The Rise of Cayley's Invariant Theory (1841–1862)

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In his pioneering papers of 1845 and 1846, Arthur Cayley (1821-1895) introduced several approaches to invariant theory, the most prominent being the method of hyperdeterminant derivation. This article discusses these papers in the light of Cayley's unpublished correspondence with George Boole, who exercised considerable influence on Cayley at this formative stage of invariant theory. In the 1850s Cayley put forward a new synthesis for invariant theory framed in terms of partial differential equations. In this period he published his memoirs on quantics, the first seven of which appeared in quick succession. This article examines the background of these memoirs and makes use of unpublished correspondence with Cayley's lifelong friend, J. J. Sylvester.

AMS 1980 subject classifications: 01A55, 15-03.

Key Words: hyperdeterminants, George Boole, partial differential equations, quantics, J. J. Sylvester, multilinear forms.

INTRODUCTION

Invariant theory attracted Arthur Cayley's almost continuous attention for more than half a century. Following George Boole's lead he published two papers...
which have since been regarded by generations of mathematicians as laying the foundations of the subject. These papers contain the seeds of the two great methods of 19th-century invariant theory. In [1845] Cayley lightly touched upon, but did not develop, the idea that an invariant (Glossary) could be considered as an algebraic solution to a set of partial differential equations. But in [1846] he based the immature theory on the hyperdeterminant derivative. The latter notion, which became a powerful tool in the hands of the German school of invariant theorists in the 1860s, was abandoned by Cayley in the 1850s when he came to write the definitive series of memoirs on quantics. In these memoirs he returned to the development of invariants from partial differential equations. With hindsight Cayley made an unfortunate choice, for the German school met with greater success as the theory gradually unfolded. But Cayley’s choice suited one of his own principal objectives which was to calculate linearly independent and irreducible invariants (Glossary) and display them in tabular form.

In this article I suggest that Cayley’s desire to calculate invariants may have had a direct influence on his choice of a basis for the subject. In conjunction with this I shall consider the background to Cayley’s papers and explain why the period from 1841 to 1862 can be justly described as encompassing the “rise” of Cayley’s theory. In doing this I seek neither to interpret Cayley’s invariant theory in the light of modern algebraic developments nor to present the old invariant theory as part of abstract “structural” algebra.

The article covers the periods when Cayley was first residing at Cambridge (1838–1846), training for the Bar (1846–1849), and practicing as a barrister at Lincoln’s Inn (1849–1862).

THE PRELUDE

During Cayley’s student days, Cambridge was generally thought to be the center of mathematics in England and for the preceding decade had displayed a particular interest in the development of algebra and in algebra applied to geometry. Even before Cayley received his bachelor’s degree in 1842 he had found that “linear transformations and analytical geometry” were his favorite subjects and had published a short but important paper on determinants. Determinants were a lifelong interest and they of course played an essential role in the development of invariant theory. His pithy remark made in later years, that had he to give fifteen lectures on the whole of mathematics he would devote one to determinants, indicates their importance in his realm of ideas [Klein 1939, 143]. In [Cayley 1843] he showed that the ordinary determinant, which he had been the first to introduce in the now familiar two-dimensional array resting between vertical lines, could be extended to a notion of more general determinants formed from multidimensional arrays. These became known as cubic determinants (Glossary).

A personal influence on the young Cayley was George Boole (1815–1864). In invariant theory Boole found a subject which presented an “ample field of research and discovery” and in his [1841] indicated that it had applications to algebraic geometry and the solution of polynomial equations. On reading this work Cayley wrote to Boole of “the pleasure afforded” by his two-part paper.
Cayley freely acknowledged Boole’s influence and in the letter added: "I . . . [am] sending you a few formulae relative to it, which were suggested to me by your very interesting paper; I should be delighted if they were to prevail upon you to resume the subject, which really appears inexhaustible" (I) [2].

Despite this request Boole left invariant theory aside and only returned to it spasmodically afterward. Cayley was astute enough to recognize the potential of the embryonic idea and, in taking its development several steps further, established his position as prime mover of the infant theory. In the course of the following two years Cayley produced [1845, 1846].

Cayley's feeling of isolation was evidently a hardship. The Cambridge mathematicians might have taken a passing interest in the new theory but, apart from Boole, no other contributor to the Cambridge Mathematical Journal published work on the subject until the early 1850s. Cayley took inspiration from Continental mathematicians writing in Crelle's Journal, where he published his own work. At home, however, his closest contact continued to be Boole. ‘‘I wish I could manage a visit to Lincoln, I should so much enjoy talking over some things with you,’’ he wrote Boole in 1845, ‘‘not to mention the temptation of your Cathedral. I think I must contrive it some time in the next six months,—in spite of there being no railroad, which one begins to consider oneself entitled to in these days’’ (8) [3]. From Cayley's letters to Boole (the other side cannot be traced) it is clear that Boole, as the more experienced mathematician, provided both help and encouragement.

THE 1845 AND 1846 PAPERS

Whereas Boole had considered homogeneous polynomials of order $n$ in $m$ variables (Glossary), Cayley considered multilinear forms. In the cases $n = 3$ and $m = 2$, for instance, Boole was concerned with the binary cubic (Glossary)

$$u = ax^3 + 3bx^2y + 3cxy^2 + dy^3,$$

whereas, for the same values of $n$ and $m$, Cayley concerned himself with the trilinear form

$$U = ax_1y_1z_1 + bx_2y_1z_1 + cx_1y_2z_1 + dx_2y_2z_1 + ex_1y_1z_2 + fx_2y_1z_2 + gx_1y_2z_2 + hx_2y_2z_2$$

in three sets of variables

$$(x_1, x_2), \quad (y_1, y_2), \quad (z_1, z_2).$$

Thus Cayley created an invariant theory for multilinear forms. He found, for example, that

$$W = (ah - cf + bg - de)^2 - 4(ag - ce)(bh - df)$$

is an invariant of the trilinear form $U$. The cubic determinants which he discovered prior to this were found useful in calculating and expressing the invariants of multilinear forms. But the multilinear theory was not the primary goal, for it was
in Boole’s specialized theory concerned only with homogeneous polynomials or *quantics* (Glossary) that Cayley saw the prospect of immediate progress. The important observation made by Cayley was that the multilinear theory shed light on the specialized theory. When certain of its variables and coefficients are identified, a multilinear form is reduced to a homogeneous polynomial while if the same identifications are performed on multilinear invariants then ordinary invariants are obtained. For example, the multilinear invariant $W$ reduces to

$$a^2d^2 - 3b^2c^2 - 6abcd + 4ac^3 + 4b^3d$$

after the identifications $b = c = e$ and $d = f = g$. While this particular invariant (the discriminant of the binary cubic) was also found by Boole, Cayley was able to obtain new invariants using his technique. For example,

$$ae - 4bd + 3e^2$$

is an invariant of the binary form of order 4, which could not be obtained using Boole’s approach [Cayley 1845].

Though [1845] is chiefly concerned with calculation Cayley did remark that the true basis for invariant theory should present an invariant as a solution of a set of partial differential equations. For example, one of the set of differential equations for $W$ (though not written in Cayley’s symbolic numeral notation) is

$$
\left(a \frac{\partial}{\partial b} + e \frac{\partial}{\partial f} + c \frac{\partial}{\partial d} + g \frac{\partial}{\partial h}\right) W = 0.
$$

He noted that “in every case it is from these equations that the form of the function [invariant] is to be investigated” [1845; CP1, 85]. The writing of [1845] did not progress smoothly owing to difficult combinatorial problems but when it was eventually finished he wrote to Boole that he was “very anxious to hear . . . [his] opinion of it” (4).

Cayley’s sequel [1846] included a statement of the theory’s objectives and the introduction of the $\Omega$-process. According to Cayley, the primary object of invariant theory should be to “find all the derivatives [invariants] of any number of functions [algebraic forms], which have the property of preserving their form unaltered after any linear transformation of the variables” [1846; CP1, 95]. To “find” was the principal motive and, perhaps as a consequence of this, Cayley’s interest in providing careful proofs was comparatively slight. Also, the goal was characteristically stated in the most general terms. It is indicative of Cayley’s insight into the problem’s difficulty that he qualified his statement and focused his attention on the specialized theory. Even here he realized that the case of the single binary form offered the only real hope of solution. This caveat was prophetic. An invariant theorist of a later generation, H. W. Turnbull, noted that the 19th-century pioneers had worked primarily with the binary form and, to a much lesser extent, with algebraic forms of three variables [Turnbull 1926]. By the 1880s, Cayley himself was still deeply concerned with the binary form of order 5—
the binary quintic—and had spent much energy over the intervening years calculating its invariants and covariants (Glossary).

In his early work Cayley found different methods for calculating invariants. He wrote to Boole about them: "I have just found a property of hyperdeterminants, which like most of the others gives another method of determining them (One would be glad not to have so many) and which seems to me perhaps the most curious of all" (9).

The "most curious" method was the hyperdeterminant derivative method, described in [1846], which came to be known as Cayley’s Ω-process. A special case of it was the precursor of the transvection operation on which the German symbolic process was based. Cayley’s typical application of this method for finding an invariant is best illustrated by a simple example, but even here the reader will notice that the calculation is lengthy. Given the quadratic form

$$\omega = ax^2 + 2bxy + cy^2,$$

suppose we wish to find its invariant. First duplicate forms $\omega_1$ and $\omega_2$ are written

$$\omega_1 = ax_1^2 + 2bx_1y_1 + cy_1^2,$$
$$\omega_2 = ax_2^2 + 2bx_2y_2 + cy_2^2.$$

Putting

$$\Omega = \left| \begin{array}{cc} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial y_1} & \frac{\partial}{\partial y_2} \end{array} \right|^2,$$

it can be verified that $\Omega \omega_1 \omega_2$ yields the invariant $ac - b^2$.

**A NEW SYNTHESIS**

In the 1850s the hyperdeterminant derivative was discarded by Cayley as he reverted to the notion of an invariant’s link to partial differential equations. The theorem forging this link, in the case of the binary form of order $n$,

$$a_0x^n + a_1 \binom{n}{1}x^{n-1}y + a_2 \binom{n}{2}x^{n-2}y^2 + \cdots + a_n y^n,$$

asserts that $I(a_0, a_1, a_2, \ldots, a_n)$ is an invariant if and only if

$$\Box I = 0$$
$$\Box I = 0,$$

where

$$\Box = a_0 \frac{\partial}{\partial a_1} + 2a_1 \frac{\partial}{\partial a_2} + \cdots + na_{n-1} \frac{\partial}{\partial a_n}$$
and

\[ \Delta = na_1 \frac{\partial}{\partial a_0} + (n - 1)a_2 \frac{\partial}{\partial a_1} + \cdots + a_n \frac{\partial}{\partial a_{n-1}} \]

[1854a; CP2, 166].

Cayley's proof relied on the fact that both \( I \) and the transformed \( I' \) must satisfy Taylor's theorem written symbolically as

\[ I' = e^{\lambda \Delta} I. \]

In this part of his work he made extensive use of the calculus of operations and thus displayed a debt to George Peacock (1791–1858), who had been Cayley's Cambridge tutor for a short time, and to the Analytical School [Koppelman 1971].

A short note to his friend and collaborator, J. J. Sylvester (1814–1897), in late 1851 showed Cayley's new commitment to founding invariants on partial differential equations. Here, without referring to his earlier allusion to differential equations in [1845], Cayley announced that the link to differential invariants would "constitute the foundation of a new theory of Invariants" (11).

Cayley was not alone in understanding the connection between invariant theory and partial differential equations. A short time after receiving the above-mentioned note from Cayley, Sylvester obtained his own derivation of the equations [1852; SPl, 352]. In addition, other independent discoverers of the relationship included Siegfried Aronhold (1819–1884) in Germany [Lampe 1901] and Francesco Brioschi (1824–1897) in Italy [1854, 112].

The famous ten memoirs on quantics which set out to encapsulate invariant theory emphasized this new synthesis and the hyperdeterminant derivative subsequently played little part in Cayley's invariant theory. The first seven memoirs were published in quick succession. The sixth memoir is known to modern mathematicians for its introduction of the Cayley projective metric but the most vital memoir for invariant theory is the second, whose centerpiece is Cayley's theorem. With the waning interest in the objectives of Cayley's research program during this century the theorem has lost its importance as an agent of calculation, but during the 19th century it formed the cornerstone of Cayley's particular approach to invariant theory.

Usually restrained in his manner, Cayley could hardly disguise his satisfaction when he wrote to Sylvester about the theorem:

Dear Sylvester,

Eureka. Let \((a, b, c, \ldots, x, y)^n\) be a quantic. I consider the coefficients \(a, b, c, \ldots\) as being of the weights [Glossary] \(-1n, 1 - 1n, \text{et cetera},\) and \(x, y\) of the weights \(-1, -1\); every covariant is of the weight 0. Write

\[ \{x\partial_x\} = nb\partial_x + (n - 1)c\partial_x + \cdots &c. = Y \]

suppose; [and]

\[ \{y\partial_y\} = a\partial_y + 2b\partial_y + \cdots &c. = X \]

and let \(A\) be a rational and integral homogeneous function of the coefficients of the weight \(-\frac{1}{2}n\). Then it is easy to see that
\[(XY - YX)A = sA,\]

and substituting for \(A \ldots\) we have [Cayley's theorem]

**Theorem.** If \(A\) be of the weight \(-\frac{1}{2}s\) and satisfy the single equation \(XA = 0\) then a covariant is

\[(A, YA, \frac{Y^2A}{1}, 2, \ldots, \xi x, y)^p.\]

Suppose that \(A\) is of the degree \(\theta\) [Glossary] in the coefficients and take for \(A\) the most general form of the degree \(\theta\) and weight \(-\frac{1}{2}s\) or what is the same thing, reckoning the weights \(a, b, c\) as 0, 1, 2, et cetera, take for \(A\) the most general form of the degree and weight \(\{n\theta - s\}.\) Then \(XA\) will be a form of the degree \(\theta\) and weight \(\{n\theta - s\} - 1;\) and putting \(XA = 0\) the coefficients of \(A\) satisfy a certain number of linear equations—there is no reason for doubting that these equations are independent—and if so the number of asyzygetic covariants [Glossary] [of degree \(\theta,\) order \(s\)] \((a, b, c, \ldots)^p \xi x, y)^p = \text{Number of terms [of] degree \(\theta,\) weight \(\{n\theta - s\}\) less Number of terms [of] degree \(\theta,\) weight \(\{n\theta - s\} + 1\)} [sic], which is I believe the law for the number of asyzygetic covariants of a given order, and degree in the coefficients. (13)

Cayley's evident pleasure in establishing the theorem is readily understood when we learn that he coupled this work in invariant theory with problems which had "resisted all . . . [his] attempts to solve" [1854a; CP2, 167]. The theorem is in two parts. The first part gives a constructive formula

\[(A, YA, \frac{Y^2A}{1}, 2, \ldots, \xi x, y)^p\]

for determining covariants as *Cartesian expressions* (Glossary). The second states a law for enumerating the linearly independent covariants of a binary form. A modern statement of this law can be found in [Springer 1977, 52].

From the first part, the procedure Cayley most likely employed to calculate a covariant was:

1. Find a trial solution for \(A\) in the form of a linear combination involving undetermined coefficients. This can be done using Arbogast’s rule [1878; CP11, 55].

2. Find the exact solution for \(A\) by solving the set of linear equations \(XA = 0\) for the undetermined coefficients. In his letter (13) Cayley merely asserted that these equations were independent. This was proved later [Sylvester 1878].

3. Apply Cayley’s formula.

The calculations could be lengthy. A typical example is illustrated in Fig. 1. The algebraic expressions \(A\) were later called semi-invariants and were studied intensively by invariant theorists during the 1880s.

**The Role of Calculation**

Cayley's early desire to "find" invariants was elaborated in his correspondence with Boole:

Do you see any way of calculating in rough, the degree of labor that would be necessary for forming tables of Elimination; Sturm's functions, our transforming functions [invariants], et
cetera. If one could get to any practical results about it, and they were not very alarming, it
would be worth while I think presenting them to the British Association: but I am afraid the
limit of possibility comes very soon: suppose one ascertained a result would take a century to
calculate, it would be rather a hopeless affair. (6)

The initial calculations presented in [1845, 1846] were fragmentary. The “highest”
invariant was of degree 4 in the case where the parent binary form was of order 9. No covariants were calculated. What Cayley found in the 1840s was that the systems of irreducible invariants and covariants for the binary forms of orders 2, 3, and 4 were straightforward to establish but the binary quintic presented an altogether different level of difficulty. It appeared to defy his speculative approach as when he suggested the existence of a certain invariant for the quintic because
“it seems so natural that the number of functions (invariants) should depend very simply upon the value of n” (5). A further instance of the quintic’s complexity was the occurrence of the unforeseen invariant of degree 18 when Cayley had previously believed that the degree of an invariant for the quintic was a multiple of 4.

The existence of this new invariant was established by Charles Hermite (1822–1901) [1854]. Yet Cayley’s affinity for calculation is best illustrated by his action following his conclusion concerning the finiteness question. It is well known that

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<th>$a^2 + 2$</th>
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<th>$a^2 + 22$</th>
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<th>$a^2 - 22$</th>
<th>$a^2 + 7$</th>
<th>$a^2 + 15$</th>
<th>$a^2 + 20$</th>
<th>$a^2 + 25$</th>
<th>$a^2 - 12$</th>
<th>$a^2 + 19$</th>
<th>$a^2 - 8$</th>
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<td>$y^2 + 2$</td>
<td>$x^2 - 5$</td>
<td>$b^2 - 2$</td>
<td>$b^2 + 8$</td>
<td>$b^2 + 19$</td>
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<td>$b^2 - 19$</td>
<td>$b^2 + 9$</td>
<td>$b^2 - 6$</td>
</tr>
</tbody>
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Fig. 1. [1856; CP2, 275]. The covariant of degree 5 and order 7 for the binary form $(a b c d e f x, y)^5$, a covariant of modest length. To calculate the coefficients in the first column, Cayley would have had to formally solve 5 linear equations in 16 variables. Because the equations are sparse this would have not been difficult. The remaining columns would have been computed using the formula of Cayley’s theorem. (The covariants chosen as exemplars in Cayley’s tabular scheme were always displayed with integer coefficients). The ± number at the foot of each column is Cayley’s check on the correctness of the result. In the first column, for instance, the sum of the positive coefficients is 26, that of negative coefficients is −26, and the sum is zero as it should be.
Cayley erroneously thought that there was not a finite number of irreducible covariants for a binary form of order greater than or equal to 5. Yet he continued with his program of calculation of these forms. In the second memoir, thirteen distinct irreducible covariants of the quintic were calculated. Perhaps the calculations were not quite so tedious for Cayley as we might now imagine. He had, after all, the insight and fluency of a man steeped in his subject. His remark about one particular calculation was that it would be "very laborious, but the forms of the results are easily foreseen, and the results can be verified by means of one or two coefficients only" [1861; CP4, 335].

That Cayley found the calculation method given by his theorem more effective than the hyperdeterminant derivative method may have been a factor in his adopting the new synthesis. He was not explicit about this, though he noted that "one finds easily the covariants by the method of undetermined coefficients" [1854a; CP2, 167] but, with the hyperdeterminant derivative method, "the application of it becomes difficult when the degree of a covariant exceeds 4" [1858; CP2, 517]. (This last remark is borne out by [1846], in which the calculated invariants are limited to degree 4.) He certainly did not abandon the hyperdeterminant derivative on suspicion of its theoretical weakness for he knew that it was possible to express any covariant in terms of it. He was not alone in his love of calculation. Both George Salmon (1819–1904) and Sylvester, the other members of the "Invariant Trinity," considered the calculation of invariants a worthwhile task.

His particular viewpoint on invariant theory during its infancy is palpably conveyed by a revealing remark found at the conclusion of the fourth memoir: "The modes of generation of a covariant are infinite in number, and it is to be anticipated that, as new theories arise, there will be frequent occasion to consider new processes of derivation, and to single out and to define and give names to new covariants" [1858; CP2, 526] (my emphasis).

Of particular interest is Cayley's intention of preparing a taxonomy. In this regard he might be compared with a typical Victorian botanist as "one who collects specimens to swell his herbarium, gives them barbarous names, and tries to arrange them in a system . . ." [Cannon 1978, 274]. The luxuriant language is certainly there and Cayley, who was extremely circumspect about the introduction of new terminology, often approved of Sylvester's spectacular choices. Throughout the writings of both Cayley and Sylvester the classificatory terms "species" and "genera" recur while there is a relative absence of the modern mathematician's "definition" and "proof." This is not to say that a taxonomic characteristic is lacking in comparable mathematics of the 20th century but by considering the classificatory aspect of Cayley's invariant theory we may better understand his motivation.

**CONCLUSION**

By the end of the 1850s, Cayley had established invariant theory and transformed it by its first synthesis. Also, the necessary first step of any science, that of classification, was in full sway. But Cayley failed to provide a sound theoretical
calculus and in the 1860s and afterward the leadership in invariant theory moved away from England, France, and Italy to Germany. While Cayley appreciated the power of the more abstract method of the German school he did not abandon his own methods. The 1850s represented the zenith of Cayley's invariant theory and the seventh memoir, published in 1861, marked a provisional end to the series. With the appearance of the seventh memoir the "rise" of Cayley's invariant theory was effectively ended.

ACKNOWLEDGMENTS

I would like to thank Dr. I. Grattan-Guinness and the anonymous referees for their help in the preparation of this article. For permission to quote from their manuscript collections, I wish to thank the Royal Society of London; the Royal Institution of Great Britain; the Master and Fellows of Trinity College, Cambridge; and the Master and Fellows of St. John's College, Cambridge. My path has been considerably eased by help given by the individual librarians of these institutions and I would like to thank them for their assistance.

NOTES

1. Cayley's and Sylvester's arcane terminology makes their work especially difficult for the modern reader. Brief explanations for some of the basic terms are given in a Glossary. These are identified in the text on their first occurrence in the form term . . . (Glossary). More technical detail than that given in the Glossary may be found in [Elliott 1913]. Sylvester's [1853; SP I, 580] and Cayley's [1860; CP4, 594] offer brief guides to the meaning of their nomenclature.

2. Letters are numbered (r) in chronological order and referred to under Index of Documents included in the References.

3. Although Cayley made a tour to the north of England in 1845, I have found no definite evidence of a meeting with Boole.

GLOSSARY

\textbf{Asyzygetic.} This term is equivalent to the modern "linearly independent." A linear relation between invariants or covariants of the same degree and order was called a syzygy. For the \textit{binary cubic,} for example, a syzygy between the (composite) covariants of degree 6 and order 6 is

\[ \Phi^2 - \mu \Omega + 4H^3 = 0. \]

\textbf{Binary cubic.} This is an algebraic form (quantic) \( u(x, y) \) with the property

\[ u = ax^3 + 3bx^2y + 3cxy^2 + dy^3 \]

and in Cayley's bracket notation by

\[ u = (a, b, c, d | x, y)^3. \]

The binary cubic possesses four \textit{irreducible} algebraic forms with the invariant property: the binary cubic \( u \) itself (degree 1, order 3); the discriminant \textit{invariant} \( \Omega \) (degree 4, order 0); the Hessian \textit{covariant} \( H \) (degree 2, order 2); and the covariant \( \Phi \) (degree 3, order 3), which is the Jacobian of \( u \) and \( H \).

\textit{Cartesian expression.} In this article the term "Cartesian expression" means an algebraic form traditionally expressed in coefficients and variables. The Cartesian expression for an algebraic form contrasts with the abbreviated notation favored by the German school of invariant theorists.

\textit{Covariant:} This is an algebraic form (quantic) \( C(u, x) \) with the property

\[ C(u, x) = KC(b, y) \]
when the parent algebraic form $F(c, x)$ is transformed by a nonsingular linear transformation of $x$ to $y$. The numerical factor $K$ involves the determinant of the transformation. Each covariant has a degree and an order. In distinction to an invariant, a covariant involves the variables of $F(c, x)$. For a particular linear transformation both the sum and product of a covariant are covariants.

Cubic determinant. A generalization of the ordinary determinant, this is the generic name given to particularly defined algebraic forms whose terms are composed of elements having $n$ subscripts. For example, in the case $n = 4$, let

$$\sigma = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\},$$

where $\sigma_i$ is a permutation of 1, 2. Define

$$\text{sgn } \sigma = \prod_{i=1}^{4} \text{sgn } (\sigma_i)$$

and the corresponding cubic determinant is

$$\sum_{\sigma} (\text{sgn } \sigma)a_{\sigma_1(1)}a_{\sigma_2(1)}a_{\sigma_3(1)}a_{\sigma_4(1)}b_{\sigma_1(2)}b_{\sigma_2(2)}b_{\sigma_3(2)}b_{\sigma_4(2)}.$$ 

It comprises eight distinct terms and is an invariant for the multilinear algebraic form with four sets of two variables.

Degree. The degree (grad) of a term in an algebraic form is the sum of the exponential indices in the product of coefficients attached to that term (as distinct from the product of variables). The usage in connection with coefficients became standard as invariant theory became established. (See Fig. 1 for an algebraic form of degree 5.)

Invariant. This is an algebraic form (quantic) $l(a)$ with the property

$$l(a) = Kl(b)$$

when the parent algebraic form $F(c, x)$ is transformed by a nonsingular linear transformation of $x$ to $y$. The numerical factor $K$ involves the determinant of the transformation.

Irreducible. This term was introduced by Cayley in the second memoir [1856]. An algebraic form is irreducible if it cannot be expressed algebraically in terms of algebraic forms of lower degree and order. For the binary cubic, for example, $u$, $v$, $H$, and $\Phi$ are the only irreducible invariants and covariants. The irreducible invariants and covariants for the binary forms of orders 2, 3, and 4 are listed in [Cayley 1856].

Order. The order (ordnung) of a term in an algebraic form is the sum of the exponential indices in the variables attached to that term. (See Fig. 1 for an algebraic form of order 7.) In his early work Cayley frequently used “order” in relation to coefficients. (See letter to Boole (5), for example.)

Quantic. This is Cayley’s term [1854b] for a homogeneous polynomial

$$F(c, x) = F(c_1, c_2, \ldots, c_r; x_1, x_2, \ldots, x_n)$$

in $n$ variables with $r$ coefficients. Cayley intended the new nomenclature to replace the earlier “rational and integral algebraical function.” Cayley made considerable use of the notation

$$(a_0, a_1, a_2, \ldots, a_n) \not x, y$$

and

$$(\not x, y)$$

to denote a general binary form of order $n$ in which the binomial coefficients are included. He used an arrowhead device as in

$$(\not x, y)$$

to denote a binary form in which the binomial coefficients are suppressed.

Weight. The weight of a term in an algebraic form is a numerical value determined by assigning values to the individual coefficients and variables. An important property of invariants and covariants
is that each of their terms had equal weight (the isobaric property). Cayley used two conventions for determining weight. As applied to the binary form of order \( n \), these are: (1) The coefficients \( a, b, c, \ldots \) are assigned weights \( 0, 1, 2, \ldots \) and the variables \( x, y \) the values 1, 0. The weight of each term for a covariant of degree \( \theta \), order \( s \) is \( \theta n + s \). (2) The coefficients \( a, b, c, \ldots \) are assigned weights \( -\frac{1}{n}, 1 - \frac{1}{n}, 2 - \frac{1}{n}, \ldots \) and the variables \( x, y \) the values \( \frac{1}{r}, -\frac{1}{r} \). The weight of each term for a covariant is zero under this convention.

REFERENCES

Index of Documents

The following abbreviations are used in the list of manuscript sources: AC, Arthur Cayley; GB, George Boole; GS, George Stokes; JB, James Booth; JJS, James Joseph Sylvester; TAH, Thomas Archer Hirst; R.I., Royal Institution of Great Britain; R.S.L., Royal Society of London; St.J., St. John’s College, Cambridge; Trin., Trinity College, Cambridge.

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Printed Works


Sylvester, J. J. 1904–1912. The collected mathematical papers of James Joseph Sylvester. 4 vols. Cambridge: Cambridge Univ. Press. (Reference to The collected mathematical papers, Vol. 1, pp. 284–363, will be given as SPI, 284–363.)


