MATHEMATICS

ORTHOGONAL POLYNOMIALS IN TWO VARIABLES WHICH ARE EIGENFUNCTIONS OF TWO ALGEBRAICALLY INDEPENDENT PARTIAL DIFFERENTIAL OPERATORS. III *)

BY

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ABSTRACT

Let the region $S=\{(x, y)|\mu(x+iy, x-iy)>0\}$ be the interior of Steiner's hypocycloid, where $\mu(z, \bar{z})=-z^2\bar{z}^2+4z^3+4\bar{z}^3-18z\bar{z}+27$. For each real $\alpha>-5/6$ an orthogonal system of polynomials $p^{\alpha}_{m,n}(z, \bar{z})$, $m, n\geqslant 0$, can be defined on this region S such that $p^{\alpha}_{m,n}(z, \bar{z})-z^m\bar{z}^n$ has degree less than $m+n$ and

$$\iint\limits_S p^{\alpha}_{m,n}(z, \bar{z})\, \overline{q(z, \bar{z})}\, (\mu(z, \bar{z}))^{\alpha}\, dx\, dy=0$$

for each polynomial q of degree less than $m+n$. If $z=e^{i(s+t/\sqrt{3})}+e^{i(-s+t/\sqrt{3})}+e^{-2it/\sqrt{3}}$ then, in terms of s and t, the functions $p^{-\frac{1}{2}}_{m,n}$ and $\mu^{\frac{1}{2}}p^{\frac{1}{2}}_{m-1,n-1}$ are the regular eigenfunctions of the operator $\partial^2/\partial s^2+\partial^2/\partial t^2$ which remain invariant or change sign, respectively, under the reflections in the edges of a certain equilateral triangle. Two explicit partial differential operators D^{α}_1 and D^{α}_2 in z and $\bar{z}$ of orders two and three, respectively, are obtained such that the polynomials $p^{\alpha}_{m,n}$ are eigenfunctions of D^{α}_1 and D^{α}_2. The operators D^{α}_1 and D^{α}_2 commute and are algebraically independent, and they generate the algebra of all differential operators for which the polynomials $p^{\alpha}_{m,n}$ are eigenfunctions. If $\alpha=0$, $\frac{1}{2}$, $3/2$ or $7/2$ then the operator D^{α}_1 expressed in terms of s and t is the radial part of the Laplace-Beltrami operator on certain compact Riemannian symmetric spaces of rank two.

1. INTRODUCTION

This paper deals with orthogonal polynomials in two variables on a region bounded by a closed three-cusped algebraic curve of fourth degree which is known as Steiner's hypocycloid. The weight function is some power of the fourth degree polynomial which vanishes on this curve. The main result in this paper is the construction of two algebraically independent partial differential operators of orders two and three, respectively, for which these orthogonal polynomials are eigenfunctions. Because of the existence of such operators these polynomials can be considered as a generalization of the classical orthogonal polynomials in one variable.

Another generalization of this type was studied in the author's previous

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paper [6], which dealt with orthogonal polynomials on a region bounded by two lines and a parabola touching these lines. The main difference between these two classes of polynomials is the method of orthogonalization. In [6] we orthogonalized the sequence $u^{n-k}v^k$, $n \geq k \geq 0$, arranged by the lexicographic ordering of the pairs (n, k). In the present paper, if $z = x+iy$, $\bar{z} = x-iy$ then the polynomial $p_{m,n}(z, \bar{z})$ is defined such that $p_{m,n}(z, \bar{z}) - z^m\bar{z}^n$ has degree less than $m+n$ and $p_{m,n}$ is orthogonal to all polynomials of degree less than $m+n$. Thus the polynomials $p_{m,n}$, $m+n=N$, form a basis for the class of all orthogonal polynomials of degree N. For the special region and class of weight functions considered here it can be proved that this is an orthogonal basis.

A special case of the orthogonal polynomials studied in [6] can be obtained by considering the functions $\cos ns \cos kt + \cos ks \cos nt$, which are eigenfunctions of the Laplace operator satisfying certain symmetry relations. Expressed in the variables $u = \cos s + \cos t$, $v = \cos s \cos t$, these functions are orthogonal polynomials with respect to the weight function $(1-u+v)^{-\frac{1}{2}}(1+u+v)^{-\frac{1}{2}}(u^2-4v)^{-\frac{1}{2}}$. Similarly, the point of departure of the present paper are the regular eigenfunctions of $\partial^2/\partial s^2 + \partial^2/\partial t^2$ which are invariant under the reflections in the edges of some equilateral triangle. Let the interior of this triangle be denoted by R. After a suitable linear transformation $(s, t) \to (\sigma, \tau)$ these eigenfunctions can be expressed as sums of at most six distinct terms $e^{i(k\sigma+l\tau)}$, k, l integers, and they constitute a complete orthogonal system on the region R. This is discussed in § 2.

The two non-constant eigenfunctions corresponding to the largest eigenvalue are the functions $z = e^{i\sigma} + e^{-i\tau} + e^{i(-\sigma+\tau)}$ and its complex conjugate $\bar{z}$. If $z = x+iy$ then the mapping $(s, t) \to (x, y)$ is bijective from R onto a region bounded by Steiner's hypocycloid. This last region will be denoted by S. In terms of z and $\bar{z}$, the eigenfunctions of $\partial^2/\partial s^2 + \partial^2/\partial t^2$ satisfying the symmetry relations mentioned above are polynomials for which the term of highest degree takes the form $z^m\bar{z}^n$. These polynomials are orthogonal on the region S with respect to the weight function $(\mu(z, \bar{z}))^{-\frac{1}{2}}$, where $\mu(z, \bar{z})$ is a fourth degree polynomial such that $\mu(z, \bar{z}) > 0$ on S and $\mu(z, \bar{z}) = 0$ on ∂S. These results are contained in § 3. It is also shown there that orthogonal polynomials on S with respect to the weight function $(\mu(z, \bar{z}))^{\frac{1}{2}}$ are related to the eigenfunctions of $\partial^2/\partial s^2 + \partial^2/\partial t^2$ which change sign under the reflections in the edges of R.

Because of the previous considerations the region S, the weight function $(\mu(z, \bar{z}))^\alpha$ and the method of orthogonalization described earlier are quite natural for the definition of a class of orthogonal polynomials. For reasons of convergence let $\alpha > -5/6$. Then $p^\alpha_{m,n}(z, \bar{z})$ is defined as a polynomial such that $p^\alpha_{m,n}(z, \bar{z}) - z^m\bar{z}^n$ has degree less than $m+n$ and

$$\iint_S p^\alpha_{m,n}(z, \bar{z})\, \overline{q(z, \bar{z})}\, (\mu(z, \bar{z}))^\alpha \, dx\, dy = 0$$

for each polynomial q of degree less than $m+n$. Some simple properties of the polynomials $p^\alpha_{m,n}$ are given in § 4. It is also pointed out in this section that the so-called disk polynomials provide a more elementary example of the method of orthogonalization used for the polynomials $p^\alpha_{m,n}$.

By the elementary interpretation of the functions $p^\alpha_{m,n}$ for $\alpha = \pm \frac{1}{2}$ one easily obtains in this case differential operators D^α_1 and D^α_2 of orders two and three, respectively, for which the functions $p^\alpha_{m,n}$ are eigenfunctions. For other values of α these operators can be generalized such that they are self-adjoint with respect to the weight function $(\mu(z, \bar{z}))^\alpha$. In § 5 and § 6 such operators D^α_1 and D^α_2, respectively, are constructed and it is proved that for all $\alpha > -5/6$ the functions $p^\alpha_{m,n}$ are eigenfunctions of D^α_1 and D^α_2. As a corollary it follows that

$$\iint_S p^\alpha_{m,n} \overline{p^\alpha_{k,l}} \mu^\alpha \, dx \, dy = 0 \text{ if } (m, n) \neq (k, l), \; m+n = k+l.$$

In [6, § 5] a partial differential operator D_2 of fourth order was obtained as the product D^+D^- of two second order operators D^- and D^+. Although the corresponding operator D^α_2 considered here has lower order, it cannot be factorized. For this reason its construction is more complicated.

For certain values of α the operator D^α_1 expressed in terms of s and t is the radial part of the Laplace-Beltrami operator on certain compact Riemannian symmetric spaces of rank two. Hence it is reasonable to expect that for such α the functions $p^\alpha_{m,n}$ are spherical functions and the operator D^α_2 is the radial part of some invariant differential operator on the corresponding symmetric space. However, this will not be proved here.

This paper concludes in § 7 with a discussion of the algebra of all partial differential operators for which the polynomials $p^\alpha_{m,n}$ are eigenfunctions. It is proved that each differential operator of this kind can be expressed in one and only one way as a polynomial in D^α_1 and D^α_2.

If all calculations in this paper would be done in a straightforward way then they would be quite long and tedious. In many cases it is indicated how a considerable gain in time and effort can be made by exploiting the symmetries in the formulas. It should be clear to the reader that due to the symmetry of the region this is a very charming class of orthogonal polynomials which can be studied in an elegant way.

2. Eigenfunctions of the Laplace operator which satisfy certain symmetry relations

Consider a regular tessellation of the Euclidean plane by equilateral triangles (cf. fig. 1). Let $\mathscr{G}$ be the group of isometries which is generated by the reflections in the edges of these triangles. Let R be a region bounded by one of these triangles, say, with vertices $O = (0, 0)$, $A_1 = (\pi, -\pi/\sqrt{3})$, $A_2 = (\pi, \pi/\sqrt{3})$. Let J_1, J_2 and J_3 denote the reflections in the edges OA_1, OA_2 and A_1A_2, respectively, of R. Observe that the isometries $J_2J_3J_2J_1$, $J_1J_3J_1J_2$ and $J_1J_2J_1J_3$ are the translations by the vectors $v_1 = (\pi, \pi\sqrt{3})$, $v_2 = (\pi, -\pi\sqrt{3})$ and $v_3 = (-2\pi, 0)$, respectively (cf. fig. 1). It follows easily that the reflections J_1, J_2 and J_3 generate the group $\mathscr{G}$. Alternatively,

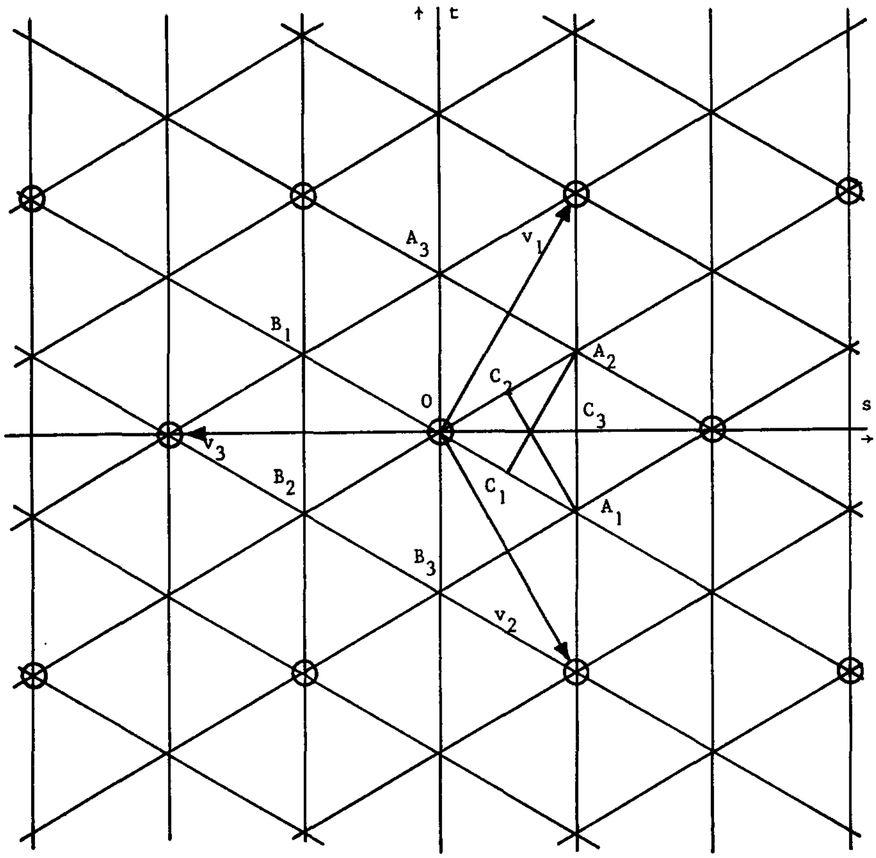

Fig. 1

the translations by v_1 and v_2 and the reflections J_1 and J_2 generate $\mathscr{G}$. Let H be the region bounded by the regular hexagon $A_1A_2A_3B_1B_2B_3$ (cf. fig. 1). The translations by v_1 and v_2 generate a translation group for which H is a fundamental region. The encircled points in fig. 1 are the midpoints of the hexagons obtained by translation of H. The reflections J_1 and J_2 generate a transformation group of the region H for which R is a fundamental region. Hence R is a fundamental region for the group $\mathscr{G}$.

In order to facilitate computations we transform the Euclidean coordinates s, t into new coordinates σ, τ defined by

$$\sigma = s + t/\sqrt{3}, \quad \tau = s - t/\sqrt{3}. \tag{2.1}$$

Note that in terms of these new coordinates $v_1 = (2\pi, 0)$ and $v_2 = (0, 2\pi)$. The reflections J_1, J_2, J_3 can be expressed by

$$\begin{cases} J_1(\sigma, \tau) = (-\sigma + \tau, \tau), \\ J_2(\sigma, \tau) = (\sigma, \sigma - \tau), \\ J_3(\sigma, \tau) = (2\pi - \tau, 2\pi - \sigma). \end{cases} \tag{2.2}$$

Let the mapping $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ be a bijection. For each function f on $\mathbf{R}^2$ let the function Tf be defined by $(Tf)(\sigma, \tau) = f(T^{-1}(\sigma, \tau))$. If X is a partial differential operator in σ and τ of order r and if the mapping T is of class C^r, then let the operator $dT(X)$ be defined such that $(dT(X))f = T(X(T^{-1}f))$ for each C^r – function f on $\mathbf{R}^2$.

In the following definition $\varrho(T)$ denotes the Jacobian determinant of an isometry T, so $\varrho(T) = \pm 1$.

DEFINITION 2.1. The function $f(\sigma, \tau)$ is called invariant (with respect to the group $\mathscr{G}$) if $Tf = f$ for each $T \in \mathscr{G}$. The function $f(\sigma, \tau)$ is called anti-invariant (with respect to $\mathscr{G}$) if $Tf = \varrho(T)f$ for each $T \in \mathscr{G}$. The partial differential operator X in σ and τ is called invariant or anti-invariant (with respect to $\mathscr{G}$) if $dT(X) = X$ or $dT(X) = \varrho(T)X$, respectively, for each $T \in \mathscr{G}$.

LEMMA 2.2. The function f is invariant if and only if f is 2π – periodic in σ and τ and $J_1 f = f = J_2 f$. The function f is anti-invariant if and only if f is 2π – periodic in σ and τ and $J_1 f = -f = J_2 f$.

This lemma is proved by using that the translations by v_1 and v_2 and the reflections J_1 and J_2 generate the group $\mathscr{G}$.

LEMMA 2.3. Let the operators $\mathscr{P}^+$ and $\mathscr{P}^-$ be defined by

$$(2.3)\quad \left\{\begin{aligned} (\mathscr{P}^\pm f)(\sigma, \tau) = \tfrac{1}{6}[f(\sigma, \tau) \pm f(\sigma, \sigma-\tau) + f(-\sigma+\tau, -\sigma) \pm f(-\tau, -\sigma) \\ + f(-\tau, \sigma-\tau) \pm f(-\sigma+\tau, \tau)]. \end{aligned}\right.$$

Then the operators $\mathscr{P}^+$ and $\mathscr{P}^-$ are projections from the class of 2π – periodic functions in σ and τ onto the class of invariant, respectively anti-invariant functions.

PROOF. By (2.2) and (2.3) we have

$$(2.4)\quad \mathscr{P}^\pm f = \tfrac{1}{6}[f \pm J_2 f + J_1 J_2 f \pm J_2 J_1 J_2 f + J_2 J_1 f \pm J_1 f].$$

The lemma follows from Lemma 2.2 by using that $J_1^2 = \text{id.} = J_2^2$ and $J_1 J_2 J_1 = J_2 J_1 J_2$. Q.e.d.

Let $\Delta = \partial^2/\partial s^2 + \partial^2/\partial t^2$ be the Laplace operator. Clearly this operator is invariant. In terms of σ and τ it is expressed by

$$(2.5)\quad \Delta = \tfrac{4}{3}\left(\frac{\partial^2}{\partial\sigma^2} + \frac{\partial^2}{\partial\tau^2} + \frac{\partial^2}{\partial\sigma\partial\tau}\right).$$

THEOREM 2.4. Let the functions $e^+_{m,n}$, $m, n \geqslant 0$, and $e^-_{m,n}$, $m, n \geqslant 1$, be defined by

$$(2.6)\quad \left\{\begin{aligned} & e^\pm_{m,n}(\sigma, \tau) = e^{i(m\sigma+n\tau)} \pm e^{i((m+n)\sigma-n\tau)} + e^{i(-(m+n)\sigma+m\tau)} \\ & \qquad \pm e^{i(-n\sigma-m\tau)} + e^{i(n\sigma-(m+n)\tau)} \pm e^{i(-m\sigma+(m+n)\tau)} && \text{if } m, n > 0, \\ & e^+_{m,0}(\sigma, \tau) = e^{im\sigma} + e^{-im\tau} + e^{i(-m\sigma+m\tau)} && \text{if } m > 0, \\ & e^+_{0,n}(\sigma, \tau) = e^{-in\sigma} + e^{in\tau} + e^{i(n\sigma-n\tau)} && \text{if } n > 0, \\ & e^+_{0,0}(\sigma, \tau) = 1. \end{aligned}\right.$$

Then the functions $e^+_{m,n}$ and $e^-_{m,n}$ are invariant, respectively anti-invariant. Both systems $\{e^+_{m,n}\}$ and $\{e^-_{m,n}\}$ are complete orthogonal systems of eigenfunctions of Δ on the region R and

$$\Delta e^\pm_{m,n} = -\tfrac{4}{3}(m^2+n^2+mn)\, e^\pm_{m,n}. \tag{2.7}$$

PROOF. Let for arbitrary integers m, n $f_{m,n}(\sigma, \tau) = e^{i(m\sigma+n\tau)}$. It follows by (2.3) and (2.6) that $\mathscr{P}^\pm f_{m,n} = 6e^\pm_{m,n}$ if $m, n > 0$, $\mathscr{P}^+ f_{m,0} = 3e^+_{m,0}$ if $m > 0$, $\mathscr{P}^+ f_{0,n} = 3e^+_{0,n}$ if $n > 0$, $\mathscr{P}^+ f_{0,0} = e^+_{0,0}$, $\mathscr{P}^- f_{m,n} = 0$ if $m = 0$ or $n = 0$. Thus the invariance of the functions $e^+_{m,n}$ and the anti-invariance of the functions $e^-_{m,n}$ follows from Lemma 2.3. Formula (2.7) is obtained from (2.4) by using that $\Delta f_{m,n} = -(4/3)(m^2+n^2+mn) f_{m,n}$ and that $dJ_1(\Delta) = \Delta = dJ_2(\Delta)$. Next we have to prove the orthogonality and the completeness of the systems $\{e^+_{m,n}\}$ and $\{e^-_{m,n}\}$. Observe that each function g on R has an invariant extension g^+ and an anti-invariant extension g^- to $\mathbf{R}^2$ and that these extensions are unique except on a set of measure zero. Let us denote by $\mathscr{H}$ the Hilbert space of 2π-periodic functions in σ and τ which are square integrable on the hexagonal region H. Then the mapping $g \to g^+$ identifies the Hilbert space $L^2(R)$ with the subspace $\mathscr{H}^+$ of $\mathscr{H}$ consisting of the invariant L^2-functions on H. Similarly, the mapping $g \to g^-$ identifies $L^2(R)$ with the subspace $\mathscr{H}^-$ of $\mathscr{H}$ consisting of the anti-invariant L^2-functions on H. Since for arbitrary $f, g \in \mathscr{H}$

$$\iint_H (\mathscr{P}^\pm f)\, \bar{g}\, d\sigma\, d\tau = 6 \iint_R (\mathscr{P}^\pm f)\, (\overline{\mathscr{P}^\pm g})\, d\sigma\, d\tau = \iint_H f (\overline{\mathscr{P}^\pm g})\, d\sigma\, d\tau$$

it follows that the projections $\mathscr{P}^+ \colon \mathscr{H} \to \mathscr{H}^+$ and $\mathscr{P}^- \colon \mathscr{H} \to \mathscr{H}^-$ are self-adjoint. Let for $m, n \geqslant 0$ $\mathscr{H}_{m,n}$ be the subspace of $\mathscr{H}$ spanned by the functions $f_{m,n}$, $J_2 f_{m,n}$, $J_2 J_1 f_{m,n}$, $J_2 J_1 J_2 f_{m,n}$, $J_1 J_2 f_{m,n}$, $J_1 f_{m,n}$, i.e., by all functions $T f_{m,n}$, $T \in \mathscr{G}$. It follows by (2.2) that $H_{m,n}$ is spanned by the functions $f_{m,n}$, $f_{m+n,-n}$, $f_{-m-n,m}$, $f_{-n,-m}$, $f_{n,-m-n}$, $f_{-m,m+n}$. Hence each function $f_{k,l}$, k, l integers, is contained in one and only one class $\mathscr{H}_{m,n}$, $m, n \geqslant 0$. The functions $f_{k,l}$, k, l integers, form an orthogonal basis for $\mathscr{H}$. So we have the orthogonal decompositions

$$\mathscr{H} = \sum_{m,n=0}^{\infty}{}^{\oplus}\, \mathscr{H}_{m,n}, \quad \mathscr{H}^+ = \sum_{m,n=0}^{\infty}{}^{\oplus}\, \mathscr{P}^+ \mathscr{H}_{m,n}, \quad \mathscr{H}^- = \sum_{m,n=0}^{\infty}{}^{\oplus}\, \mathscr{P}^- \mathscr{H}_{m,n}.$$

By (2.4) $\mathscr{P}^+ \mathscr{H}_{m,n}$ is spanned by $\mathscr{P}^+ f_{m,n} = \text{const.}\ e^+_{m,n}$ with non-zero constant, $\mathscr{P}^- \mathscr{H}_{m,n}$ is spanned by $\mathscr{P}^- f_{m,n} = 6e^-_{m,n}$ if $m, n \geqslant 1$ and $\mathscr{P}^- \mathscr{H}_{m,n} = \{0\}$ if $m = 0$ or $n = 0$. This proves the orthogonality and the completeness of the systems $\{e^+_{m,n}\}$ and $\{e^-_{m,n}\}$. Q.e.d.

A function $g(\sigma, \tau)$ which is a finite linear combination of the functions $e^{i(k\sigma+l\tau)}$, k, l integers, will be called a trigonometric polynomial in σ and τ. The functions $e^+_{m,n}$ and $e^-_{m,n}$, defined by (2.6), are clearly trigonometric polynomials in σ and τ. Note that both $e^+_{m,n}$ and $e^-_{m,n}$ are expressed by a sum $\sum c_{k,l} e^{i(k\sigma+l\tau)}$, such that $c_{m,n} = 1$ and $c_{k,l} = 0$ if $k, l \geqslant 0$ and $(k, l) \neq \neq (m, n)$. This property is applied in the following useful lemma.

LEMMA 2.5. Let $g(\sigma, \tau) = \sum c_{m,n} e^{i(m\sigma+n\tau)}$ be a trigonometric polynomial. Then $g = \sum_{m \geqslant 0} \sum_{n \geqslant 0} c_{m,n} e^{+}_{m,n}$ if g is invariant and $g = \sum_{m \geqslant 1} \sum_{n \geqslant 1} c_{m,n} e^{-}_{m,n}$ if g is anti-invariant.

PROOF. Let g be invariant. It follows from the proof of Theorem 2.4 that $g = \mathscr{P}^{+} g = \sum c_{m,n} \mathscr{P}^{+} f_{m,n} = \sum_{m \geqslant 0} \sum_{n \geqslant 0} b_{m,n} e^{+}_{m,n}$ for certain coefficients $b_{m,n}$. Hence $\sum c_{m,n} e^{i(m\sigma+n\tau)} = \sum_{m \geqslant 0} \sum_{n \geqslant 0} b_{m,n} e^{+}_{m,n}(\sigma, \tau)$. This implies that $c_{m,n} = b_{m,n}$ if $m, n \geqslant 0$. If g is anti-invariant then a similar proof can be given. Q.e.d.

Let the lines A_2C_1, A_1C_2 and OC_3 bisect the angles of the triangle OA_1A_2 (cf. fig. 1) and let I_1, I_2 and I_3 denote the reflections in the lines A_2C_1, A_1C_2 and OC_3, respectively. Then these reflections generate the group of isometries which map R onto itself. This group is isomorphic to the permutation group in three letters. It is also generated by the reflection I_3 and by the rotation $I_2I_1 = I_3I_2 = I_1I_3$. Observe that

$$(2.8) \quad \begin{cases} I_1(\sigma, \tau) = (\sigma - \tau + 2\pi/3,\ -\tau + 4\pi/3), \\ I_2(\sigma, \tau) = (-\sigma + 4\pi/3,\ -\sigma + \tau + 2\pi/3), \\ I_3(\sigma, \tau) = (\tau, \sigma). \end{cases}$$

It follows by inspection from (2.6) that

$$(2.9) \quad \begin{cases} (I_3 e^{\pm}_{m,n})(\sigma, \tau) = e^{\pm}_{m,n}(\tau, \sigma) = \pm e^{\pm}_{m,n}(-\sigma, -\tau) \\ \qquad = e^{\pm}_{n,m}(\sigma, \tau) = \pm \overline{e^{\pm}_{m,n}(\sigma, \tau)}, \end{cases}$$

$$(2.10) \quad \begin{cases} (I_1 I_2 e^{\pm}_{m,n})(\sigma, \tau) = e^{\pm}_{m,n}(-\sigma + \tau + 2\pi/3,\ -\sigma + 4\pi/3) \\ \qquad = e^{i(m-n)2\pi/3} e^{\pm}_{m,n}(\sigma, \tau). \end{cases}$$

Let us introduce the first order differential operators

$$(2.11) \quad \begin{cases} X_1 = \dfrac{1}{i}\left(-\tfrac{3}{2}\dfrac{\partial}{\partial s} + \tfrac{1}{2}\sqrt{3}\dfrac{\partial}{\partial t}\right) = \dfrac{1}{i}\left(-\dfrac{\partial}{\partial \sigma} - 2\dfrac{\partial}{\partial \tau}\right), \\ X_2 = \dfrac{1}{i}\left(\tfrac{3}{2}\dfrac{\partial}{\partial s} + \tfrac{1}{2}\sqrt{3}\dfrac{\partial}{\partial t}\right) = \dfrac{1}{i}\left(2\dfrac{\partial}{\partial \sigma} + \dfrac{\partial}{\partial \tau}\right), \\ X_3 = -\dfrac{1}{i}\sqrt{3}\dfrac{\partial}{\partial t} = \dfrac{1}{i}\left(-\dfrac{\partial}{\partial \sigma} + \dfrac{\partial}{\partial \tau}\right). \end{cases}$$

Observe that

$$(2.12) \quad X_1 + X_2 + X_3 = 0.$$

It follows from (2.2) and (2.8) that for each permutation (i, j, k) of $(1, 2, 3)$ we have

$$(2.13) \quad \begin{cases} dJ_i(X_i) = X_i,\ dJ_i(X_j) = X_k,\ dJ_i(X_k) = X_j, \\ dI_i(X_i) = -X_i,\ dI_i(X_j) = -X_k,\ dI_i(X_k) = -X_j. \end{cases}$$

THEOREM 2.6. *Let Q be a symmetric polynomial in three variables. Then*

$$Q(X_1, X_2, X_3)\, e^{\pm}_{m,n} = Q(-m-2n,\ 2m+n,\ -m+n)\, e^{\pm}_{m,n}. \tag{2.14}$$

PROOF. By the symmetry property of Q and by (2.13) the operator $Q(X_1, X_2, X_3)$ is invariant. Hence the function $Q(X_1, X_2, X_3)\, e^{+}_{m,n}$ is invariant and the function $Q(X_1, X_2, X_3)\, e^{-}_{m,n}$ is anti-invariant. By (2.11) we have

$$Q(X_1, X_2, X_3)\, e^{i(k\sigma+l\tau)} = Q(-k-2l,\ 2k+l,\ -k+l)\, e^{i(k\sigma+l\tau)}.$$

The theorem follows by using Lemma 2.5. Q.e.d.

Two particular cases of (2.14) are the differential equations

$$(X_1^2+X_2^2+X_3^2)\, e^{\pm}_{m,n} = 6(m^2+n^2+mn)\, e^{\pm}_{m,n}, \tag{2.15}$$

$$(X_1 X_2 X_3)\, e^{\pm}_{m,n} = (m-n)(2m+n)(m+2n)\, e^{\pm}_{m,n}. \tag{2.16}$$

Note that $\Delta = -(2/9)(X_1^2+X_2^2+X_3^2)$. It follows from the lemma below that (2.15) and (2.16) generate all differential equations of type (2.14).

LEMMA 2.7. *Let D be an invariant differential operator in σ and τ with constant coefficients. Then there exists a symmetric polynomial Q in three variables and a polynomial P in two variables such that $D = Q(X_1, X_2, X_3) = P(X_1^2+X_2^2+X_3^2,\ X_1X_2X_3)$. The polynomial P is uniquely determined by D.*

PROOF. Clearly there is a unique polynomial F in two variables such that $D = F(X_1, X_2)$. By (2.13) and by the invariance of D it follows that $D = (1/6) \sum F(X_i, X_j)$, where the summation runs over all permutations (i, j, k) of $(1, 2, 3)$. Hence there exists a symmetric polynomial Q such that $D = Q(X_1, X_2, X_3)$. According to VAN DER WAERDEN [9, § 33] the symmetric polynomial $Q(X_1, X_2, X_3)$ can be expressed as a polynomial in the three elementary symmetric polynomials $X_1+X_2+X_3$, $X_1X_2+X_2X_3+X_3X_1$ and $X_1X_2X_3$. But $X_1+X_2+X_3=0$, hence $X_1^2+X_2^2+X_3^2 = -2(X_1X_2+X_2X_3+X_3X_1)$, so there exists a polynomial P in two variables such that $D = Q(X_1, X_2, X_3) = P(X_1^2+X_2^2+X_3^2,\ X_1X_2X_3)$. In order to prove the uniqueness of P, suppose that $P(X_1^2+X_2^2+X_3^2,\ X_1X_2X_3) = 0$ and that the polynomial P is non-zero. Substituting $X_3 = -X_1 - X_2$ we obtain that $P(2(X_1^2+X_2^2+X_1X_2),\ -X_1^2X_2 - X_1X_2^2) = 0$. The polynomial $P(x, y)$ is a sum of terms $c_{k,l}(\frac{1}{2}x)^{\frac{1}{2}k-l}(-y)^l$, where $\frac{1}{2}k$ and l are integers and $\frac{1}{2}k \geqslant l \geqslant 0$. Among the pairs of integers (k, l) such that $c_{k,l} \neq 0$ there is a maximal element (m, n) with respect to lexicographic ordering. Then $c_{m,n}$ is the coefficient of $X_1^m X_2^n$ in the operator $P(2(X_1^2+X_2^2+X_1X_2),\ -X_1^2X_2 - X_1X_2^2)$ expressed as a polynomial in X_1 and X_2. Hence $c_{m,n} = 0$. This is a contradiction. Q.e.d.

3. A GENERALIZATION OF THE CHEBYSHEV POLYNOMIALS

The classes of functions $\cos ns$, $n=0, 1, 2, \ldots$, and $\sin ns$, $n=1, 2, \ldots$, are both complete orthogonal systems of eigenfunctions of the operator d^2/ds^2 on the interval $(0, \pi)$. These functions satisfy the symmetry relations $f(-s)=f(s)=f(2\pi-s)$ and $f(-s)=-f(s)=f(2\pi-s)$, respectively. Let the functions T_n and U_n be defined by the identities $T_n(\cos s)=\cos ns$ and $U_n(\cos s)=(\sin(n+1)s)/\sin s$. Then $T_n(x)$ and $U_n(x)$ are both polynomials of degree n, the so-called Chebyshev polynomials of the first and of the second kind, respectively. They satisfy the orthogonality relations

$$\int_{-1}^{1} T_m(x)\,T_n(x)(1-x^2)^{-\frac{1}{2}}\,dx=0,\quad m\neq n,$$

and

$$\int_{-1}^{1} U_m(x)\,U_n(x)(1-x^2)^{\frac{1}{2}}\,dx=0,\quad m\neq n.$$

The functions $e^+_{m,n}(\sigma, \tau)$ and $e^-_{m,n}(\sigma, \tau)$ can be considered as generalizations of the functions $\cos ns$ and $\sin ns$, respectively. It will be proved in this section that the functions $e^+_{m,n}(\sigma, \tau)$ and $e^-_{m+1,n+1}(\sigma, \tau)/e^-_{1,1}(\sigma, \tau)$ can be expressed as polynomials in $e^+_{1,0}(\sigma, \tau)$ and $e^+_{0,1}(\sigma, \tau)$ and that both classes of polynomials obtained in this way are orthogonal systems on a region bounded by Steiner's hypocycloid. These orthogonal systems are a natural generalization of the Chebyshev polynomials.

Let us write

$$(3.1)\qquad \begin{cases} z(\sigma, \tau)=e^+_{1,0}(\sigma, \tau)=e^{i\sigma}+e^{-i\tau}+e^{i(-\sigma+\tau)}, \\ \bar{z}(\sigma, \tau)=e^+_{0,1}(\sigma, \tau)=e^{-i\sigma}+e^{i\tau}+e^{i(\sigma-\tau)}. \end{cases}$$

Note that $\bar{z}(\sigma, \tau)$ is the complex conjugate of $z(\sigma, \tau)$.

LEMMA 3.1. Let Q be a polynomial in two variables such that $Q(z(\sigma, \tau), \bar{z}(\sigma, \tau))=0$ for all σ, τ. Then Q is the zero polynomial.

PROOF. Suppose that Q is non-zero and has degree N. Then we can write $Q(u, v)=\sum c_{k,l}u^k v^l$, where $c_{m,n}\neq 0$ for some pair (m, n), $m+n=N$. It follows from (3.1) that $c_{m,n}$ is the coefficient of $e^{i(m\sigma+n\tau)}$ in the trigonometric polynomial $Q(z(\sigma, \tau), \bar{z}(\sigma, \tau))$. Hence $c_{m,n}=0$. This is a contradiction. Q.e.d.

Using (2.6) and Lemma 2.5 we derive the recurrence relations

$$(3.2)\qquad \begin{cases} e^+_{m+1,n}=z\,e^+_{m,n}-A_m\,e^+_{m-1,n+1}-A_n\,e^+_{m,n-1} & \text{if } m>0 \text{ or } n>1, \\ e^+_{m,n+1}=\bar{z}\,e^+_{m,n}-A_n\,e^+_{m+1,n-1}-A_m\,e^+_{m-1,n} & \text{if } m>1 \text{ or } n>0, \\ e^+_{1,1}=z\,\bar{z}-3, & \end{cases}$$

where $A_n=1$ if $n\neq 1$, $A_1=2$, $e^+_{m,-1}=0=e^+_{-1,n}$.

From now on $\pi_n(z, \bar{z})$ will denote an arbitrary polynomial in z and $\bar{z}$ of degree $\leqslant n$.

THEOREM 3.2. For each pair (m, n) of nonnegative integers there is a unique polynomial in two variables, denoted by $p_{m,n}^{-\frac{1}{2}}$, such that

$$p_{m,n}^{-\frac{1}{2}}(z(\sigma, \tau), \bar{z}(\sigma, \tau)) = e_{m,n}^{+}(\sigma, \tau). \tag{3.3}$$

Then

$$p_{m,n}^{-\frac{1}{2}}(z, \bar{z}) = z^m \bar{z}^n + \pi_{m+n-1}(z, \bar{z}).$$

PROOF. The uniqueness part follows from Lemma 3.1. The existence part and the last statement of the theorem follow from (3.2) by using complete induction with respect to $m+n$. Q.e.d.

By (2.9) there is the symmetry relation

$$p_{m,n}^{-\frac{1}{2}}(z, \bar{z}) = p_{n,m}^{-\frac{1}{2}}(\bar{z}, z). \tag{3.4}$$

It can be derived from (3.2) that, for instance,

$$\begin{cases} p_{0,0}^{-\frac{1}{2}}(z, \bar{z}) = 1, \ p_{1,0}^{-\frac{1}{2}}(z, \bar{z}) = z, \\ p_{2,0}^{-\frac{1}{2}}(z, \bar{z}) = z^2 - 2\bar{z}, \ p_{1,1}^{-\frac{1}{2}}(z, \bar{z}) = z\bar{z} - 3, \\ p_{3,0}^{-\frac{1}{2}}(z, \bar{z}) = z^3 - 3z\bar{z} + 3, \ p_{2,1}^{-\frac{1}{2}}(z, \bar{z}) = z^2\bar{z} - 2\bar{z}^2 - z, \\ p_{4,0}^{-\frac{1}{2}}(z, \bar{z}) = z^4 - 4z^2\bar{z} + 2\bar{z}^2 + 4z, \\ p_{3,1}^{-\frac{1}{2}}(z, \bar{z}) = z^3\bar{z} - 3z\bar{z}^2 - z^2 + 5\bar{z}, \\ p_{2,2}^{-\frac{1}{2}}(z, \bar{z}) = z^2\bar{z}^2 - 2z^3 - 2\bar{z}^3 + 4z\bar{z} - 3. \end{cases} \tag{3.5}$$

In a similar way as (3.2) one can derive the recurrence relations

$$\begin{cases} e_{m+1,n}^{-} = z\, e_{m,n}^{-} - e_{m-1,n+1}^{-} - e_{m,n-1}^{-}, & m, n \geqslant 1, \\ e_{m,n+1}^{-} = \bar{z}\, e_{m,n}^{-} - e_{m+1,n-1}^{-} - e_{m-1,n}^{-}, & m, n \geqslant 1, \end{cases} \tag{3.6}$$

where $e_{m,n}^{-} = 0$ if $m = 0$ or $n = 0$.

The following theorem can be proved in a similar way as Theorem 3.2.

THEOREM 3.3. For each pair (m, n) of nonnegative integers there is a unique polynomial in two variables, denoted by $p_{m,n}^{\frac{1}{2}}$, such that

$$p_{m,n}^{\frac{1}{2}}(z(\sigma, \tau), \bar{z}(\sigma, \tau)) = \frac{e_{m+1,n+1}^{-}(\sigma, \tau)}{e_{1,1}^{-}(\sigma, \tau)}. \tag{3.7}$$

Then

$$p_{m,n}^{\frac{1}{2}}(z, \bar{z}) = z^m \bar{z}^n + \pi_{m+n-1}(z, \bar{z}).$$

Again we have by (2.9) a symmetry relation

$$p_{m,n}^{\frac{1}{2}}(z, \bar{z}) = p_{n,m}^{\frac{1}{2}}(\bar{z}, z). \tag{3.8}$$

It can be derived from (3.6) that, for instance,

$$\begin{cases} p_{0,0}^{\frac{1}{2}}(z, \bar{z}) = 1, \ p_{1,0}^{\frac{1}{2}}(z, \bar{z}) = z, \\ p_{2,0}^{\frac{1}{2}}(z, \bar{z}) = z^2 - \bar{z}, \ p_{1,1}^{\frac{1}{2}}(z, \bar{z}) = z\bar{z} - 1, \\ p_{3,0}^{\frac{1}{2}}(z, \bar{z}) = z^3 - 2z\bar{z} + 1, \ p_{2,1}^{\frac{1}{2}}(z, \bar{z}) = z^2\bar{z} - \bar{z}^2 - z. \end{cases} \tag{3.9}$$

LEMMA 3.4. Let the coordinate transformation $(s, t) \to (x, y)$ be defined by (2.1), (3.1) and

$$x = \tfrac{1}{2}(z+\bar{z}), \quad y = \tfrac{1}{2}i(-z+\bar{z}). \tag{3.10}$$

Then the Jacobian determinant of this transformation equals

$$\left\{\begin{aligned} \frac{\partial(x, y)}{\partial(s, t)} &= -(i/\sqrt{3})\, e^{-}_{1,1}(\sigma, \tau) \\ &= (8/\sqrt{3}) \sin s \sin(\tfrac{1}{2}s + \tfrac{1}{2}\sqrt{3}\, t) \sin(\tfrac{1}{2}s - \tfrac{1}{2}\sqrt{3}\, t), \end{aligned}\right. \tag{3.11}$$

and it is non-zero on the region R.

PROOF. We have

$$\frac{\partial(x, y)}{\partial(s, t)} = \frac{\partial(\sigma, \tau)}{\partial(s, t)} \frac{\partial(z, \bar{z})}{\partial(\sigma, \tau)} \frac{\partial(x, y)}{\partial(z, \bar{z})} = -\frac{i}{\sqrt{3}} \frac{\partial(z, \bar{z})}{\partial(\sigma, \tau)}.$$

The function $\partial(z, \bar{z})/\partial(\sigma, \tau)$ is anti-invariant and

$$\frac{\partial(z, \bar{z})}{\partial(\sigma, \tau)} = \frac{\partial z}{\partial \sigma}\frac{\partial \bar{z}}{\partial \tau} - \frac{\partial z}{\partial \tau}\frac{\partial \bar{z}}{\partial \sigma} = e^{-}_{1,1}(\sigma, \tau)$$

by (3.1) and Lemma 2.5. It follows from the explicit expression (2.6) of $e^{-}_{1,1}$ that

$$e^{-}_{1,1}(\sigma, \tau) = -8i \sin(\tfrac{1}{2}\sigma + \tfrac{1}{2}\tau) \sin(\sigma - \tfrac{1}{2}\tau) \sin(-\tfrac{1}{2}\sigma + \tau).$$

This proves (3.11). The zero lines of the function $e^{-}_{1,1}$ are just the edges of the triangles in the tessellation of fig. 1. Q.e.d.

By the previous lemma the mapping $(s, t) \to (x, y)$ is a diffeomorphism from R onto a certain region S in the (x, y)-plane and the boundary ∂R of R is mapped onto the boundary ∂S of S. In terms of the coordinates σ, τ the edges of the triangle ∂R have the parameter representations

$$\left\{\begin{aligned} OA_1 &= \{(\theta, 2\theta) \mid 0 \leqslant \theta \leqslant 2\pi/3\}, \\ A_1A_2 &= \{(\theta, 2\pi - \theta) \mid 2\pi/3 \leqslant \theta \leqslant 4\pi/3\}, \\ A_2O &= \{(4\pi - 2\theta, 2\pi - \theta) \mid 4\pi/3 \leqslant \theta \leqslant 2\pi\}, \end{aligned}\right.$$

cf. fig. 1. Hence, by (3.1), ∂S has a parameter representation

$$z = x + iy = 2e^{i\theta} + e^{-2i\theta}, \quad 0 \leqslant \theta < 2\pi, \tag{3.12}$$

where the images of O, A_1 and A_2 correspond with the values $\theta = 0$, $2\pi/3$, $4\pi/3$, respectively. It follows easily from (3.12) that if a circle of radius 1 rolls on the inside of a fixed circle of radius 3 then ∂S is the orbit of a point on the smaller circle. The resulting curve (cf. fig. 2) has three cusps and it is known as Steiner's hypocycloid, see for instance LORIA [7, §§ 73, 74] and HILTON [4, Chap. 17, §§ 2, 5]. Then S is the region inside this curve.

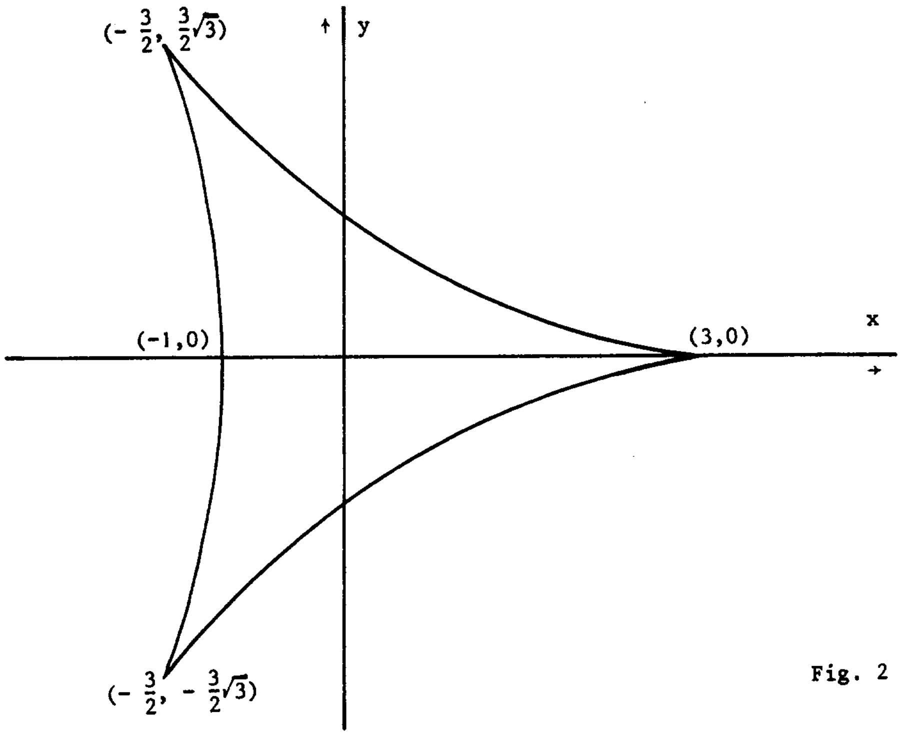

Fig. 2

By elimination of θ in (3.12) it can be shown that Steiner's hypocycloid is an algebraic curve of the fourth degree. This can also be proved in the following way. The function $(e^-_{1,1})^2$ is invariant. It follows from (2.6) and Lemma 2.5 that

$$(e^-_{1,1})^2 = e^+_{2,2} - 2e^+_{3,0} - 2e^+_{0,3} + 2e^+_{1,1} - 6e^+_{0,0}.$$

Hence, by (3,3), (3.5) and (3.4)

$$(e^-_{1,1}(\sigma, \tau))^2 = z^2\bar{z}^2 - 4z^3 - 4\bar{z}^3 + 18z\bar{z} - 27. \tag{3.13}$$

By putting $e^-_{1,1}(\sigma, \tau) = 0$ the equation for Steiner's hypocycloid takes the form

$$(x^2 + y^2 + 9)^2 + 8(-x^3 + 3x^2 y) - 108 = 0. \tag{3.14}$$

Instead of x, y we shall often use the coordinates $z = x + iy$, $\bar{z} = x - iy$ on the region S. Let

$$w(\sigma, \tau) = \sin(\tfrac{1}{2}\sigma + \tfrac{1}{2}\tau) \sin(\sigma - \tfrac{1}{2}\tau) \sin(-\tfrac{1}{2}\sigma + \tau), \tag{3.15}$$

$$\mu(z, \bar{z}) = -z^2\bar{z}^2 + 4z^3 + 4\bar{z}^3 - 18z\bar{z} + 27. \tag{3.16}$$

Then w is positive on R, μ is positive on S, and by (3.11) and (3.13) we have

$$\mu(z, \bar{z}) = -(e^-_{1,1}(\sigma, \tau))^2 = 64(w(\sigma, \tau))^2. \tag{3.17}$$

THEOREM 3.5. The polynomials $p_{m,n}^{-\frac{1}{2}}$ are orthogonal on S with respect to the weight function $\mu^{-\frac{1}{2}}$. The polynomials $p_{m,n}^{\frac{1}{2}}$ are orthogonal on S with respect to the weight function $\mu^{\frac{1}{2}}$.

PROOF. By (3.11) and (3.17) $(\mu(z, \bar{z}))^{-\frac{1}{2}} dx\, dy = \text{const.}\, d\sigma\, d\tau$. It follows that

$$\iint_R e_{m,n}^{+}(\sigma, \tau) \overline{e_{k,l}^{+}(\sigma, \tau)}\, d\sigma\, d\tau$$
$$= \text{const.} \iint_S p_{m,n}^{-\frac{1}{2}}(z, \bar{z}) \overline{p_{k,l}^{-\frac{1}{2}}(z, \bar{z})}\, (\mu(z, \bar{z}))^{-\frac{1}{2}}\, dx\, dy$$

and

$$\iint_R e_{m+1,n+1}(\sigma, \tau) \overline{e_{k+1,l+1}^{-}(\sigma, \tau)}\, d\sigma\, d\tau$$
$$= \text{const.} \iint_R p_{m,n}^{\frac{1}{2}}(z, \bar{z}) \overline{p_{k,l}^{\frac{1}{2}}(z, \bar{z})}\, (e_{1,1}^{-}(\sigma, \tau))^2\, d\sigma\, d\tau$$
$$= \text{const.} \iint_S p_{m,n}^{\frac{1}{2}}(z, \bar{z}) \overline{p_{k,l}^{\frac{1}{2}}(z, \bar{z})}\, (\mu(z, \bar{z}))^{\frac{1}{2}}\, dx\, dy.$$

The theorem is then proved by using Theorem 2.4. Q.e.d.

(To be continued)