ELSEVIER
Available online at www.sciencedirect.com
ScienceDirect
International Journal of Approximate Reasoning
48 (2008) 314–331
INTERNATIONAL JOURNAL OF APPROXIMATE REASONING
www.elsevier.com/locate/ijar

Fuzzy region connection calculus: Representing vague topological information

Steven Schockaert *, Martine De Cock, Chris Cornelis, Etienne E. Kerre

Department of Applied Mathematics and Computer Science, Ghent University, Krijgslaan 281 (S9), B-9000 Gent, Belgium

Received 16 May 2007; received in revised form 1 October 2007; accepted 2 October 2007
Available online 9 October 2007

Abstract

Qualitative spatial information plays a key role in many applications. While it is well-recognized that all but a few of these applications deal with spatial information that is affected by vagueness, relatively little work has been done on modelling this vagueness in such a way that spatial reasoning can still be performed. This paper presents a general approach to represent vague topological information (e.g., A is a part of B, A is bordering on B), using the well-known region connection calculus as a starting point. The resulting framework is applicable in a wide variety of contexts, including those where space is used in a metaphorical way. Most notably, it can be used for representing, and reasoning about, qualitative relations between regions with vague boundaries.

Keywords: Spatial reasoning; Region connection calculus; Fuzzy relation

1. Introduction

There is an increasing interest in formalisms that describe properties of space in a qualitative way. Usually such a qualitative representation takes the form of topological relations between regions [2,4] (e.g., region A is bordering on region B), orientation and distance relations between points [8,10] (e.g., place p is located north of place q, p is located far from q), or even information about the size and shape of objects (e.g., region A is smaller than region B; see [22] for an overview). In the context of geographical information systems (GISs), qualitative relations are useful to express spatial queries, while route planners and GPS systems benefit from using qualitative descriptions as they are often easier to understand by humans than quantitative descriptions (e.g., compare *turn right immediately after the bridge* with *turn right in 673 meters*). Another important area in which qualitative spatial relations can play an important role is geographical information retrieval [38,40]. The goal of a geographical information retrieval system is to pinpoint information in a large document collection that is both relevant to a general query, and to a given geographical context (e.g., web pages about movie

* Corresponding author. Tel.: +32 9 264 4772.
E-mail address: steven.schockaert@ugent.be (S. Schockaert).

0888-613X/$ - see front matter
doi:10.1016/j.ijar.2007.10.001

theatres near Gent, Belgium). On one hand, this could be achieved by finding addresses, transforming these addresses to geographical coordinates, and comparing these coordinates with available (structured) information. However, there is also a lot of relevant geographical information available in the form of qualitative relations, either extracted from natural language texts, or a priori available in geo-ontologies [40].

In most existing work, qualitative relations are crisp relations, e.g., p is either far from q or not far from q, and regions are assumed to have precisely defined boundaries. These assumptions stand in stark contrast to the nature of real-world geographical information. For example, most non-political geographical regions, such as Western Europe, Downtown Seattle, or the Alps, have vague boundaries [18,26,28,29]. Also the concept of nearness of places is generally perceived as a vague property, where a proposition like p is close to q is often considered true to some degree in a given context [3,6,19,20,27]. Hence, there is a clear need for formalisms that describe qualitative spatial properties in a graded way.

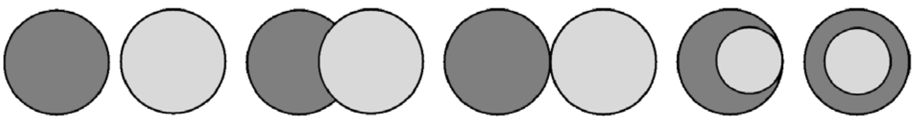
In this paper, we will focus on topological relations. Usually, information such as *A is a part of B* is formally expressed using either the Region Connection Calculus (RCC) [4] or the 9-intersection model [2]. We will focus on the former, since it is more tailored towards reasoning. In the RCC, spatial relations are defined using a primitive dyadic relation C which expresses the notion of connection between regions. For example, we may think of regions as sets of points, and define C such that for two regions a and b, $C(a,b)$ holds iff a and b have a point in common. Other topological relations are defined in terms of the relation C, as shown in Table 1. The intuitive meaning of some of these relations is shown in Fig. 1. Throughout this paper, we will use upper case letters like $A, B, C, \ldots$ to denote specific regions, and lower case letters like $a, b, c, \ldots$ to denote variables that take values from the universe of regions U.

Clearly, the crisp nature of the RCC relations is a major limitation in many application domains. For example, while the relations EC and DC are mutually exclusive, in practical applications it is often difficult, or even undesirable, to differentiate between situations where two regions are very close to each other, but disconnected, and situations where two regions are connected. For example, it is commonplace to say that a cabinet is located against a wall even if there is a gap of a few millimeters between the cabinet and the wall. When modelling such a spatial configuration using the RCC relations, EC would hold if the cabinet is actually located against the wall, while DC would hold as soon as there is a gap, irrespective of its size. A cognitively more adequate approach would be to define relations like EC and DC such that EC holds to the extent that the

Table 1
Definition of topological relations in the RCC

Name	Relation	Definition
Disconnected from	$DC(a,b)$	$\neg C(a,b)$
Part of	$P(a,b)$	$(\forall c \in U)(C(c,a) \Rightarrow C(c,b))$
Proper part of	$PP(a,b)$	$P(a,b) \wedge \neg P(b,a)$
Equal to	$EQ(a,b)$	$P(a,b) \wedge P(b,a)$
Overlaps with	$O(a,b)$	$(\exists c \in U)(P(c,a) \wedge P(c,b))$
Discrete from	$DR(a,b)$	$\neg O(a,b)$
Partially overlaps with	$PO(a,b)$	$O(a,b) \wedge \neg P(a,b) \wedge \neg P(b,a)$
Externally connected to	$EC(a,b)$	$C(a,b) \wedge \neg O(a,b)$
Non-tangential part of	$NTP(a,b)$	$P(a,b) \wedge \neg(\exists c \in U)(EC(c,a) \wedge EC(c,b))$
Tangential proper part of	$TPP(a,b)$	$PP(a,b) \wedge \neg NTP(a,b)$
Non-tangential proper part of	$NTPP(a,b)$	$\neg P(b,a) \wedge NTP(a,b)$

a and b denote regions, i.e., elements of the universe of regions U.



(a) DC (b) PO (c) EC (d) TPP (e) $NTPP$

Fig. 1. Intuitive meaning of some RCC relations.

cabinet is located against, or close to the wall, and *DC* holds to the extent that the cabinet is not close to the wall, where closeness is defined as a gradual, vague property. In this way, the aforementioned problem does not occur anymore, since the transition from *DC* to *EC* becomes gradual, rather than abrupt, and we can express knowledge like *a* and *b* are more or less externally connected, or *a* and *b* are definitely disconnected. Moreover, in many geographical contexts, regions are not well-defined sets of points, but, ill-defined areas with vague, gradual boundaries (e.g., London's West End, the Ardennes, etc.). The topological relations between such vague regions are most naturally represented as graded relations, rather than crisp relations such as those of the RCC.

It is important to keep in mind that the RCC does not impose a particular representation of regions, nor a particular interpretation of connection. The only restriction imposed by the RCC is that the relation *C* is reflexive and symmetric. For example, in [32] the RCC relations are used to dynamically structure information from distributed hypermedia systems such as the web. In this context, regions are represented as vectors of attributes describing information units (e.g., paragraphs in a document), and two regions are connected if the degree of similarity of the corresponding information units exceeds a given threshold. Again, a graded approach may be more natural, in which two information units could be connected to the degree that they are similar to each other. Another interpretation of connection is introduced in [42] in the context of image processing, where regions are defined as black-and-white images and *C* is defined using dilations. Dilations are morphological operators that are often used in image processing for segmentation of images, boundary detection, etc. Using this interpretation of *C*, the RCC relations can be used for processing black-and-white images. The idea of dilations has been extended to gray-scale images and even color images, using dilation operators defined as fuzzy relations [34]. Interestingly, using these fuzzy dilation operators, it is possible to extend the idea from [42] to gray-scale images or color images in a generalization of the RCC that can cope with fuzzy relations. Although outside the context of the RCC, this idea has already been pursued to some extent in [37].

The aim of this paper is to introduce such a generalization of the RCC, based on an arbitrary reflexive and symmetric *fuzzy* relation *C*. In the spirit of the RCC, we do not impose any constraints on how regions are represented, or how connection should be interpreted. Therefore, our fuzzy relations can be used in contexts where space is used in a metaphorical way (e.g., regions as information units or images), as well as in, for example, geographical applications. Moreover, in the special case where *C* is a crisp relation, our definitions coincide with the original definitions of the RCC relations. In the next section, we recall some basic notions from fuzzy set theory, while Section 3 reviews related work on the modelling of vague regions and imprecise topological relations. Next, in Section 4, we introduce the definitions of our generalized RCC relations. Since there are many ways to generalize the original definitions, we show a number of interesting properties of our generalized definitions to justify the choices we made. Many of these properties are also useful in practice; most notably, the transitivity properties of our generalized definitions support spatial reasoning (i.e., the inference of new information from given spatial relations). Finally, some conclusions are presented in Section 5.

In a follow-up paper [46], we focus on the specific case where regions are represented as fuzzy sets of points and two regions are called connected to the degree that they are *close*. We provide a characterization of the generalized RCC relations under this interpretation, revealing their semantics, and providing a way to evaluate the fuzzy spatial relations in applications. A preliminary version of the results in this paper and [46] appeared in [39].

2. Preliminaries from fuzzy set theory

A fuzzy set [1] A in a universe X is defined as a mapping from X to the unit interval $[0,1]$. For x in X, $A(x)$ is called the membership degree of x in A. For α in $[0,1]$, the set $A_\alpha = \{x | x \in X \text{ and } A(x) \geqslant \alpha\}$ is called the α-level set of A.

A fuzzy set R in $X \times X$ is called a fuzzy relation in X. R is called reflexive iff $R(x,x) = 1$ for all x in X, and irreflexive iff $R(x,x) = 0$ for all x in X. It is symmetric iff $R(x,y) = R(y,x)$ for all x and y in X. The inverse of a fuzzy relation R in X is the fuzzy relation R^{-1} in X defined for all x and y in X by $R^{-1}(y,x) = R(x,y)$; the complement $co\ R$ of R is defined as $(co\ R)(x,y) = 1 - R(x,y)$ for all x and y in X.

A t-norm T is defined as a symmetric, associative, increasing $[0,1]^2 - [0,1]$ mapping satisfying the boundary condition $T(x,1) = x$ for all x in $[0,1]$. Some common t-norms are the minimum T_M, the product T_P and the Łukasiewicz t-norm T_W, defined by:

$$T_M(x,y) = \min(x,y)$$
$$T_P(x,y) = x \cdot y$$
$$T_W(x,y) = \max(0, x+y-1)$$

for all x and y in $[0,1]$. It is possible to define an ordering relation $\leqslant$ for t-norms as follows. If T_1 and T_2 are two t-norms, then

$$T_1 \leqslant T_2 \iff (\forall x,y \in [0,1])(T_1(x,y) \leqslant T_2(x,y)) \tag{1}$$

For example, it is easy to verify that $T_W \leqslant T_P \leqslant T_M$.

Similarly, a t-conorm is defined as a symmetric, associative, increasing $[0,1]^2 - [0,1]$ mapping S satisfying $S(0,x) = x$ for all x in $[0,1]$. Common t-conorms are the maximum S_M, the probabilistic sum S_P, and the Łukasiewicz t-conorm S_W, defined by

$$S_M(x,y) = \max(x,y)$$
$$S_P(x,y) = x + y - x \cdot y$$
$$S_W(x,y) = \min(1, x+y)$$

for all x and y in $[0,1]$. The negation of an element x in $[0,1]$ is commonly defined by $1-x$. Finally, a $[0,1]^2 - [0,1]$ mapping I which is decreasing in the first and increasing in the second argument and which satisfies $I(0,0) = I(0,1) = I(1,1) = 1$ and $I(1,0) = 0$ is called an implicator. For an arbitrary t-conorm S, the mapping I_S, defined for x and y in $[0,1]$ by

$$I_S(x,y) = S(1-x, y) \tag{2}$$

is called the strong implicator of S. For example, the strong implicator corresponding to S_M is defined by

$$I_{S_M}(x,y) = \max(1-x, y) \tag{3}$$

for all x and y in $[0,1]$. Let T be an arbitrary t-norm; the mapping I_T, defined for x and y in $[0,1]$ by

$$I_T(x,y) = \sup\{\lambda | \lambda \in [0,1] \text{ and } T(x,\lambda) \leqslant y\} \tag{4}$$

is called the residual implicator of T. For example, the residual implicators corresponding to T_M, T_P, and T_W are defined by:

$$I_{T_M}(x,y) = \begin{cases} 1 & \text{if } x \leqslant y \\ y & \text{otherwise} \end{cases}$$
$$I_{T_P}(x,y) = \begin{cases} 1 & \text{if } x \leqslant y \\ \frac{y}{x} & \text{otherwise} \end{cases}$$
$$I_{T_W}(x,y) = \min(1, 1-x+y)$$

for all x and y in $[0,1]$. We will mainly use residual implicators in this paper. For convenience, we will sometimes write I_M, I_P, and I_W instead of I_{T_M}, I_{T_P}, and I_{T_W}. If T is a left-continuous t-norm (i.e., a t-norm whose partial mappings are left-continuous), it can be shown that for all x, y and z in $[0,1]$, J an arbitrary index set, and $(x_j)_{j \in J}$ and $(y_j)_{j \in J}$ families in $[0,1]$, it holds that (see e.g., [17])

$$I_T(x,y) = 1 \iff x \leqslant y \tag{5}$$
$$T(x, I_T(x,y)) \leqslant y \tag{6}$$
$$I_T(T(x,y), z) = I_T(x, I_T(y,z)) \tag{7}$$
$$I_T\left(x, \inf_{j \in J} y_j\right) = \inf_{j \in J} I_T(x, y_j) \tag{8}$$
$$T\left(\inf_{j \in J} x_j, y\right) \leqslant \inf_{j \in J} T(x_j, y) \tag{9}$$

Moreover, it is easy to see that for an arbitrary t-norm T it holds that

$$I_T(1,x) = x \tag{10}$$

Note that for any implicator I, it holds that $I(x,1) = 1$, for every x in $[0,1]$.

3. Related work

It has been widely recognized that in the real world, geographical regions tend to be vague (e.g., [11,18,26,28,29]). Several formalisms to represent such vague regions have already been proposed, including supervaluation semantics [23,26,28], pairs of crisp sets [7,9,29], and fuzzy sets [14,16].

Most definitions of topological relations between vague regions extend either the RCC or the 9-intersection model by treating a vague region a as a pair of two crisp regions: one region $\underline{a}$ which consists of the points that definitely belong to the vague region, and one region $\bar{a}$ whose complement consists of the points that definitely do not belong to the vague region. The region defined by $\bar{a} \setminus \underline{a}$ (provided $\underline{a}$ is a proper part of $\bar{a}$) consists of the points for which it is hard to tell whether they are in the vague region or not. A well-known example is the egg-yolk calculus [7], which is based on the RCC. In [9], a similar approach, based on the notion of a thick boundary, is proposed as an extension of the 9-intersection model. Both models cause a significant increase in the number of possible relations: 46 and 44 relations, respectively. For example, instead of specifying that two regions a and b overlap, we may specify that $\bar{a}$ and $\underline{b}$ overlap (but not $\underline{a}$ and $\underline{b}$), or that $\bar{a}$ and $\bar{b}$ overlap, or that $\underline{a}$ and $\underline{b}$ overlap, etc. where $\underline{a}$ and $\bar{a}$ (respectively, $\underline{b}$ and $\bar{b}$) represent the yolk and the egg of a (respectively, b). In [43], spatial relations between such vague regions are represented as 4-tuples of classical topological relations, facilitating the study of, for instance, transitivity rules. Another possibility, which is adopted in [24], is to stay with the spatial relations of the RCC, but to use three-valued relations instead of classical two-valued relations.

Other approaches have been concerned with defining (fuzzy) spatial relations between vague regions represented as fuzzy sets. For example, in [15,25], generalizations of the 9-intersection model based on α-levels of fuzzy sets are suggested. In [31], a generalization of the 9-intersection model is introduced using concepts from fuzzy topology, yielding a set of 44 crisp spatial relations. Another generalization of the 9-intersection model, using similar fuzzy topological concepts, is proposed in [41], again obtaining 44 relations between fuzzy sets. On the other hand, [33] uses the RCC as a starting point to define crisp spatial relations between fuzzy sets. However, this approach can only be used when the membership values of the fuzzy sets are taken from a finite universe. The total number of relations is dependent on the cardinality of the finite set of membership values. In [30], degrees of appropriateness are assigned to RCC relations, modelling possibilistic uncertainty. These degrees could be interpreted as encoding, for instance, preferences or possibilistic uncertainty. Finally, [35,36] discuss fuzzy topological relations with the goal of modelling position uncertainty of region boundaries.

Approaches based on supervaluation semantics, like [11], mainly deal with a different kind of vagueness. A typical example is the definition of a forest [28]: should a forest be self-connected or can it consist of several disjoint parts; are roads and paths going through a forest parts of the forest? Approaches based on fuzzy sets, on the other hand, are more concerned with indeterminacy resulting from the fact that the transition from satisfying a certain condition to not satisfying is gradual. Typical examples from geography are concepts like mountains, or regions like Western Europe, downtown Seattle, or the Alps.

All the aforementioned approaches have in common that certain assumptions are made on how vague regions are represented. Moreover, they are mainly applicable to geographical contexts, and can usually not be used in situations where, for example, RCC relations are used in a metaphorical way. The generality and much of the elegance of the RCC is lost in this way. A different possibility, which we adopt in this paper, is to generalize the RCC relations directly, without making any assumptions on how regions should be represented. This idea has already been pursued, to some extent, in [12], where the starting point is to define connection as an arbitrary symmetric fuzzy relation C in the universe U of regions, satisfying a weak reflexivity property, namely $C(a,a) > 0.5$ for every region a in U. The fuzzy relation P (part of), for example, is defined by

$$P(a,b) = \inf_{z \in U} I_{S_M}(C(z,a), C(z,b)) \tag{11}$$

where a and b are regions in U. However, many properties of the original RCC relations are lost in this approach. For example, in correspondence with the reflexivity of P in the RCC, it would be desirable that $P(a,a) = 1$ for any region a in U. Unfortunately, this is in general not the case when (11) is used to define P, due to the choice of I_{S_M} to generalize logical implication. Similarly, many interesting transitivity properties are also lost, which makes the fuzzy relations unsuitable for spatial reasoning.

Finally, note that apart from generalizing topological relations to deal with vagueness, it is also possible to extend classical formalisms with the aim of modelling (probabilistic) uncertainty. For example, in [21], a probabilistic extension of the 9-intersection model is introduced to deal with uncertainty arising from imprecise measurements of region boundaries.

4. Fuzzy spatial relations

4.1. Definition

Henceforth, let T denote a left-continuous t-norm and I_T its residual implicator. Let C be a reflexive and symmetric fuzzy relation, where for two regions a and b, $C(a,b)$ expresses the degree to which a and b are connected. Table 2 proposes our generalization of the spatial relations of the RCC, expressing the degree $P(a,b)$ to which a is a part of b, the degree $O(a,b)$ to which a overlaps with b, etc.

Most of these expressions are straightforward generalizations of the definitions in Table 1, where logical operators are generalized using their corresponding fuzzy logic operators, and universal and existential quantification is generalized using the infimum and supremum, respectively. Note, however, that logical conjunction '$\wedge$' is sometimes modelled by min (e.g., in $EQ(a,b)$) and sometimes by T (e.g., in $O(a,b)$). This is because in the former case, the joint satisfaction of two independent constraints is evaluated, hence idempotency is desirable (recall that min is the only idempotent t-norm). However, in the latter case, this idempotency is not required, and other choices of T than the minimum should not be excluded a priori.

It is well-known that fuzzifying two formulas that are equivalent in binary logic, does not necessarily yield two equivalent formulas in fuzzy logic. Hence, it may be desirable to generalize formulas that are equivalent to the original definitions of some of the RCC relations, rather than the original definitions themselves. This is the case for NTP, where our definitions are simpler to manipulate than the definitions resulting from a straightforward generalization, and, moreover, yield a generalization that satisfies more interesting properties.

When C is a crisp relation, our definitions coincide with the original definitions of the RCC. To see why this is also true for NTP, we consider the following lemma.

Lemma 1

$$P(a,b) \wedge \neg(\exists c \in U)(EC(c,a) \wedge EC(c,b)) \equiv (\forall c \in U)(C(c,a) \Rightarrow O(c,b)) \tag{12}$$

Table 2
Generalized definitions of the spatial relations of the RCC

Relation	Definition
$DC(a,b)$	$1 - C(a,b)$
$P(a,b)$	$\inf_{c\in U} I_T(C(c,a), C(c,b))$
$PP(a,b)$	$\min(P(a,b), 1 - P(b,a))$
$EQ(a,b)$	$\min(P(a,b), P(b,a))$
$O(a,b)$	$\sup_{c\in U} T(P(c,a), P(c,b))$
$DR(a,b)$	$1 - O(a,b)$
$PO(a,b)$	$\min(O(a,b), 1 - P(a,b), 1 - P(b,a))$
$EC(a,b)$	$\min(C(a,b), 1 - O(a,b))$
$NTP(a,b)$	$\inf_{c\in U} I_T(C(c,a), O(c,b))$
$TPP(a,b)$	$\min(PP(a,b), 1 - NTP(a,b))$
$NTPP(a,b)$	$\min(1 - P(b,a), NTP(a,b))$

U is the universe of all regions, while a and b are variables denoting arbitrary elements of U, i.e., regions.

Proof. First, we prove

$$P(a,b) \wedge \neg(\exists c \in U)(EC(c,a) \wedge EC(c,b)) \Rightarrow (\forall c \in U)(C(c,a) \Rightarrow O(c,b))$$

or, equivalently,

$$P(a,b) \Rightarrow (\neg(\exists c \in U)(EC(c,a) \wedge EC(c,b)) \Rightarrow (\forall c \in U)(C(c,a) \Rightarrow O(c,b)))$$

Assuming $P(a,b)$, i.e., $C(c,a) \Rightarrow C(c,b)$ for all u in U, we obtain

$$\begin{aligned}\neg(\exists c \in U)(EC(c,a) \wedge EC(c,b)) &\equiv (\forall c \in U)(\neg EC(c,a) \vee \neg EC(c,b))\\ &\equiv (\forall c \in U)(\neg C(c,a) \vee O(c,a) \vee \neg C(c,b) \vee O(c,b))\end{aligned}$$

From $C(c,a) \Rightarrow C(c,b)$, we obtain $\neg C(c,a) \vee \neg C(c,b) \equiv \neg C(c,a)$. Moreover, we can show that, under the assumption that $P(a,b)$, it holds that $O(c,a) \Rightarrow O(c,b)$, and hence $O(c,a) \vee O(c,b) \equiv O(c,b)$. Thus we find

$$\begin{aligned}&(\forall c \in U)(\neg C(c,a) \vee O(c,a) \vee \neg C(c,b) \vee O(c,b))\\ &\quad \equiv (\forall c \in U)(\neg C(c,a) \vee O(c,b)) \equiv (\forall c \in U)(C(c,a) \Rightarrow O(c,b))\end{aligned}$$

Conversely, we immediately have that $(\forall c \in U)$ $(C(c,a) \Rightarrow O(c,b)) \Rightarrow P(a,b)$, since $O(u,v) \Rightarrow C(u,v)$ for all u and v in U. Finally, we show that also $(\forall c \in U)(C(c,a) \Rightarrow O(c,b)) \Rightarrow \neg(\exists c \in U)(EC(c,a) \wedge EC(c,b))$

$$\begin{aligned}(\forall c \in U)(C(c,a) \Rightarrow O(c,b)) &\equiv (\forall c \in U)(\neg C(c,a) \vee O(c,b))\\ &\Rightarrow (\forall c \in U)(\neg C(c,a) \vee O(c,a) \vee \neg C(c,b) \vee O(c,b))\\ &\equiv (\forall c \in U)(\neg EC(c,a) \vee \neg EC(c,b))\\ &\equiv \neg(\exists c \in U)(EC(c,a) \wedge EC(c,b)) \qquad \square\end{aligned}$$

Note that the right-hand side of (12) is the alternative definition of *NTP* which we have used for our generalization.

4.2. Properties

Next, we show some properties of our generalized RCC relations which are desirable in practice. They also serve as a justification of some of the decisions we made regarding the definitions of the fuzzy spatial relations, e.g., the use of residual implicators, and the somewhat peculiar definitions of *TPP* and *NTPP*. The first proposition shows that the (ir)reflexivity of the original RCC relations carries over to our generalizations.

Proposition 1. *The fuzzy relations P, O and EQ are reflexive, while the fuzzy relations DC, PP, DR, PO, EC, TPP and NTPP are irreflexive.*

Proof. Using (5), we find

$$P(a,a) = \inf_{z \in U} I_T(C(z,a), C(z,a)) = \inf_{z \in U} 1 = 1$$

For the fuzzy relation O, we obtain

$$O(a,a) = \sup_{z \in U} T(P(z,a), P(z,a)) \geqslant T(P(a,a), P(a,a)) = T(1,1) = 1$$

The reflexivity of *EQ* immediately follows from the reflexivity of P, while the irreflexivity of *DC* follows from the reflexivity of C. The irreflexivity of *PP*, *PO*, *TPP*, and *NTPP* follows from the reflexivity of P, and the irreflexivity of *DR* and *EC* follows from the reflexivity of O. $\square$

The relations of the RCC are not independent of each other. For example, if $TPP(a,b)$ holds, then also $PP(a,b)$. The following proposition generalizes such dependencies.

Proposition 2

1. $PO(a,b) \leqslant O(a,b)$	6. $TPP(a,b) \leqslant PP(a,b)$
2. $NTPP(a,b) \leqslant PP(a,b)$	7. $PP(a,b) \leqslant P(a,b)$
3. $EQ(a,b) \leqslant P(a,b)$	8. $P(a,b) \leqslant O(a,b)$
4. $O(a,b) \leqslant C(a,b)$	9. $EC(a,b) \leqslant C(a,b)$
5. $EC(a,b) \leqslant DR(a,b)$	10. $DC(a,b) \leqslant DR(a,b)$

Proof. First, we show that $O(a,b) \leqslant C(a,b)$:

$$O(a,b) = \sup_{z\in U} T(P(z,a),P(z,b)) = \sup_{z\in U} T(\inf_{u\in U} I_T(C(u,z),C(u,a)), \inf_{u\in U} I_T(C(u,z),C(u,b)))$$
$$\leqslant \sup_{z\in U} T(I_T(C(z,z),C(z,a)), I_T(C(a,z),C(a,b))) = \sup_{z\in U} T(I_T(1,C(z,a)), I_T(C(a,z),C(a,b)))$$

By (10), the symmetry of C, and (6), we obtain

$$= \sup_{z\in U} T(C(z,a), I_T(C(a,z),C(a,b))) = \sup_{z\in U} T(C(z,a), I_T(C(z,a),C(a,b))) \leqslant C(a,b)$$

As a corollary, we also have $DC(a,b) \leqslant DR(a,b)$ and $NTPP(a,b) \leqslant PP(a,b)$.

Next, we show that $P(a,b) \leqslant O(a,b)$:

$$O(a,b) = \sup_{z\in U} T(P(z,a),P(z,b)) \geqslant T(P(a,a),P(a,b)) = T(1,P(a,b)) = P(a,b)$$

where we made use of the reflexivity of P. The remaining inequalities follow straightforwardly from the definition of the minimum. □

Lemma 2 [44]. *Let* x, y, $z \in [0,1]$. *It holds that*

$$S_W(\min(x,y),\min(x,z)) \geqslant \min(x, S_W(y,z))$$

In the original RCC, if $PP(a,b)$ holds, then we know that either $TPP(a,b)$ or $NTPP(a,b)$. The following proposition presents a generalization of this observation.

Proposition 3

$$S_W(TPP(a,b), NTPP(a,b)) \geqslant PP(a,b) \tag{13}$$
$$S_W(PP(a,b), EQ(a,b)) \geqslant P(a,b) \tag{14}$$
$$S_W(PO(a,b), P(a,b), PP^{-1}(a,b)) \geqslant O(a,b) \tag{15}$$
$$S_W(O(a,b), EC(a,b)) \geqslant C(a,b) \tag{16}$$
$$S_W(EC(a,b), DC(a,b)) \geqslant DR(a,b) \tag{17}$$
$$S_W(C(a,b), DC(a,b)) = 1 \tag{18}$$
$$S_W(O(a,b), DR(a,b)) = 1$$

Proof. As an example, we show (13). We obtain

$$S_W(TPP(a,b), NTPP(a,b)) = S_W(\min(PP(a,b), 1-NTP(a,b)), \min(1-P(b,a), NTP(a,b)))$$
$$\geqslant S_W(\min(PP(a,b), 1-NTP(a,b)), \min(1-P(b,a), P(a,b), NTP(a,b)))$$
$$= S_W(\min(PP(a,b), 1-NTP(a,b)), \min(PP(a,b), NTP(a,b)))$$

By Lemma 2, and the fact that $S_W(x, 1-x) = 1$ for every x in [0, 1], we obtain

$$\geqslant \min(PP(a,b), S_W(NTP(a,b), 1-NTP(a,b))) = \min(PP(a,b), 1) = PP(a,b) \qquad \square$$

Note that the Łukasiewicz t-conorm is used in the previous proposition, regardless of the choice for T in the definitions of the fuzzy spatial relations. t-Conorms such as S_M or S_P cannot be used since they do not satisfy the law of the excluded middle, i.e., for a in $[0,1]$, it does not hold that $S_M(1-a,a)=1$ or $S_P(1-a,a)=1$ in general.

Most applications use only a subset of the RCC relations. Two subsets of RCC relations, called the RCC-8 relations and the RCC-5 relations, are particularly popular. The set of RCC-8 relations consists of the relations DC, EQ, EC, PO, TPP, $NTPP$, TPP^{-1} and $NTPP^{-1}$, while the RCC-5 relations are DR, EQ, PO, PP and PP^{-1}. In other words, when using the RCC-5 relations, DC and EC are taken together (DR), as well as TPP and $NTPP$ (PP) and their inverses. These two subsets of RCC relations have the important property that they are jointly exhaustive and pairwise disjoint (JEPD), i.e., for any two regions, exactly one of the RCC-8 relations holds, and exactly one of the RCC-5 relations. In the following propositions, we show that a generalization of this property remains valid for our definitions. Again the Łukasiewicz connectives are used in these properties to express the joint exhaustivity and the mutual exclusiveness.

Proposition 4. *Let R and Q be two of the fuzzy relations DC, EQ, EC, PO, TPP, $NTPP$, TPP^{-1} and $NTPP^{-1}$. If $R \neq Q$, it holds that*

$$T_W(R(a,b),Q(a,b))=0$$

Proof. As an example, we show that $T_W(EC(a,b),\ DC(a,b))=0$:

$$T_W(EC(a,b),DC(a,b))=T_W(\min(1-O(a,b),C(a,b)),1-C(a,b))\leqslant T_W(C(a,b),1-C(a,b))=0$$

where we used the fact that $T_W(x,1-x)=0$ for every x in $[0,1]$. □

Note that Proposition 4 does not hold in general for t-norms such as T_M and T_P. For example, let a, b and c be regions for which $NTP(a,b)=0.6$, $P(a,b)=0.8$ and $P(b,a)=0$. It holds that

$$NTPP(a,b)=\min(1-0,0.6)=0.6$$
$$TPP(a,b)=\min(0.8,1-0,1-0.6)=0.4$$

Hence we find

$$T_M(NTPP(a,b),TPP(a,b))=0.4>0$$
$$T_P(NTPP(a,b),TPP(a,b))=0.24>0$$

Proposition 5

$$S_W(DC(a,b),EQ(a,b),EC(a,b),PO(a,b),TPP(a,b),NTPP(a,b),TPP^{-1}(a,b),NTPP^{-1}(a,b))=1$$

Proof

$$\begin{aligned}&S_W(DC(a,b),EQ(a,b),EC(a,b),PO(a,b),TPP(a,b),NTPP(a,b),TPP^{-1}(a,b),NTPP^{-1}(a,b))\\&\quad\geqslant S_W(DC(a,b),EQ(a,b),EC(a,b),PO(a,b),PP(a,b),PP^{-1}(a,b))\\&\quad\geqslant S_W(DC(a,b),EC(a,b),PO(a,b),P(a,b),PP^{-1}(a,b))\geqslant S_W(DC(a,b),EC(a,b),O(a,b))\\&\quad\geqslant S_W(DC(a,b),C(a,b))=S_W(1-C(a,b),C(a,b))=1\end{aligned}$$

Where we used (13)–(16), the definition of DC, and the fact that $S_W(1-x,x)=1$ for all x in $[0,1]$. □

Analogously, we can show the following two propositions about the generalized RCC-5 relations.

Proposition 6. *Let R and Q be two of the fuzzy relations DR, EQ, PO, PP and PP^{-1}. If $R \neq Q$, it holds that*

$$T_W(R(a,b),Q(a,b))=0$$

Proposition 7

$$S_W(DR(a,b), EQ(a,b), PO(a,b), PP(a,b), PP^{-1}(a,b)) = 1$$

4.3. Transitivity

To facilitate spatial reasoning with the RCC-8 relations, a composition table (or transitivity table) has been introduced in [5]. The purpose of such a table is to specify, for each pair R, S of RCC-8 relations, the union of all RCC-8 relations F for which $F \cap (R \circ S) \neq \emptyset$, where the composition $R \circ S$ is defined for a and c in U as

$$(R \circ S)(a,c) \equiv (\exists b \in U)(R(a,b) \wedge S(b,c))$$

In other words, the composition table specifies which RCC-8 relations may hold between the regions a and c, given that $R(a,b)$ and $S(b,c)$ for some region b in U.

For example, as can be seen from Table 3, when $DC(a,b)$ and $EC(b,c)$ holds, either $DC(a,c)$, $EC(a,c)$, $PO(a,c)$, $TPP(a,c)$, or $NTPP(a,c)$ must hold. Therefore, the RCC-8 composition table contains $\{DC, EC, PO, TPP, NTPP\}$ in the entry on the row corresponding to DC and the column corresponding to EC. However, from the fact that the RCC-8 relations are JEPD, we easily obtain that the relations DC, EC, PO, TPP, $NTPP$ and P^{-1} are also JEPD; hence we have that

$$DC \cup EC \cup PO \cup TPP \cup NTPP = coP^{-1}$$

Therefore, the entry in the composition table could equivalently be $\neg P^{-1}$ instead of $\{DC, EC, PO, TPP, NTPP\}$. Similarly, all unions of RCC relations in the RCC-8 composition table can equivalently be formulated as intersections of C, P, P^{-1}, O, NTP, NTP^{-1}, DC, $\neg P$, $\neg P^{-1}$, DR, $\neg NTP$, and $\neg NTP^{-1}$. A similar observation was made in [13]. The resulting composition table is shown in Table 4.

To show that Table 4 is indeed equivalent to Table 3, we need the following lemma.

Lemma 3

$$(\exists z \in U)(EC(z,b)) \Rightarrow (NTP(a,b) \equiv NTPP(a,b))$$

Table 3
Original RCC-8 composition table (where EQ is omitted) [5]

	DC	EC	PO	TPP	$NTPP$	TPP^{-1}	$NTPP^{-1}$
DC	1	DC, EC, PO, TPP, $NTPP$	DC, EC, PO, TPP, $NTPP$	DC, EC, PO, TPP, $NTPP$	DC, EC, PO, TPP, $NTPP$	DC	DC
EC	DC, EC, PO, TPP^{-1}, $NTPP^{-1}$	DC, EC, PO, TPP, TPP^{-1}, EQ	DC, EC, PO, TPP, $NTPP$	EC, PO, TPP, $NTPP$	PO, TPP, $NTPP$	DC, EC	DC
PO	DC, EC, PO, TPP^{-1}, $NTPP^{-1}$	DC, EC, PO, TPP^{-1}, $NTPP^{-1}$	1	PO, TPP, $NTPP$	PO, TPP, $NTPP$	DC, EC, PO, TPP^{-1}, $NTPP^{-1}$	DC, EC, PO, TPP^{-1}, $NTPP^{-1}$
TPP	DC	DC, EC	DC, EC, PO, TPP, $NTPP$	TPP, $NTPP$	$NTPP$	DC, EC, PO, TPP, TPP^{-1}, EQ	DC, EC, PO, TPP^{-1}, $NTPP^{-1}$
$NTPP$	DC	DC	DC, EC, PO, TPP, $NTPP$	$NTPP$	$NTPP$	DC, EC, PO, TPP, $NTPP$	1
TPP^{-1}	DC, EC, PO, TPP^{-1}, $NTPP^{-1}$	EC, PO, TPP^{-1}, $NTPP^{-1}$	PO, TPP^{-1}, $NTPP^{-1}$	PO, EQ, TPP, TPP^{-1}	PO, TPP, $NTPP$	TPP^{-1}, $NTPP^{-1}$	$NTPP^{-1}$
$NTPP^{-1}$	DC, EC, PO, TPP^{-1}, $NTPP^{-1}$	PO, TPP^{-1}, $NTPP^{-1}$	PO, TPP^{-1}, $NTPP^{-1}$	PO, TPP^{-1}, $NTPP^{-1}$	PO, TPP^{-1}, TPP, $NTPP$, $NTPP^{-1}$, EQ	$NTPP^{-1}$	$NTPP^{-1}$

Table entries that contain more than one RCC-8 relation correspond to the *union* of the given relations; 1 denotes the union of all RCC-8 relations, i.e., the universal relation in the universe of regions U.

Table 4
Alternative formulation of the RCC-8 composition table (where EQ is omitted)

	DC	EC	PO	TPP	$NTPP$	TPP^{-1}	$NTPP^{-1}$
DC	1	coP^{-1}	coP^{-1}	coP^{-1}	coP^{-1}	DC	DC
EC	coP	$coNTP$, $coNTP^{-1}$	coP^{-1}	C, coP^{-1}	O, coP^{-1}	DR	DC
PO	coP	coP	1	O, coP^{-1}	O, coP^{-1}	coP	coP
TPP	DC	DR	coP^{-1}	P, coP^{-1}	NTP, coP^{-1}	$coNTP$, $coNTP^{-1}$	coP
$NTPP$	DC	DC	coP^{-1}	NTP, coP^{-1}	NTP, coP^{-1}	coP^{-1}	1
TPP^{-1}	coP	C, coP	O, coP	O, $coNTP$, $coNTP^{-1}$	O, coP^{-1}	P^{-1}, coP	NTP^{-1}, coP
$NTPP^{-1}$	coP	O, coP	O, coP	O, coP	O	NTP^{-1}, coP	NTP^{-1}, coP

Table entries containing more than one relation correspond to the *intersection* of the given relations; 1 denotes the universal relation in the universe of regions U.

Proof. Assume that for some z it holds that $EC(z,b)$, i.e., $C(z,b)$ and $\neg O(z,b)$. To show that, under this assumption, $NTP(a,b) \equiv NTPP(a,b)$, we only need to show that $NTP(a,b) \Rightarrow \neg P(b,a)$. To this end, we show that $\neg P(b,a)$ holds under the assumption $NTP(a,b)$

$$\neg P(b,a) \equiv \neg(\forall c \in U)(C(c,b) \Rightarrow C(c,a)) \equiv (\exists c \in U)(C(c,b) \wedge \neg C(c,a))$$

Using our alternative definition of $NTP(a,b)$, we find that $C(c,a) \Rightarrow O(c,b)$ holds, and hence also $\neg O(c,b) \Rightarrow \neg C(c,a)$. We obtain

$$\begin{aligned} &\Leftarrow (\exists c \in U)(C(c,b) \wedge \neg O(c,b)) \\ &\Leftarrow (C(z,b) \wedge \neg O(z,b)) \end{aligned}$$

The latter right hand side corresponds to our initial assumption $EC(z,b)$. □

Proposition 8. *The unions of the RCC-*8 *relations in the entries of* Table 3 *are equal to the corresponding intersections of the RCC relations in* Table 4.

Proof. Above we have already shown that $coP^{-1} = DC \cup EC \cup PO \cup TPP \cup NTPP$. Most equalities can analogously be obtained using the fact that, beside the RCC-8 and RCC-5 relations, the following sets of RCC relations are also JEPD (which easily follows from the fact that the RCC-8 and RCC-5 relations are JEPD):

$$\begin{aligned} &\{DC, EC, PO, TPP, NTPP, P^{-1}\} \\ &\{DC, EC, PO, TPP^{-1}, NTPP^{-1}, P\} \\ &\{DR, PO, TPP, NTPP, P^{-1}\} \\ &\{DR, PO, TPP^{-1}, NTPP^{-1}, P\} \\ &\{DR, PO, TPP, NTPP, TPP^{-1}, NTPP^{-1}, EQ\} \end{aligned}$$

To show the equality corresponding to the entry on the second row, second column, we need to show that

$$\begin{aligned} (EC(a,b) \wedge EC(b,c) &\Rightarrow (DC \cup EC \cup PO \cup TPP \cup TPP^{-1} \cup EQ)(a,c)) \\ &\equiv (EC(a,b) \wedge EC(b,c) \\ &\Rightarrow \neg NTP(a,c) \wedge \neg NTP^{-1}(a,c)) \end{aligned}$$

or, equivalently, using the fact that the RCC-8 relations are JEPD

$$\begin{aligned} (EC(a,b) \wedge EC(b,c) &\Rightarrow \neg NTPP(a,c) \wedge \neg NTPP^{-1}(a,c)) \\ &\equiv (EC(a,b) \wedge EC(b,c) \\ &\Rightarrow \neg NTP(a,c) \wedge \neg NTP^{-1}(a,c)) \end{aligned}$$

which is equivalent to showing

$$(NTPP(a,c) \wedge \neg NTPP^{-1}(a,c)) \equiv (\neg NTP(a,c) \wedge \neg NTP^{-1}(a,c))$$

under the assumption that $EC(a,b)$ and $EC(b,c)$ hold. This assumption implies that $(\exists z \in U)(EC(z,a))$ and $(\exists z \in U)(EC(z,c))$. Using Lemma 3, we conclude from this that

$$NTP(c,a) \equiv NTPP(c,a)$$
$$NTP(a,c) \equiv NTPP(a,c)$$

Finally, the equivalences corresponding to the entry on the fourth row, sixth column and the entry on the sixth row, fourth column, can be proven entirely analogously. □

Generalizations of Tables 4 and 3, using our generalized RCC relations, are not equivalent anymore. However, we still have

$$1 - P^{-1}(a,c) \leqslant S_W(DC(a,c), EC(a,c), PO(a,c), TPP(a,c), NTPP(a,c)) \tag{19}$$

Indeed, using Proposition 3 and the symmetry of DR and PO, we find

$$S_W(DC(a,c), EC(a,c), PO(a,c), TPP(a,c), NTPP(a,c), P^{-1}(a,c))$$
$$\geqslant S_W(DR(a,c), PO(a,c), PP(a,c), P^{-1}(a,c)) \geqslant S_W(DR(a,c), O(a,c)) = 1$$

which is equivalent to (19).

Transitivity properties of fuzzy relations generally take the form of inequalities of the form $T(R(a,b), S(b,c)) \leqslant Q(a,c)$ where R, S and Q are fuzzy relations in a suitable universe. As a consequence of (19),

$$T(DC(a,b), EC(b,c)) \leqslant 1 - P^{-1}(a,c)$$

is a stronger statement than

$$T(DC(a,b), EC(b,c)) \leqslant S_W(DC(a,c), EC(a,c), PO(a,c), TPP(a,c), NTPP(a,c))$$

Therefore, our aim is to generalize Table 4 rather than Table 3. However, as the entries of this table are formulated in terms of C, DC, O, DR, etc. we will provide a generalized transitivity table (shown in Table 5) where rows and columns correspond to fuzzy relations such as C, DC, O, or DR, rather than generalized RCC-8 relations. Below, we will introduce a spatial reasoning algorithm which can, among others, be used to reason about generalized RCC-8 relations using the generalized transitivity rules from Table 5. As we will show, a direct generalization of Table 4 can easily be obtained using this spatial reasoning algorithm.

Proposition 9. *Let R and S be two generalized RCC-8 relations, and let Q be the fuzzy relation in the entry of Table 5 on the row corresponding to R and the column corresponding to S. Furthermore, assume that the t-norm T used in the generalized definitions of the RCC relations satisfies $T_W \leqslant T$. For every region a, b, and c, it holds that*

$$T_W(R(a,b), S(b,c)) \leqslant Q(a,c) \tag{20}$$

For example, the entry on the second row, first column should be interpreted as

$$T_W(DC(a,b), C(b,c)) \leqslant (coP^{-1})(a,c) \tag{21}$$

Proof. See Appendix A. □

Recall that T_M and T_P are greater than T_W, i.e., the generalized transitivity rules hold when T_W, T_P, or T_W is used in the definition of the generalized RCC relations. Note that when the Łukasiewicz t-norm in (20) is replaced by T_M or T_P, the corresponding proposition is not valid anymore, even when T_M or T_P is used in the definition of the generalized RCC relations. To see this, consider the following counterexample.

Example 1. Let $U = \{a,b,c\}$, i.e., U only consists of three regions. Using the reflexivity of C, (5) and (10), we obtain

Table 5
Transitivity table for the generalized RCC relations

	C	DC	P	P^{-1}	coP	coP^{-1}	O	DR	NTP	NTP^{-1}	$coNTP$	$coNTP^{-1}$
C	1	coP	C	1	1	1	1	$coNTP$	O	1	1	1
DC	coP^{-1}	1	coP^{-1}	DC	1	1	coP^{-1}	1	coP^{-1}	DC	1	1
P	1	DC	P	1	1	coP^{-1}	1	DR	NTP	1	1	$coNTP^{-1}$
P^{-1}	C	coP	O	P^{-1}	coP	1	O	coP	O	NTP^{-1}	$coNTP$	1
coP	1	1	1	coP	1	1	1	1	1	coP	1	1
coP^{-1}	1	1	coP^{-1}	1	1	1	1	1	coP^{-1}	1	1	1
O	1	coP	O	1	1	1	1	coP	O	1	1	1
DR	$coNTP^{-1}$	1	coP^{-1}	DR	1	1	coP^{-1}	1	coP^{-1}	DC	1	1
NTP	1	DC	NTP	1	1	coP^{-1}	1	DC	NTP	1	1	coP^{-1}
NTP^{-1}	O	coP	O	NTP^{-1}	coP	1	O	coP	O	NTP^{-1}	coP	1
$coNTP$	1	1	1	$coNTP$	1	1	1	1	1	coP	1	1
$coNTP^{-1}$	1	1	$coNTP^{-1}$	1	1	1	1	1	coP^{-1}	1	1	1

Note that the transitivity rules summarized in this table only hold when the t-norm T in the definition of the fuzzy relations satisfies $T_W \leqslant T$.

$$
\begin{aligned}
P(c,a) &= \min(I_T(C(a,c),C(a,a)), I_T(C(b,c),C(b,a)), I_T(C(c,c),C(c,a))) \\
&= \min(I_T(C(a,c),1), I_T(C(b,c),C(b,a)), I_T(1,C(c,a))) = \min(1, I_T(C(b,c),C(b,a)), C(c,a)) \\
&= \min(I_T(C(b,c),C(b,a)), C(c,a))
\end{aligned}
$$

Furthermore, assume that C satisfies $C(c,a) = 0.9$, $C(b,c) = 0.2$, and $C(b,a) = 0.4$. When T_M and I_M are used in the definition of the generalized RCC relations, we obtain (using the symmetry of C):

$$(coP^{-1})(a,c) = 1 - P(c,a) = 1 - \min(1, 0.9) = 1 - 0.9 = 0.1$$
$$T_M(DC(a,b), C(b,c)) = \min(1 - C(a,b), C(b,c)) = \min(0.6, 0.2) = 0.2$$

Hence

$$T_M(DC(a,b), C(b,c)) > (coP^{-1})(a,c)$$

Similarly, when T_P and I_P are used in the definition of the generalized RCC relations, we have

$$(coP^{-1})(a,c) = 1 - P(c,a) = 1 - \min(1, 0.9) = 1 - 0.9 = 0.1$$
$$T_P(DC(a,b), C(b,c)) = (1 - C(a,b))C(b,c) = 0.6 \cdot 0.2 = 0.12$$

and thus

$$T_P(DC(a,b), C(b,c)) > (coP^{-1})(a,c)$$

Many of the generalized RCC relations from Table 2 are defined as the minimum of some of the fuzzy relations from Table 5. To derive transitivity rules for these fuzzy relations, based on the transitivity rules from Table 5, we can use the fact that (x, y, and z in $[0,1]$)

$$T_W(\min(x,y), z) \leqslant \min(T_W(x,z), T_W(y,z)) \tag{22}$$

which tells us how the minimum from the definition of the generalized RCC-8 relations interacts with the Łukasiewicz t-norm from the transitivity rules. Note that (22) is a special case of (9).

For example, using (22) we obtain, for regions a,b and c in U,

$$
\begin{aligned}
T_W(DC(a,b), EC(b,c)) &= T_W(DC(a,b), \min(C(b,c), DR(b,c))) \\
&\leqslant \min(T_W(DC(a,b), C(b,c)), T_W(DC(a,b), DR(b,c)))
\end{aligned}
$$

From Table 5 we have

$$\leqslant \min((coP^{-1})(a,c), 1) = (coP^{-1})(a,c)$$

This corresponds to the RCC-8 transitivity rule that from $DC(a,b)$ and $EC(b,c)$, it follows that $coP^{-1}(a,c)$ (see Table 4). In general, we can apply the following algorithm:

(1) Assume two fuzzy spatial relations R and Q are given that can be written as

$$R = \min(r_1, \ldots, r_n)$$
$$Q = \min(q_1, \ldots, q_m)$$

where r_i and q_j ($1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m$) are C, DC, P, P^{-1}, coP, coP^{-1}, O, DR, NTP, NTP^{-1}, $coNTP$, or $coNTP^{-1}$. This applies, among others, to all RCC-8 and RCC-5 relations.

(2) Repeatedly applying (22) yields

$$T_W(R(a,b), Q(b,c)) = T_W\left(\min_{i=1}^{n} r_i(a,b), \min_{j=1}^{m} q_j(b,c)\right) \leqslant \min_{i=1}^{n} \min_{j=1}^{m} T_W(r_i(a,b), q_j(b,c))$$

(3) For each i and each j, use Table 5 to obtain a conclusion of the form

$$T_W(r_i(a,b), q_j(b,c)) \leqslant t_{ij}(a,c) \tag{23}$$

Hence we obtain

$$T_W(R(a,b), Q(b,c)) \leqslant \min_{i=1}^{n} \min_{j=1}^{m} t_{ij}(a,c) \tag{24}$$

(4) Use Proposition 2 to obtain a minimal subset A of $\{t_{ij} | 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m\}$ for which it holds that

$$\min_{i=1}^{n} \min_{j=1}^{m} t_{ij}(a,c) = \min_{t \in A} t(a,c) \tag{25}$$

(5) We conclude

$$T_W(R(a,b), Q(b,c)) \leqslant \min_{t \in A} t(a,c) \tag{26}$$

Finally we show that applying this algorithm is a sound generalization of applying RCC-8 transitivity rules.

Proposition 10. *If C is a crisp relation, the deductions made for the RCC-8 relations using the spatial reasoning algorithm above are equivalent to the deductions made using the composition table introduced in [5] (i.e., Table 3).*

Proof. Each entry of the RCC-8 composition table (Table 3) corresponds to a transitivity rule of the form $R(a,b) \wedge S(b,c) \Rightarrow Q(a,c)$, where R and S are RCC-8 relations and Q is the union of some RCC-8 relations. We need to show that a conclusion equivalent to $Q(a,c)$ is obtained by our algorithm when $R(a,b)$ and $S(b,c)$ are known to hold. As an example, we show this for the entry on the second row, second column. Applying our spatial reasoning algorithm, we obtain

$$\begin{aligned}
T_W(EC(a,b), EC(b,c)) &= T_W(\min(C(a,b), 1 - O(a,b)), \min(C(b,c), 1 - O(b,c))) \\
&= T_W(\min(C(a,b), DR(a,b)), \min(C(b,c), DR(b,c))) \\
&\leqslant \min(T_W(C(a,b), C(b,c)), T_W(C(a,b), DR(b,c)), \\
&\quad T_W(DR(a,b), C(b,c)), T_W(DR(a,b), DR(b,c))) \\
&\leqslant \min(1, 1 - NTP(a,c), 1 - NTP^{-1}(a,c), 1) \\
&= \min(1 - NTP(a,c), 1 - NTP^{-1}(a,c))
\end{aligned}$$

If C is a crisp relation, then EC and NTP are crisp relations as well. Hence, we have established that from $EC(a,b)$ and $EC(b,c)$ it follows that $\neg NTP(a,c)$ and $\neg NTP^{-1}(a,c)$, which is equivalent to $DC(a,c) \vee EC(a,c) \vee PO(a,c) \vee TPP(a,c) \vee TPP^{-1}(a,c) \vee EQ(a,c)$ by Proposition 8. □

Note how in the proof of Proposition 10, a generalization is obtained of the transitivity rule $EC(a,b) \wedge EC(b,c) \Rightarrow \neg NTP(a,c) \wedge \neg NTP^{-1}(a,c)$, which corresponds to the entry on the second row, second column of Table 4. In general, we can show that applying the algorithm above to generalized RCC-8 relations is always equivalent to a generalization of the corresponding transitivity rule from Table 4.

Proposition 10 demonstrates that the transitivity rules from Table 5 behave intuitively when applied to crisp spatial information. It furthermore provides a means to deduce new information from given assertions about fuzzy topological relations. However, it does not provide any guarantees on the completeness of the inferences made. While it is possible to derive complete fuzzy spatial reasoning algorithms based on the transitivity rules from Table 5, a detailed discussion of this is outside the scope of this paper. We refer to [45] for more details.

5. Conclusions

We have introduced a generalization of the region connection calculus. The key idea is that the primitive relation C from the RCC is replaced by a fuzzy relation. The definitions of the other RCC relations are generalized accordingly, using fuzzy logic connectives instead of the original first-order logic formulation. As we make no assumptions on how regions are represented, and only require of C that it is reflexive and symmetric, the resulting framework can be used in a wide variety of contexts, including contexts where space is used in a metaphorical way. We have shown a number of interesting properties of our generalized RCC relations that demonstrate the potential of our approach. In particular, we have introduced a transitivity table revealing that generalizations of all the transitivity properties of the original RCC are valid for our definitions. These transitivity rules are important for applications, as they can be used as a basis to perform spatial reasoning.

Appendix A. Proof of the generalized RCC transitivity table

To prove the transitivity rules summarized in Table 5, the following characterizations are very useful.

Lemma 4. *Let a and b be arbitrary regions from U. It holds that*

$$P(a,b) = \inf_{z\in U} I_T(P(z,a),P(z,b)) \tag{A.1}$$

$$P(a,b) \leqslant \inf_{z\in U} I_T(O(z,a),O(z,b)) \tag{A.2}$$

$$P(a,b) = \inf_{z\in U} I_T(P(b,z),P(a,z)) \tag{A.3}$$

$$P(a,b) = \inf_{z\in U} I_T(NTP(z,a),NTP(z,b)) \tag{A.4}$$

$$P(a,b) = \inf_{z\in U} I_T(NTP(b,z),NTP(a,z)) \tag{A.5}$$

$$NTP(a,b) = \inf_{z\in U} I_T(P(z,a),NTP(z,b)) \tag{A.6}$$

$$NTP(a,b) = \inf_{z\in U} I_T(P(b,z),NTP(a,z)) \tag{A.7}$$

$$O(a,b) = \inf_{z\in U} I_T(P(a,z),O(b,z)) \tag{A.8}$$

Proof. As an example, we show (A.1). Using (8), we find

$$\begin{aligned}\inf_{z\in U} I_T(P(z,a),P(z,b)) &= \inf_{z\in U} I_T\left(\inf_{u\in U} I_T(C(u,z),C(u,a)), \inf_{u\in U} I_T(C(u,z),C(u,b))\right)\\ &= \inf_{z\in U}\inf_{u\in U} I_T\left(\inf_{u'\in U} I_T(C(u',z),C(u',a)), I_T(C(u,z),C(u,b))\right)\\ &\geqslant \inf_{z\in U}\inf_{u\in U} I_T(I_T(C(u,z),C(u,a)), I_T(C(u,z),C(u,b)))\end{aligned}$$

and by (7) and (6)

$$\begin{aligned}&= \inf_{z\in U}\inf_{u\in U} I_T(T(C(u,z),I_T(C(u,z),C(u,a))),C(u,b)) \geqslant \inf_{z\in U}\inf_{u\in U} I_T(C(u,a),C(u,b))\\ &= \inf_{u\in U} I_T(C(u,a),C(u,b)) = P(a,b)\end{aligned}$$

which already shows that $P(a,b) \leqslant \inf_{z\in U} I_T(P(z,a),P(z,b))$. Conversely we find, using the reflexivity of P, and (10)

$$\inf_{z\in U} I_T(P(z,a),P(z,b)) \leqslant I_T(P(a,a),P(a,b)) = I_T(1,P(a,b)) = P(a,b) \qquad \square$$

The following lemma relates the ordering of t-norms, as defined in (1), to an ordering of their corresponding residual implicators.

Lemma 5. *Let T_1 and T_2 be two t-norms satisfying $T_1 \leqslant T_2$. For every x and y in $[0,1]$, it holds that*

$$I_{T_1}(x,y) \geqslant I_{T_2}(x,y) \tag{A.9}$$

Proof. Let x and y be elements of $[0,1]$. Because $T_1 \leqslant T_2$, we have that for any $\lambda \in [0,1]$, it holds that

$$T_2(x,\lambda) \leqslant y \Rightarrow T_1(x,\lambda) \leqslant y$$

Hence

$$\{\lambda|\lambda \in [0,1] \text{ and } T_2(x,\lambda) \leqslant y\} \subseteq \{\lambda|\lambda \in [0,1] \text{ and } T_1(x,\lambda) \leqslant y\}$$

From the monotonicity of the supremum, we conclude

$$\sup\{\lambda|\lambda \in [0,1] \text{ and } T_2(x,\lambda) \leqslant y\} \leqslant \sup\{\lambda|\lambda \in [0,1] \text{ and } T_1(x,\lambda) \leqslant y\}$$

which is equivalent to (A.9) by the definition (4) of residual implicator. $\square$

Table 5 summarizes a number of transitivity rules that should be interpreted as explained in Proposition 9. As an example, we show how to prove that

$$T_W((coP^{-1})(a,b),P(b,c)) \leqslant (coP^{-1})(a,c)$$

Using (A.3), we find

$$\begin{aligned} T_W((coP^{-1})(a,b),P(b,c)) &= T_W(1-P(b,a),P(b,c)) = T_W(1-P(b,a),\inf_{z\in U} I_T(P(c,z),P(b,z))) \\ &\leqslant T_W(1-P(b,a),I_T(P(c,a),P(b,a))) \end{aligned}$$

Using the fact that $T_W \leqslant T$ and Lemma 5, we obtain

$$\begin{aligned} \leqslant T_W(1-P(b,a),I_W(P(c,a),P(b,a))) &= T_W(1-P(b,a),\min(1,1-P(c,a)+P(b,a))) \\ &= T_W(1-P(b,a),\min(1,1-(1-P(b,a))+(1-P(c,a)))) \\ &= T_W(1-P(b,a),I_W(1-P(b,a),1-P(c,a))) \end{aligned}$$

And by (6)

$$\leqslant 1-P(c,a) = (coP^{-1})(a,c)$$

References

[1] L.A. Zadeh, Fuzzy sets, Information and Control 8 (3) (1965) 338–353.
[2] M.J. Egenhofer, R. Franzosa, Point-set topological spatial relations, International Journal of Geographical Information Systems 5 (2) (1991) 161–174.
[3] S. Dutta, Approximate spatial reasoning: integrating qualitative and quantitative constraints, International Journal of Approximate Reasoning 5 (3) (1991) 307–330.
[4] D.A. Randell, Z. Cui, A.G. Cohn, A spatial logic based on regions and connection, in: Proceedings of the 3rd International Conference on Knowledge Representation and Reasoning, 1992, pp. 165–176.
[5] Z. Cui, A.G. Cohn, D.A. Randell, Qualitative and topological relationships, in: Proceedings of the Third International Symposium on Advances in Spatial Databases, LNCS, vol. 692, 1993, pp. 296–315.
[6] M. Gahegan, Proximity operators for qualitative spatial reasoning, in: Proceedings of the International Conference on Spatial Information Theory: A Theoretical Basis for GIS (COSIT 1995), LNCS, vol. 988, 1995, pp. 31–44.

[7] A.G. Cohn, N.M. Gotts, The 'egg-yolk' representation of regions with indeterminate boundaries, in: P.A. Burrough, A.U. Frank (Eds.), Geographic Objects with Indeterminate Boundaries, Taylor and Francis Ltd., 1996, pp. 171–187.
[8] A.U. Frank, Qualitative spatial reasoning: cardinal directions as an example, International Journal of Geographical Information Systems 10 (3) (1996) 269–290.
[9] E. Clementini, P. Di Felice, Approximate topological relations, International Journal of Approximate Reasoning 16 (2) (1997) 173–204.
[10] E. Clementini, P. Di Felice, D. Hernández, Qualitative representation of positional information, Artificial Intelligence 95 (2) (1997) 317–356.
[11] M. Erwig, M. Schneider, Vague regions, in: Proceedings of the 5th International Symposium on Advances in Spatial Databases, LNCS, vol. 1262, 1997, pp. 298–320.
[12] A. Esterline, G. Dozier, A. Homaifar, Fuzzy spatial reasoning, in: Proceedings of the 1997 International Fuzzy Systems Association Conference, 1997, pp. 162–167.
[13] B. Bennett, Determining consistency of topological relations, Constraints 3 (2–3) (1998) 213–225.
[14] M.F. Goodchild, D.R. Montello, P. Fohl, J. Gottsegen, Fuzzy spatial queries in digital spatial data libraries, in: Proceedings of the IEEE World Congress on Computational Intelligence, 1998, pp. 205–210.
[15] F.B. Zhan, Approximate analysis of binary topological relations between geographic regions with indeterminate boundaries, Soft Computing 2 (2) (1998) 28–34.
[16] L.L. Hill, J. Frew, Q. Zheng, Geographic names: the implementation of a gazetteer in a georeferenced digital library, D-Lib Magazine 5 (1) (1999).
[17] V. Novák, I. Perfilieva, J. Močkoř, Mathematical Principles of Fuzzy Logic, Kluwer Academic Publishers, 1999.
[18] P. Fisher, Sorites paradox and vague geographies, Fuzzy Sets and Systems 113 (1) (2000) 7–18.
[19] H.W. Guesgen, J. Albrecht, Imprecise reasoning in geographic information systems, Fuzzy Sets and Systems 113 (1) (2000) 121–131.
[20] V.B. Robinson, Individual and multipersonal fuzzy spatial relations acquired using human–machine interaction, Fuzzy Sets and Systems 113 (1) (2000) 133–145.
[21] S. Winter, Uncertain topological relations between imprecise regions, International Journal of Geographical Information Science 14 (5) (2000) 411–430.
[22] A.G. Cohn, S.M. Hazarika, Qualitative spatial representation and reasoning: an overview, Fundamenta Informaticae 46 (1–2) (2001) 1–29.
[23] L. Kulik, A geometric theory of vague boundaries based on supervaluation, in: Proceedings of the International Conference on Spatial Information Theory: Foundations of Geographic Information Science (COSIT 2001), LNCS, vol. 2205, 2001, pp. 44–59.
[24] A.J. Roy, J.G. Stell, Spatial relations between indeterminate regions, International Journal of Approximate Reasoning 27 (3) (2001) 205–234.
[25] M. Schneider, A design of topological predicates for complex crisp and fuzzy regions, in: Proceedings of the 20th International Conference on Conceptual Modeling, LNCS, vol. 2224, 2001, pp. 103–116.
[26] A.C. Varzi, Vagueness in geography, Philosophy and Geography 4 (1) (2001) 49–65.
[27] M.F. Worboys, Nearness relations in environmental space, International Journal of Geographical Information Science 15 (7) (2001) 633–651.
[28] B. Bennett, What is a forest? On the vagueness of certain geographic concepts, Topoi 20 (2) (2001) 189–201.
[29] T. Bittner, J.G. Stell, Vagueness and rough location, Geoinformatica 6 (2) (2002) 99–121.
[30] H.W. Guesgen, From the egg-yolk to the scrambled-egg theory, in: Proceedings of the 15th International FLAIRS Conference, 2002, pp. 476–480.
[31] X. Tang, W. Kainz, Analysis of topological relations between fuzzy regions in a general fuzzy topological space, in: Proceedings of the Symposium on Geospatial Theory, Processing and Applications, 2002, <http://www.isprs.org/commission4/proceedings>.
[32] C.G. Ralha, J.C.L. Ralha, Intelligent mapping of hyperspace, in: Proceedings of the IEEE/WIC International Conference on Web Intelligence, 2003, pp. 454–457.
[33] Y. Li, S. Li, A fuzzy sets theoretic approach to approximate spatial reasoning, IEEE Transactions on Fuzzy Systems 12 (6) (2004) 745–754.
[34] V. De Witte, S. Schulte, M. Nachtegaal, D. Van der Weken, E.E. Kerre, Vector morphological operators for colour images, in: Proceedings of the 2nd International Conference on Image Analysis and Recognition, LNCS, vol. 3656, 2005, pp. 667–675.
[35] S. Du, Q. Qin, Q. Wang, B. Li, Fuzzy description of topological relations I: a unified fuzzy 9-Intersection model, Advances in Natural Computation, LNCS, vol. 3612, 2005, pp. 1261–1273.
[36] S. Du, Q. Wang, Q. Qin, Y. Yang, Fuzzy description of topological relations II: computation methods and examples, Advances in Natural Computation, LNCS, vol. 3612, 2005, pp. 1274–1279.
[37] I. Bloch, Spatial reasoning under imprecision using fuzzy set theory, formal logics and mathematical morphology, International Journal of Approximate Reasoning 41 (2) (2006) 77–95.
[38] S. Schockaert, M. De Cock, E.E. Kerre, Towards fuzzy spatial reasoning in geographic IR systems, Workshop on Geographic Information Retrieval, at the 29th Annual International ACM SIGIR Conference on Research & Development in Information Retrieval, 2006, pp. 34–36.
[39] S. Schockaert, C. Cornelis, M. De Cock, E.E. Kerre, Fuzzy spatial relations between vague regions, in: Proceedings of the 3rd IEEE Conference on Intelligent Systems, 2006, pp. 221–226.
[40] A.I. Abdelmoty, P. Smart, B.A. El-Geresy, Towards the practical use of qualitative spatial reasoning in geographic information retrieval, in: Proceedings of the 3rd IEEE Conference on Intelligent Systems, 2006, pp. 71–73.

[41] K. Liu, W. Shi, Computing the fuzzy topological relations of spatial objects based on induced fuzzy topology, International Journal of Geographical Information Science 20 (8) (2006) 857–883.
[42] M. Aiello, B. Ottens, The mathematical morpho-logical view on reasoning about space, in: Proceedings of the 20th International Joint Conference on Artificial Intelligence, 2007, pp. 205–211.
[43] S. Du, Q. Qin, Q. Wang, H. Ma, The reasoning about topological relations between regions with broad boundaries, International Journal of Approximate Reasoning, in press, doi:10.1016/j.ijar.2007.05.002.
[44] S. Schockaert, M. De Cock, E.E. Kerre, Fuzzifying Allen's Temporal Interval Relations, IEEE Transactions on Fuzzy Systems, in press.
[45] S. Schockaert, M. De Cock, Reasoning about vague topological information, Proceedings of the ACM Conference on Information and Knowledge Management, in press.
[46] S. Schockaert, M. De Cock, C. Cornelis, E.E. Kerre, Fuzzy region connection calculus: an interpretation based on closeness, International Journal of Approximate Reasoning, accepted for publication, doi:10.1016/j.ijar.2007.10.002.