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RESTRICTIONS ON HARMONIC MAPS OF SURFACES

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Dedicated to Professors S.-S. Chern and H. Whitney.

§1. STATEMENT OF THE MAIN RESULTS

LET X and Y be closed orientable smooth surfaces, and $\varphi: X \to Y$ a smooth map. Relative to Riemannian metrics g, h on X, Y we say that *φ is harmonic* if the divergence of its differential vanishes identically[3]. In isothermal charts, writing $h = \sigma^2(w)\,dw\,d\bar{w}$ and representing φ in the form $z \to w(z)$, that condition is expressed by

$$w_{z\bar{z}} + (2\sigma_w/\sigma)w_z w_{\bar{z}} = 0, \tag{1}$$

being the Euler–Lagrange equation associated to the energy functional

$$E(\varphi) = \frac{1}{2}\int |d\varphi(x)|^2\,dx = \frac{1}{2}\int_X \sigma^2(w(z))[|w_z|^2 + |w_{\bar{z}}|^2]\,dx\,dy.$$

The purpose of this note† is to prove the

THEOREM. *If $\varphi: X \to Y$ is a harmonic map relative to Riemannian metrics g and h, and if $e(X) + |d_\varphi e(Y)| > 0$, then φ is $\pm$ holomorphic relative to the complex structures determined by g and h.*

Here $e(X) = 2 - 2p$ and $e(Y) = 2 - 2q$ denote Euler characteristics; and d_φ is the degree of φ. φ is $\pm$ holomorphic means that φ is either holomorphic or anti-holomorphic.

The following consequence was first established by Wood[12] and Lemaire[8]:

COROLLARY 1. *If $p = 0$, then harmonic maps $\varphi: (X, g) \to (Y, h)$ are $\pm$ holomorphic, regardless of g and h. (In fact, for $q = 0$ they are the Möbius transformations; and for $q \geq 1$ they are the constant maps).*

COROLLARY 2. *If $q = 0$ and $\varphi: (X, g) \to (Y, h)$ is harmonic with $|d_\varphi| \geq p$, then φ is $\pm$ holomorphic. In particular, if the Riemann surface X has no meromorphic function of degree p, then there is no harmonic map $\varphi: X \to S$ of degree p.*

Remark. There are many examples of essential harmonic maps $\varphi: X \to S$ for every $p \geq 2$ which are not $\pm$ holomorphic[8].

Example. Take $p = 1$ and $q = 0$. We know that there are holomorphic maps from a torus T to the sphere S of all degrees ≥ 2. These provide harmonic maps $\varphi: (T, g) \to (S, h)$ with $|d_\varphi| \geq 2$, whatever the metrics g and h; and are essentially the only ones with those degrees. In contrast, Smith[10] has constructed examples of harmonic maps of degree 0 from a flat torus to the Euclidean sphere which are surjective; and others whose images are proper subsets with interior. Finally, Corollary 2 implies that *there is no harmonic map $\varphi: (T, g) \to (S, h)$ of degree ± 1, whatever the metrics g, h.* (Earlier it had been shown by Lemaire[8] and Uhlenbeck that there was no such map giving an absolute minimum of the energy).

Example. Let X be a hyperelliptic Riemann surface of odd genus $p \geq 3$. Then there is no meromorphic function $\varphi: X \to S$ of degree p, since every such φ is a rational function of a meromorphic function $\theta: X \to S$ of degree 2[1]. Therefore *there is no harmonic map $\varphi: (X, g) \to (S, h)$ of degree p, whatever the metrics (compatible with the complex structures).*

By way of contrast, A. M. Macbeath has shown us that meromorphic functions of degree p do exist on all other Riemann surfaces X of genus $p \geq 2$.

†A special (and the most important) case of that result was anounced at the Summer Course in Complex Analysis, Trieste 1975[14]. It is a pleasure to record our thanks to M. J. Field, L. Lemaire, and A. M. Macbeath for their comments during the preparation of this note.

Application. We give an analytical proof of the following topological theorem of H. Kneser[7]:

If $q \geqslant 2$, then for any continuous map $\psi: X \to Y$, we have $|d_\psi| e(Y) \geqslant e(X)$. Namely, we introduce an arbitrary metric g on X, and a metric h of negative curvature on Y, and appeal to the existence theorem[3] to obtain a harmonic map $\varphi: (X, g) \to (Y, h)$ which is homotopic to ψ; thus $d_\varphi = d_\psi$. Now if $|d_\varphi| e(Y) < e(X)$, then $e(X) + |d_\varphi e(Y)| > 0$, so that φ is $\pm$ holomorphic. Applying Hurwitz' formula [15, Ch. II] for a holomorphic map we obtain $|d_\varphi| e(Y) = e(X) + r$ with $r \geqslant 0$, a contradiction which establishes Kneser's inequality. [A differential geometric proof is provided by G. Lusztig, based on the two main theorems in [9]. Namely, take a flat $GL^+(R^2)$-bundle η over Y with Euler class $W(\eta) = q - 1$. Then $\varphi^*\eta$ is a flat $GL^+(R^2)$-bundle over X, so that $|W(\varphi^{-1}\eta)| \leqslant p - 1$. Thus $|d_\varphi(q-1)| = |\varphi^* W(\eta)| = |W\varphi^{-1}\eta| \leqslant p - 1$].

Remark. Lemaire notes that with our present knowledge—especially his existence theorems[8]—we can respond to the question:

Given closed orientable surfaces X, Y and a homotopy class of maps $\psi: X \to Y$, can we find metrics g, h on X, Y relative to which there is a harmonic map φ homotopic to ψ?

The answer is yes in all cases except when $p = 1$, $q = 0$, and $|d_\psi| = 1$. In that case the answer is no.

§2. PROOF OF THE THEOREM

Given a smooth map $\varphi: X \to Y$, its differential $d\varphi: T(X) \to T(Y)$ extends to a complex linear map $d^c\varphi: T^c(X) \to T^c(Y)$ of the complexified tangent bundle $T^c(X) = T(X) \otimes_R C$; see [6, Ch. IX]. Corresponding to the decompositions $T^c(X) = T^{1,0}(X) \oplus T^{0,1}(X)$ we have

$$\partial_{1,0}\varphi: T^{1,0}(X) \to T^{1,0}(Y), \qquad \partial_{0,1}\varphi: T^{1,0}(X) \to T^{0,1}(Y).$$

These can be interpreted as sections of the complex line bundles

$$T^*_{1,0}(X) \otimes_C \varphi^{-1} T^{1,0}(Y) \quad \text{and} \quad T^*_{1,0}(X) \otimes_C \varphi^{-1} T^{0,1}(Y),$$

respectively. We note that

$$\partial_{1,0}\varphi \equiv 0 \text{ iff } \varphi \text{ is anti-holomorphic,}$$
$$\partial_{0,1}\varphi \equiv 0 \text{ iff } \varphi \text{ is holomorphic.}$$

The following result is standard:

PROPOSITION 1. *If the sections $\partial_{1,0}\varphi$ and $\partial_{0,1}\varphi$ have only finitely many zeros, then*

$$\text{Index}\,(\partial_{1,0}\varphi) = -e(X) + d_\varphi e(Y);$$
$$\text{Index}\,(\partial_{0,1}\varphi) = -e(X) - d_\varphi e(Y). \tag{2}$$

Here Index(s) = the sum of the zeros of the section s (having only finitely many zeros) of the complex line bundle ξ over X, counted at each point x according to the local degree of s at x. Then Index$(s) = c(\xi)$, the (first) Chern class of ξ; see [11, Part III].

For the first assertion of the Proposition, we take $T^*_{1,0}(X) \otimes_C \varphi^{-1} T^{1,0}(Y)$ for ξ. Now [4, Chapter I]

$$c(\xi_1 \otimes \xi_2) = c(\xi_1) + c(\xi_2), \qquad c(\xi^*) = -c(\xi), \qquad c(\varphi^{-1}\xi) = d_\varphi c(\xi).$$

Since we have canonical isomorphisms $T^*_{1,0}(X) \approx T^*(X)$ and $T^{1,0}(Y) \approx T(Y)$ as complex line bundles, we conclude that

$$\begin{aligned} \text{Index}\,(\partial_{1,0}\varphi) &= c(T^*_{1,0}(X) \otimes_C \varphi^{-1} T(Y)) \\ &= c(T^*_{1,0}(X) + c(\varphi^{-1} T^{1,0}(Y)) \\ &= -e(X) + d_\varphi e(Y). \end{aligned}$$

Similarly for Index $(\partial_{0,1}\varphi)$.

Our next step is to make a local study of the zeros of $\partial_{1,0}\varphi$ and $\partial_{0,1}\varphi$ when $\varphi: (X, g) \to (Y, h)$ is a harmonic map. The following properties have been known for some time (e.g., to Gerstenhaber–Rauch[5] in a special case, and to J. Sampson in general); see [2, 12, 13].

PROPOSITION 2. *If $\varphi: X \to Y$ is harmonic, then the* (2,0)*-part of the tensor field φ^*h is a holomorphic quadratic differential on X; we denote it by η_φ. Furthermore, $\eta_\varphi \equiv 0$ iff φ is $\pm$ holomorphic.*

Here

$$\eta_\varphi = a(z)\,dz^2 = \sigma^2(w(z))\frac{\partial w}{\partial z}\frac{\partial \bar{w}}{\partial z}. \tag{3}$$

The partial differentials $\partial_{1,0}\varphi$ and $\partial_{0,1}\varphi$ are represented by $\partial w/\partial z$ and $\partial \bar{w}/\partial z$. Let us say that $\partial_{1,0}\varphi$ *has a zero of infinite order* at a point if

$$\partial w/\partial z = o(|z|^m) \text{ as } z \to 0 \text{ for all } m \geq 0. \tag{4}$$

COROLLARY. *If $\varphi: X \to Y$ is harmonic but not $\pm$ holomorphic, then neither $\partial_{1,0}\varphi$ nor $\partial_{0,1}\varphi$ has a zero of infinite order.*

For otherwise the holomorphic function $a(z) = o(|z|^m)$ as $z \to 0$ for all $m \geq 0$; but that would imply that $a \equiv 0$, so that by Proposition 2, φ would be $\pm$ holomorphic.

LEMMA. *If $\varphi: X \to Y$ is harmonic but not $\pm$ holomorphic, then in isothermal charts $\partial_{1,0}\varphi$ and $\partial_{0,1}\varphi$ have the forms*

$$\begin{aligned} &\partial w/\partial z = Az^m + o(|z|^m) \textit{ for some } m \geq 0 \textit{ and complex number } A \neq 0;\\ &\partial \bar{w}/\partial z = Bz^n + o(|z|^n) \textit{ for some } n \geq 0 \textit{ and } B \neq 0. \end{aligned} \tag{5}$$

Proof. Taylor's expansion to m^{th} order for the first gives

$$\partial w/\partial z = Q_m(z, \bar{z}) + R(z, \bar{z}), \tag{6}$$

where Q_m is a homogeneous polynomial of degree $m \geq 0$, and R is $o(|z|^m)$ as $z \to 0$.

The preceding Corollary shows that $Q_m \neq 0$ for some m. Substituting (6) in (1) gives

$$\frac{\partial}{\partial \bar{z}}(Q_m + R) + \frac{2\sigma_w}{\sigma}(Q_m + R)\frac{\partial w}{\partial \bar{z}} = 0.$$

Now $\partial Q_m/\partial \bar{z}$ is either a homogeneous polynomial of degree $m-1$ or is identically 0. Each of the other terms is $o(|z|^{m-1})$. It follows that $\partial Q_m/\partial \bar{z} \equiv 0$. Therefore Q_m is a holomorphic homogeneous polynomial of degree m, and consequently has the form Az^m. The case of $\partial_{0,1}\varphi$ is handled similarly.

Remark. A local expansion theorem for harmonic maps, their differentials, and Jacobians was given by Wood [12, Th 1.4.8; 13], based on work of Hartman–Wintner and Heinz. Our needs here are more modest, and the above direct argument suffices.

PROPOSITION 3. *If $\varphi: X \to Y$ is harmonic and not $\pm$ holomorphic, then the zeros of $\partial_{1,0}\varphi$ and $\partial_{0,1}\varphi$ are isolated and of strictly positive index.*

Proof. Taking an isothermal chart centered at a zero x of $\partial_{1,0}\varphi$, we obtain from (5) $\partial w/\partial z = Az^m + o(|z|^m)$ with $m > 0$; thus $\partial_{1,0}\varphi$ has a zero of index m at x. Similarly for the zeros of $\partial_{0,1}\varphi$.

Proof of the theorem. Assume that φ is not $\pm$ holomorphic. Proposition 3 shows that we can apply Proposition 1. But if $e(X) + |d_\varphi e(Y)| > 0$, one of the right members of (2) is negative, contradicting Proposition 3.

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