $V_0(D)$. Recall that $a \in D$ is compact iff the principal filter $\uparrow a = \{b \in D \mid a \sqsubseteq b\}$ is Scott-open.

Lemma 3.3 *Suppose $\eta = \sum_{a \in A} r_a \delta_a \in V_0(D)$. Then there is a finite set $\mathcal{F} \subseteq \sigma_{\sqsubseteq}$ such that*

$$d(\eta, \mu) = \sup_{O \in \mathcal{F}} (\eta(O) \dot{-} \mu(O))$$

for all valuations μ.

Proof. Define $\mathcal{F} = \{\uparrow B \mid B \subseteq A\}$ then $\mathcal{F} \subseteq \sigma_{\sqsubseteq}$ as A is a finite set of compact elements. Hence

$$\begin{aligned} d(\eta, \mu) &= \sup_{O \in \sigma_{\sqsubseteq}} (\eta(O) \dot{-} \mu(O)) \\ &\geq \sup_{O \in \mathcal{F}} (\eta(O) \dot{-} \mu(O)) \qquad (20) \end{aligned}$$

But if $O \in \sigma_{\sqsubseteq}$ then certainly $\uparrow(A \cap O) \in \mathcal{F}$ and $\uparrow(A \cap O) \subseteq O$, thus $\mu(\uparrow(A \cap O)) \leq \mu(O)$. As $\eta(O) = \eta(\uparrow(A \cap O))$ this implies $\eta(O) \dot{-} \mu(O) \leq \eta(\uparrow(A \cap O)) \dot{-} \mu(\uparrow(A \cap O))$. Therefore, equality holds in (20). □

Theorem 3.4 *If $(D, \sqsubseteq)$ is an algebraic domain then $(V(D), d)$, $(V_{\leq 1}(D), d)$ and $(V_{=1}(D), d)$ are algebraic qm-domains. More specifically, all finite simple valuations based on compact elements are compact and $V_0(D)$ generates $V(D)$.*

Proof. Suppose that η is a finite simple valuation based on compact elements. Furthermore, let $(\mu_i)_{i \in I}$ be a Cauchy net of valuations. We have to verify $d(\eta, \lim_{i \in I} \mu_i) = \lim_{i \in I} d(\eta, \mu_i)$. By Proposition 3.1, the limit $\lim_{i \in I} \mu_i$ is calculated pointwise. Employing the finite set $\mathcal{F} \subseteq \sigma_{\sqsubseteq}$ from Lemma 3.3 we get

$$\begin{aligned}\lim_{i\in I} d(\eta,\mu_i) &= \lim_{i\in I}\sup_{O\in\mathcal{F}}\Big(\eta(O)\dot{-}\mu_i(O)\Big)\\ &= \sup_{O\in\mathcal{F}}\lim_{i\in I}\Big(\eta(O)\dot{-}\mu_i(O)\Big)\\ &= \sup_{O\in\mathcal{F}}\Big(\eta(O)\dot{-}\lim_{i\in I}\mu_I(O)\Big)\\ &= d(\eta,\lim_{i\in I}\mu_i),\end{aligned}$$

Where the limit and the supremum commute since the set $\mathcal{F}$ is finite and continuity of the subtraction is ensured as $\eta(O) < \infty$. This proves that η is compact.

Now every continuous valuation in $V(D)$ is the directed supremum of elements from $V_0(D)$, so in particular the limit of a forward Cauchy net on $V_0(D)$. Hence $(V(D), d)$ is algebraic. The same argument works for the other two spaces where $V_{\leq 1}(D)\cap V_0(D)$ and $V_{=1}(D)\cap V_0(D)$, respectively, are bases.□

4 V(X) as quasimetric powerdomain

4.1 More quasimetric domain theory

In [15] a theory of powerspaces for quantitative domains was developed which carried over the theme of replacing the 2-valued logic by a $[0,\infty]$-valued logic (for the case of quasimetric spaces) to the interpretation of the *element relation* $\in$. Consequently, powerdomains were defined to be *modules*. Let us briefly give the relevant definitions for the case of quasimetric spaces. A quasimetric space (X, d) is *pointed* by $b: X \to [0,\infty]$ (the *bottom-predicate*) if $b(x) \geq d(x, y)$ for all $x, y \in X$. A map $f: (X, d, b) \to (X', d', b')$ between pointed quasimetric spaces is *strict* if $b(x) \geq b'(f(x))$ holds for all $x \in X$.

A *algebraic* $[0,\infty]$*-module* $(X, d; +, \cdot, 0)$ consists of an algebraic qm-domain (X, d) with a Scott-continuous scalar multiplication $\cdot : [0,\infty] \times X \to X$, a Scott-continuous addition $+ : X \times X \to X$, and a special element $0 \in X$ satisfying the usual algebraic axioms of modules, i.e. addition is commutative and associative and has 0 as neutral element, $m \cdot 0 = 0 \cdot x = 0$ and $1 \cdot x = x$ for all $m \in [0,\infty]$ and $x \in X$, the distributivity laws $(m+n)\cdot x = m\cdot x + n\cdot x$ and $m\cdot(x+y) = m\cdot x + m\cdot y$ as well as $(mn)\cdot x = m\cdot(n\cdot x)$ for all $x, y \in X$ and $m, n \in [0,\infty]$. Moreover, for $x, y, z \in X$ and $m, n \in [0,\infty]$ the following are required:

$$\begin{aligned} d(x,y) &\geq d(x+z, y+z)\\ md(x,y) &\geq d(mx, my)\\ m \leq n &\Longrightarrow d(mx, nx) = 0.\end{aligned}$$

A *morphism of algebraic* $[0,\infty]$*-modules* X and Y is a nonexpansive Scott-continuous map $f: X \to Y$ such that $f(x+y) = f(x) + f(y)$ and $f(mx) = m \cdot f(x)$ hold for all $x, y \in X$ and $m \in [0,\infty]$. Every algebraic $[0,\infty]$-module is implicitly pointed by the function $b(x) = d(x, 0)$. Morphisms of algebraic

$[0,\infty]$-modules are strict with respect to this bottom-predicate.

The *powerdomain* $P(X)$ exists for pointed algebraic qm-domains X and is characterized by the following universal property: the space X is embedded into $P(X)$ and whenever Y is an algebraic $[0,\infty]$-module and $f\colon X \to Y$ is a strict nonexpansive Scott continuous function, then there is a unique extension of f to $P(X)$ which is a morphism of algebraic $[0,\infty]$-modules. The paper [15] does also give a construction of the powerdomain. We do not give technical details here but instead just note that $P(X)$ is constructed in four stages. First, the free algebra in the purely algebraic sense of signature of $[0,\infty)$-modules (i.e. $(0, 2, (1)_{[0,\infty)})$) over X_0, the set of compact elements of X is considered. Then, a pseudo-quasimetric is defined on this set (*pseudo* refers to not requiring the T_0-axiom) in a way similar to the definition of $\hat{d}$ in Section 2.2 of the present paper. This distance function factors through the quotient which yields the free $[0,\infty]$-module $P_0(X_0)$ in the sense of universal algebra. For the last step of the construction, we need to introduce some further machinery.

An *ideal* on the quasimetric space X is a map $\varphi\colon X \to [0,\infty]$ such that (1) $d(x,y)+\varphi(y) \geq \varphi(x)$ for all $x, y \in X$, (2) there is an x such that $\infty > \varphi(x)$ and (3) whenever $\epsilon_1 > \varphi(x_1)$, $\epsilon_2 > \varphi(x_2)$ and $\delta > 0$, there is an x such that $\delta > \varphi(x)$, $\epsilon_1 > d(x_1, x)$, and $\epsilon_2 > d(x_2, x)$. The theory of quasimetric domains can be developed using ideals instead of forward Cauchy nets as was done in [6]. Ideals were independently introduced as *flat modules* in [17]. The *ideal completion* of X is the set $\mathrm{Idl}(X)$ of all ideals on X supplied with the sup-distance $\hat{d}(\varphi,\psi) = \sup_{x\in X} d(\varphi(x), \psi(x))$. The ideal completion is an algebraic qm-domain and contains an isomorphic copy of X via the map that sends an element x to the ideal $\iota_X(x)$ defined by $\iota_X(x)(y) = d(y, x)$ for all $y \in X$. If X is an algebraic qm-domain with X_0 its set of compact elements and $Y \subseteq X_0$ generates X then $\mathrm{Idl}(Y)$ and X are isometric.

The last step of the construction of the powerdomain of X is the definition $P(X) = \mathrm{Idl}(P_0(X_0))$. It is shown in [15] that the operations of modules lift to the ideal completion and that this space has the desired properties.

4.2 Ordinary domains as quasimetric domains

The following proposition enables us to consider ordinary posets as quasimetric spaces and ordinary domains as qmdcpos.

Proposition 4.1 *Let $(D, \sqsubseteq)$ be a partially ordered set. Define*

$$d(x,y) = \begin{cases} 0 & x \sqsubseteq y \\ 1 & x \not\sqsubseteq y. \end{cases}$$

Then (D, d) is a quasimetric space. It is a qmdcpo iff $(D, \sqsubseteq)$ is a dcpo. This qmdcpo is algebraic iff $(D, \sqsubseteq)$ is. In particular, an element of (D, d) is compact iff it is compact in the usual sense. Moreover, $\mathrm{Idl}(D, d)$ is the usual ideal completion of $(D, \sqsubseteq)$ with 0-1-valued distance as above.

Proof. It is clear that the above d is a quasimetric. Suppose $(x_i)_{i\in I}$ is a Cauchy net on (D,d). Applying the Cauchy-condition for $\varepsilon = \frac{1}{2}$ yields an index $i_0 \in I$ such that $x_j \sqsubseteq x_k$ holds for all $j,k \in I$ with $i_0 \le j \le k$. This means that the net is directed from i_0 onwards. Furthermore, a potential limit point $x \in D$ may be characterized as follows:

$$\begin{aligned} x = \lim_{i\in I} x_i &\iff \forall y \in X.\ d(x,y) = \inf_{i\in I} \sup_{j\ge i} d(x_i,y) \\ &\iff \forall y \in X.\ \Big(x \sqsubseteq y \iff \exists i \in I \forall j \ge i.\ x_i \sqsubseteq y\Big) \\ &\iff x = \bigvee_{i\ge i_0}^{\uparrow} x_i. \end{aligned}$$

So (D,d) is a qmdcpo iff the poset $(D,\sqsubseteq)$ is directed complete. In a similar fashion, one concludes that compactness coincides with compactness in the usual sense. Thus the claim on algebraicity follows.

Finally the ideal completion. We first prove that every ideal $\varphi\colon D \to [0,\infty]$ only takes values in $\{0,1\}$. Fix $x \in D$ and assume first that $\varphi(x) < 1$. As φ is an ideal, for every $\delta > 0$ there is $y \in D$ with $1 > d(x,y)$ and $\delta > \varphi(y)$. But $d(x,y) \in \{0,1\}$, so this implies $d(x,y) = 0$. Thus $\delta > \varphi(y) = d(x,y) + \varphi(y) \ge \varphi(x)$ in this case. From the arbitrary nature of δ we conclude $\varphi(x) = 0$. It remains to consider the case where $\varphi(x) \ge 1$. For every ideal φ and every $\delta > 0$ there is a point y with $\delta > \varphi(y)$. In our case, picking $\delta = 1$ yields $y \in D$ with $\varphi(y) = 0$ from what we proved so far. This implies $1 = 1 + 0 \ge d(x,y) + \varphi(y) \ge \varphi(x)$. Thus we proved $\varphi(x) \in \{0,1\}$ for all $x \in D$. Therefore, the ideals of (D,d) are via $\varphi \mapsto \{x \in D \mid \varphi(x) = 0\}$ in bijective correspondence to the ideals of $(D,\sqsubseteq)$. It is easy to see that sup-distance on $\mathrm{Idl}(D,d)$ corresponds exactly to the 0-1-valued distance on $\mathrm{Idl}(D,\sqsubseteq)$ derived from subset inclusion. □

Theorem 4.2 *Suppose that $(D,\sqsubseteq)$ is an (ordinary) algebraic domain. Endow D with the 0-1-valued distance from Proposition 4.1 and constant bottom-predicate $b(x) = 1$ for all $x \in D$. Then the quasimetric powerdomain of this algebraic qm-domain is given by the space of valuations $(V(D),d)$.*

Proof. We gave in Section 4.1 a sketch of the construction of the quasimetric powerdomain $P(D)$ from [15]. The (algebraically) free module $P_0(D_0)$ constructed in the third step is in natural bijection to the set $V_0(D)$ of all finite simple valuations based on compact elements: the bijection sends a point valuation δ_a to the generator of $P_0(D_0)$ corresponding to $a \in X_0$. The distance on $P_0(D_0)$ coincides with $\hat{d}$ considered in Section 2.2 as both are constructed in exactly the same way. By the Generalized Splitting Lemma, $(V_0(D),\hat{d})$ is isometric to $(V_0(D),d)$. By Theorem 3.4, all finite simple valuations based on compact elements are compact in $(V(D),d)$, and, moreover, they generate $V(D)$. Thus $(V(D),d) \cong \mathrm{Idl}(V_0(D),d)$. Hence $(V(D),\hat{d}) \cong \mathrm{Idl}(V_0(D),\hat{d}) \cong \mathrm{Idl}(P_0(D_0),d) = (P(D),d)$. □

In [15], it was shown that the quantitative powerdomain coincides with the *lower* or *Hoare* powerdomain for the case of ordinary domains, i.e. with two-valued logic. Thus, in a nutshell, Theorem 4.2 says that $V(D)$ is the lower powerdomain of D when the latter is regarded as a quasimetric domain.

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