Uniformly Bounded Representations. IV

Analytic Continuation of the Principal Series for Complex
Classical Groups of Types $B_n$, $C_n$, $D_n$

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INTRODUCTION

This paper is a continuation of our work begun in [1]–[3]. In this series of papers we are concerned with the problem of constructing uniformly bounded representations of semi-simple Lie groups and the significance of these representations for various questions in analysis.

The particular representations in question are obtained by analytic continuation from a certain normalization of the non-degenerate principal series of unitary representations. As seen in [2], which treats $SL(n, \mathbb{C})$, the problem of constructing the normalized principal series and its analytic continuation is intimately connected with the construction and properties of the intertwining operators between various equivalent members of the principal series. In this paper we utilize the general properties of intertwining operators on complex semi-simple Lie groups, which were obtained in [3], to carry out the analytic continuation for the complex classical groups $O(2n + 1, \mathbb{C})$, $Sp(n, \mathbb{C})$, $O(2n, \mathbb{C})$. In the sequel we shall refer to these groups as the groups of types $B_n$, $C_n$, and $D_n$.

In Section 1, we prove some general results on the complete irreducibility of representations of an arbitrary locally compact group on a Banach space which depend analytically on complex parameters.

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Roughly speaking we show that essentially all of the representations are completely irreducible or that none of them are.

In Section 2, we prove results, again valid in a general context, concerning the construction of uniformly bounded representations by analytic continuation. These results are applicable to any semi-simple Lie group and, e.g., to the problem of constructing complementary series associated with degenerate principal series. The specific theorems for the complex classical groups of types $B_n$, $C_n$, and $D_n$ are developed in Sections 3–7 except for Section 6 where we prove some general results concerning analytic continuation of functions of several complex variables.

The main problem we deal with can be reduced to the construction of an appropriate normalizing operator, $W(\lambda)$, which among other things satisfies the relation $W(\lambda) = W(p\lambda) A(p, \lambda)$, where the $A(p, \lambda)$ are the intertwining operators. Two comments are in order concerning this construction. First, although it is general in the sense that the overall procedure is the same for the three classes of groups, it depends at a crucial stage (Section 4) on a result that requires a separate proof for each of the three cases. Second, an analysis of this part of the proof makes it seem likely that a suitable modification of the construction could be carried out for any simple group other than $G_2$, and also for their corresponding real normal forms (as in the case of $SL(2, \mathbb{R})$ in [1]).

The results obtained here allow us to prove the following theorem. Suppose $G$ is a complex classical group (i.e., that $G$ is of type $A_n$, $B_n$, $C_n$, or $D_n$). Let $g \in L^p(G)$, $1 \leq p < 2$. Then the convolution operator $f \to f \ast g$ is bounded on $L^2(G)$.

Our proof, which will not be presented here, is similar to the argument we gave in [1] for $SL(2, \mathbb{R})$. It is based on the explicit Plancherel formula for $G$ and the properties of the analytic families of uniformly bounded representations constructed here. An analogous argument, which proves the result for $SL(n, \mathbb{C})$, has already been published by R. L. Lipsman [4].

1. Irreducibility of Analytic Families of Representations

In this section we prove some general results concerning the complete irreducibility of representations that depend analytically on certain complex parameters. To formulate these results we need to introduce the notion of a negligible set.

A real form of $\mathbb{C}^n$ is a real linear subspace $V$ of dimension $n$ over $\mathbb{R}$.
such that $V + iV = \mathbb{C}^n$. We say that a Borel subset $S$ of $\mathbb{C}^n$ is \textit{negligible} if for every $z$ in $\mathbb{C}^n$ and every real form $V$, the set $S \cap (z + V)$ is of measure 0 with respect to the measure on $z + V$ which is inherited from $n$-dimensional Lebesgue measure on $V$.

The following properties of negligible sets are basic for our purpose.

\textbf{Lemma 1.} Every Borel subset of a negligible set is negligible. A countable union of negligible sets is again negligible. If $f$ is a non-zero complex-analytic function defined on an open connected subset of $\mathbb{C}^n$, then

$$\{z : f(z) = 0\}$$

is negligible.

\textit{Proof.} The first two statements are evident consequences of the properties of sets of measure 0. To prove the third we argue as follows. Assume the contrary. Then there exists a non-zero analytic function $f$ on an open connected set and a pair $z', V$ such that

$$\{z : f(z) = 0\} \cap (z' + V)$$

has positive measure. By an affine transformation on $\mathbb{C}^n$, we may assume that $z' = 0$ and that $V = \mathbb{R}^n$. Then $f$ is a non-zero holomorphic function defined on an open connected subset $D$ of $\mathbb{C}^n$, and there is a Borel set $E$ in $\mathbb{R}^n$ of positive Lebesgue measure such that $E \subseteq D$ and $f(x_1, \ldots, x_n) = 0$ for all $(x_1, \ldots, x_n)$ in $E$. The connected components of $D \cap \mathbb{R}^n$ are open and at least one of them, say $C$, meets $E$ in a set of positive measure. A real-analytic function defined on an open connected subset of $\mathbb{R}^n$ that vanishes on a set of positive measure is identically 0. Thus $f(x_1, \ldots, x_n) = 0$ for all $(x_1, \ldots, x_n)$ in $C$. It follows easily from this that $f(z) = 0$ for all $z$ in $D$, which contradicts our original assumption.

We now consider representations. Let $G$ be a locally compact group, $\Omega$ an open connected subset of $\mathbb{C}^n$, and $\mathcal{H}$ a complex Banach space. We assume that for each $s$ in $\Omega$, there is given a continuous representation

$$x \rightarrow R(x, s) \quad (x \in G)$$

of $G$ on $\mathcal{H}$ such that for each $x$ in $G$, the map

$$s \rightarrow R(x, s) \quad (s \in \Omega)$$

is holomorphic in $\Omega$. 
Let $K$ be a compact subgroup of $G$ and $\mathcal{E}$ the set of equivalence classes of irreducible unitary representations of $K$. Let $\mathcal{D}$ be an element of $\mathcal{E}$ and $s$ a point in $\Omega$. If $m$ is a non-negative integer and $\mathcal{D}$ occurs with multiplicity $m$ in the restriction of $R(\cdot, s)$ to $K$, we set $N^s(\mathcal{D}) = m$. If $\mathcal{D}$ occurs with infinite multiplicity, we set $N^s(\mathcal{D}) = \infty$.

**Theorem 1.** The quantity $N^s(\mathcal{D})$ is independent of $s$.

**Proof.** Let $\chi_\mathcal{D}$ be the character of $\mathcal{D}$ and set

$$E_\mathcal{D}^s = d_\mathcal{D} \int_K R(k, s) \overline{\chi_\mathcal{D}(k)} \, dk$$

where $d_\mathcal{D}$ is the degree of $\mathcal{D}$. Then, as is well known, $E_\mathcal{D}^s$ is a projection of $\mathcal{H}$ on the subspace that transforms according to $\mathcal{D}$ (if $\mathcal{H}$ is a Hilbert space and $R(\cdot, s)$ is a unitary representation, then $E_\mathcal{D}^s$ is the orthogonal projection of $\mathcal{H}$ on $E_\mathcal{D}^s(\mathcal{H})$). When $E_\mathcal{D}^s(\mathcal{H})$ is of finite dimension, we also have

$$\dim(E_\mathcal{D}^s(\mathcal{H})) = d_\mathcal{D} N^s(\mathcal{D}).$$

This also holds more generally if we set $\dim(E_\mathcal{D}^s(\mathcal{H})) = \infty$ when $E_\mathcal{D}^s(\mathcal{H})$ is not finite dimensional. Thus it suffices to show that $\dim(E_\mathcal{D}^s(\mathcal{H}))$ is independent of $s$. Since $\Omega$ is connected, it is enough to show that $s \mapsto \dim(E_\mathcal{D}^s)$ is continuous in $\Omega$. Approximating $E_\mathcal{D}^s$ by appropriate Riemann sums and using the analyticity of the representations, we find that the function $s \mapsto E_\mathcal{D}^s$ is also analytic in $\Omega$ for each fixed $\mathcal{D}$. Since holomorphic operator valued functions are, in particular, continuous in the operator norm topology, each point $s$ of $\Omega$ has a neighborhood, say $U_s$ such that $\|E_\mathcal{D}^s - E_\mathcal{D}^t\| < 1$ for all $t$ in $U_s$. Thus, to complete the proof it suffices to prove the following result.

**Lemma 2.** If $E_1$ and $E_2$ are projections on $\mathcal{H}$ such that $\|E_1 - E_2\| < 1$, then $\dim E_1(\mathcal{H}) = \dim E_2(\mathcal{H})$.

**Proof.** Let $x \in E_2(\mathcal{H})$. Then

$$E_2x = x \quad \text{and} \quad \|E_1x - x\| = \|E_1x - E_2x\| < \|x\|.$$ 

If $x \neq 0$ it follows that $E_1x \neq 0$. Thus the restriction of $E_1$ to $E_2(\mathcal{H})$ is a one-one mapping $E_2(\mathcal{H})$ into $E_1(\mathcal{H})$. This implies $\dim E_1(\mathcal{H}) \geq \dim E_2(\mathcal{H})$. Similarly, $\dim E_2(\mathcal{H}) \geq \dim E_1(\mathcal{H})$, so that $\dim E_1(\mathcal{H}) = \dim E_2(\mathcal{H})$. 

We recall that the representation $R(\cdot, s)$ of $G$ is **completely irreducible** if the linear span of the $R(y, s)$ ($y \in G$) is dense in the set of all bounded operators on $\mathcal{H}$ in the strong operator topology. If a representation is completely irreducible, it is also topologically irreducible in the sense that $\mathcal{H}$ has no proper closed subspaces invariant under all operators of the representation. It is well known that for unitary representations complete irreducibility is the same as topological irreducibility.

**Theorem 2.** Suppose $\mathcal{H}$ is a separable Hilbert space, and assume that the common value $N(D)$ of the $N^*(D)$ is finite for each $D$ in $\mathcal{E}$. Then only one of the following possibilities holds.

(a) There are no points $s$ in $\Omega$ such that $R(\cdot, s)$ is completely irreducible.
(b) $R(\cdot, s)$ is completely irreducible for all $s$ in the complement of some negligible set.

The proof is based on the following simple result.

**Lemma 3.** Let $\mathcal{B}$ be an arbitrary complex Banach space and $f_1, \ldots, f_N$ $\mathcal{B}$-valued holomorphic functions defined on $\Omega$. Suppose that there is a point $t$ in $\Omega$ such that the vectors $f_1(t), \ldots, f_N(t)$ are linearly independent. Then the vectors $f_1(s), \ldots, f_N(s)$ are linearly independent for all $s$ in the complement of some negligible set.

**Proof.** Let $L_1, \ldots, L_N$ be the linear functionals on the subspace spanned by $f_1(t), \ldots, f_N(t)$ such that

$$L_k(f_i(t)) = \delta_{ik}.$$

Now extend the linear functionals $L_1, \ldots, L_N$ to bounded linear functionals on $\mathcal{B}$ by the Hahn–Banach theorem. Let $M(s)$ ($s \in \Omega$) be the $N \times N$ matrix with entries

$$M_{kj}(s) = L_k(f_j(s)).$$

Then, $\det M(t) = 1$, and hence the complex-analytic function $s \mapsto \det M(s)$ vanishes only on a negligible set. If $s$ does not belong to this negligible set, the vectors $f_1(s), \ldots, f_N(s)$ must be linearly independent, because any relation of linear dependence implies a corresponding relation of linear dependence on the columns of the matrix $M(s)$.

**Proof of the theorem.** Suppose there is some $t$ in $\Omega$ such that $R(\cdot, t)$ is completely irreducible. Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be any elements of $\mathcal{E}$ and consider
the linear space spanned by the operators $E^s_{\mathcal{D}_1} R(y, s) E^s_{\mathcal{D}_2}$ for a fixed $s$ in $\Omega$ as $y$ varies over $G$. Since $E^s_{\mathcal{D}_1}$ is a projection of $\mathcal{H}$ on the finite dimensional subspace $E^s_{\mathcal{D}_1}(\mathcal{H})$ (of dimension $d^s_{\mathcal{D}_1} N(\mathcal{D}_1)$), the space spanned by the operators is at most of dimension

$$N = d^s_{\mathcal{D}_1} d^s_{\mathcal{D}_2} N(\mathcal{D}_1) N(\mathcal{D}_2).$$

Since the closure, in the strong operator topology, of the linear span of the $R(y, t)$ is the space $\mathcal{D}$ of all bounded operators on $\mathcal{H}$, it is clear that the linear span of the $E^s_{\mathcal{D}_1} R(y, t) E^t_{\mathcal{D}_2} (y \in G)$ is isomorphic to the space of all linear maps from $E^s_{\mathcal{D}_1}(\mathcal{H})$ to $E^t_{\mathcal{D}_2}(\mathcal{H})$. We can therefore find $N$ elements $y_1, \ldots, y_N$ of $G$ such that the operators

$$E^s_{\mathcal{D}_1} R(y_1, s) E^s_{\mathcal{D}_2}, \ldots, E^s_{\mathcal{D}_1} R(y_N, s) E^s_{\mathcal{D}_2}$$

are linearly independent when $s = t$. Since the functions

$$s \rightarrow E^s_{\mathcal{D}_1} R(y_i, s) E^s_{\mathcal{D}_2} \quad (1 \leq i \leq N)$$

are analytic in $\Omega$, it follows from Lemma 3 that the above sequence of operators is independent for all $s$ in the complement of some negligible set. For any such $s$, the linear span of the $E^s_{\mathcal{D}_1} R(y, s) E^s_{\mathcal{D}_2}$, as $y$ varies over $G$, is precisely $E^s_{\mathcal{D}_1} \mathcal{D} E^s_{\mathcal{D}_2}$. Let $\mathcal{S}^*$ be the set of all $\mathcal{D}$ in $\mathcal{S}$ such that $N(\mathcal{D}) \neq 0$ and $S$ the union of the above negligible sets attached to each of the pairs $\mathcal{D}_1, \mathcal{D}_2$ where $\mathcal{D}_1$ and $\mathcal{D}_2$ range over $\mathcal{S}^*$. Since $\mathcal{H}$ is separable, $\mathcal{S}^*$ is countable, and therefore $S$ is negligible, by Lemma 1. Moreover, $E^s_{\mathcal{D}_1} \mathcal{D} E^s_{\mathcal{D}_2}$ is the linear span of the $E^s_{\mathcal{D}_1} R(y, s) E^s_{\mathcal{D}_2} (y \in G)$ for every pair $\mathcal{D}_1, \mathcal{D}_2$ in $\mathcal{S}^*$ and every $s$ in the complement of $S$.

We claim that $y \rightarrow R(y, s)$ is completely irreducible for every $s$ in the complement of $S$. The argument is as follows: Suppose $s$ lies in the complement of $S$. Then there is a bounded linear operator $A(s)$ on $\mathcal{H}$ with a bounded inverse such that the representation

$$T(y, s) = A(s) R(y, s) A(s)^{-1}$$

is unitary on $K$. It is easy to check that $R(\cdot, s)$ is completely irreducible if and only if $T(\cdot, s)$ is completely irreducible. Now for an arbitrary $\mathcal{D}$ in $\mathcal{S}$, let

$$P_\mathcal{D}^s = \frac{1}{\mathcal{D}} \int_K T(k, s) \tilde{\chi}(k) dk.$$
Then \( P_{\mathcal{D}}^s \) lies in the uniform closure of the linear space spanned by the operators \( T(k, s) \) \((k \in K)\). Moreover, \( P_{\mathcal{D}}^s = A(s) E_{\mathcal{D}}^s A(s)^{-1} \), and \( P_{\mathcal{D}_1}^s B P_{\mathcal{D}_2}^s \) is the linear span of the \( P_{\mathcal{D}_1}^s T(y, s) P_{\mathcal{D}_2}^s \) \((y \in G)\) for all \( \mathcal{D}_1, \mathcal{D}_2 \) in \( \mathcal{E}^* \). Therefore, \( P_{\mathcal{D}_1}^s B P_{\mathcal{D}_2}^s \) lies in the uniform closure of the linear space spanned by the operators \( T(y, s) \), as \( y \) varies over \( G \). Since \( T(\cdot, s) \) is unitary on \( K \) the family of projections

\[
\{ P_{\mathcal{D}}^s \} \quad (\mathcal{D} \in \mathcal{E}^*)
\]
defines an orthogonal direct sum decomposition of \( \mathcal{H} \). Let \( \mathcal{D}_1, \mathcal{D}_2, \ldots \) be an enumeration of \( \mathcal{E} \) and set

\[
P_n = P_{\mathcal{D}_1}^s + \cdots + P_{\mathcal{D}_n}^s
\]

for \( n = 1, 2, \ldots \). Then, clearly, \( P_n \to \text{Id} \) as \( n \to \infty \). If \( A \) is an arbitrary operator on \( \mathcal{H} \), it follows that \( P_n A P_n \to A \) in the strong operator topology. This proves the complete irreducibility.

In certain circumstances, Theorem 2 admits a converse; the complete irreducibility of just one of the representations \( R(\cdot, s) \) is enough to insure that the common value \( N(\mathcal{D}) \) of the \( N^s(\mathcal{D}) \) is finite for every \( \mathcal{D} \).

**Theorem 3.** Let \( G \) be a connected semi-simple Lie group with a faithful finite dimensional representation and \( K \) a maximal compact subgroup of \( G \). Suppose \( \mathcal{H} \) is a separable Hilbert space and that there is a point \( s_0 \) in \( \Omega \) such that \( R(\cdot, s_0) \) is completely irreducible. Then

\[
N(\mathcal{D}) \leq \text{degree}(\mathcal{D})
\]

for every \( \mathcal{D} \) in \( \mathcal{E} \), and \( R(\cdot, s) \) is completely irreducible for all \( s \) in the complement of some negligible set. If \( f \) is a bounded Baire function on \( G \) with compact support, the operator

\[
R(f, s) = \int_G f(x) R(x, s) \, dx \quad (s \in \Omega)
\]
is of Hilbert-Schmidt class, and the map

\[
s \to R(f, s)
\]
is analytic in \( \Omega \).

**Proof.** It follows from a result of Godement [5, Theorem 2] that

\[
N^s(\mathcal{D}) \leq \text{degree}(\mathcal{D})
\]
for every $\mathcal{D}$ in $\mathcal{E}$. By Theorem 1, $N^s(\mathcal{D}) = N^s(\mathcal{D})$ for all $s$ in $\Omega$. Hence, $N(\mathcal{D}) \leq \text{degree}(\mathcal{D})$, $N(\mathcal{D})$ being the common value of the $N^s(\mathcal{D})$. Thus, $R(\cdot, s)$ is completely irreducible for all $s$ in the complement of some negligible set, by Theorem 2.

Now let $f$ be a bounded Baire function with compact support $S$. Since each of the representations $R(\cdot, s)$ is strongly continuous, it follows from the uniform boundedness principle that the quantity

$$M(s) = \sup_{x \in S} \| R(x, s) \|$$

is finite for every $s$ in $\Omega$. At this point, the balance of the proof is quite similar to that of [2, Lemma 28], and we shall only sketch the argument. For each $s$ in $\Omega$, there is a bounded operator $B(s)$ with a bounded inverse such that the representation

$$P(x, s) = B(s) R(x, s) B(s)^{-1}$$

is unitary on $K$. Because $P(\cdot, s)$ is equivalent to $R(\cdot, s)$, $N(\mathcal{D})$ is also the number of times $\mathcal{D}$ occurs in the reduction of the restriction of $P(\cdot, s)$ to $K$. Setting

$$P(f, s) = \int_{G} f(x) P(x, s) \, dx$$

integrating over $K$, and denoting the Hilbert-Schmidt norm by $\| \cdot \|_2$ and the usual operator norm by $\| \cdot \|_\infty$, we then have

$$\| P(f, s) \|_2 \leq \sup_{x \in S} \| P(x, s) \|_\infty \int_{G} \left\| \int_{G} f(kx) P(k, s) \, dk \right\|_2 \, dx.$$ 

Since $N(\mathcal{D}) \leq \text{degree}(\mathcal{D})$, it follows from the Peter–Weyl theorem that

$$\left\| \int_{K} f(kx) P(k, s) \, dk \right\|_2 \leq \| f \|_\infty.$$ 

Therefore, $\| P(f, s) \|_2 \leq \text{meas}(KS)\| B(s) \|_\infty M(s) \| B(s)^{-1} \|_\infty \| f \|_\infty$. On the other hand, $R(f, s) \to B(s)^{-1} P(f, s) B(s)$, so that

$$\| R(f, s) \|_2 \leq \| B(s)^{-1} \| \| P(f, s) \|_2 \| B(s) \|_\infty.$$ 

Thus $\| R(f, s) \|_2 \leq A(S, s) \| f \|_\infty$ where $A(S, s)$ is finite and depends only on $s$ and the support $S$ of $f$. 

If $\varphi$ and $\psi$ are arbitrary vectors in $\mathcal{H}$, it follows by straightforward and familiar arguments that

$$
\psi \rightarrow \int_{G} f(x)(R(x, s)\varphi \mid \psi) \, dx
$$

is analytic in $\Omega$. This implies the required analyticity and completes the proof.

The proof of Lemma 29 in [2] applies without change to the present situation and yields the following result.

**Corollary.** Under the assumptions of the theorem, suppose that $f$ is the convolution of two bounded Baire functions with compact support. For each $s$ in $\Omega$, let

$$
R(f, s) = \int_{G} f(x) R(x, s) \, dx.
$$

Then $R(f, s)$ is of trace class, and if the family $\{R(x, \cdot) : x \in G\}$ is uniformly bounded in the operator norm on compact subsets of $\Omega$, then

$$
\psi \rightarrow \text{Tr} R(f, s)
$$

is analytic in $\Omega$.

2. **Analytic Continuation of Representations**

In this section we prove some general results concerning the construction of uniformly bounded representations by analytic continuation.

For this purpose it is convenient to make a definition. If $\Omega$ is an open connected subset of $\mathbb{C}^n$, any set of the form

$$
\Omega \cap (w + V)
$$

where $w \in \Omega$ and $V$ is a real form of $\mathbb{C}^n$, will be called a proper hyperplane section of $\Omega$.

Throughout this section it will be assumed that $G$ is a locally compact group with a countable base for open sets.

**Theorem 4.** Let $\Omega$ be an open connected subset of $\mathbb{C}^n$, $S$ a proper hyperplane section of $\Omega$, $m$ a positive integer, and $A$ a subset of $G$ such that $A^m = G$. Suppose $R$ is a map from $G \times S$ to the bounded linear operators on a complex Banach space $\mathcal{H}$ such that
(1) for each \( s \) in \( S \), \( y \mapsto R(y, s) \) is a continuous uniformly bounded representation of \( G \)

(2) for each \( a \) in \( A \), \( s \mapsto R(a, s) \) (\( s \in S \)) has an analytic continuation into \( \Omega \)

(3) the family of continuations

\[ \{R(a, \cdot) : a \in A\} \]

is uniformly bounded (in the operator norm) on compact subsets of \( \Omega \). Then \( R \) has a unique extension to \( G \times \Omega \) (the extension also being denoted by \( R \)) such that

(a) for each \( s \) in \( \Omega \), \( y \mapsto R(y, s) \) is a continuous uniformly bounded representation of \( G \)

(b) for each \( y \) in \( G \), \( s \mapsto R(y, s) \) is analytic in \( \Omega \).

**Proof.** By Lemma 1, there is at most one extension of \( R \) to \( G \times \Omega \) which satisfies (b).

To prove the existence, suppose \( y \in G \). Then there are elements \( a_i \) in \( A \) such that \( y = a_1 \cdots a_m \). It follows from (1) that

\[ R(y, s) = R(a_1, s) \cdots R(a_m, s) \quad (2.1) \]

for all \( s \) in \( S \). Since a product of analytic operator valued functions is again analytic, it follows from (2) that the right side of (2.1) is analytic in \( \Omega \). Thus

\[ s \rightarrow R(y, s), \quad s \in S \]

has an analytic continuation into \( \Omega \) for each \( y \) in \( G \).

Next let \( M(s) = \sup_{a \in A} \| R(a, s) \| \). Then \( M(s) \) is finite and by (2.1)

\[ \| R(y, s) \| \leq (M(s))^m \quad (2.2) \]

for all \( (y, s) \) in \( G \times \Omega \). If \( y_1 \) and \( y_2 \) are any elements of \( G \), then

\[ R(y_1 y_2, s) = R(y_1, s) R(y_2, s) \quad (2.3) \]

for all \( s \) in \( S \). Since both sides of this equation are analytic in \( \Omega \), it follows from Lemma 1 that (2.3) holds for all \( s \) in \( \Omega \).

Now suppose there is a point \( t \) in \( \Omega \) at which \( R(\cdot, t) \) fails to be continuous. Then there is a vector \( \varphi \), an \( \varepsilon > 0 \), and, since \( G \) is separable, a sequence \( \{y_n\} \) in \( G \) such that \( y_n \rightarrow e \) as \( n \rightarrow \infty \), and

\[ \| R(y_n, t) \varphi - \varphi \| \geq \varepsilon \]
for all \( n \). Let \( F_n(s) = R(y_n, s) \varphi - \varphi \). Then \( F_n \) is analytic on \( \Omega \), and since \( M(s) \) is bounded on compact sets, it follows from (2.2) that the sequence \( \{F_n\} \) is uniformly bounded on compact subsets of \( \Omega \). By the theory of normal families, there is a subsequence \( \{F_{n_k}\} \) which converges uniformly on compact subsets of \( \Omega \) to an analytic function \( F \). Since \( R(\cdot, s) \) is continuous when \( s \in \mathcal{S} \), \( F(s) = 0 \) for all \( s \) in \( \mathcal{S} \). Hence \( F(s) = 0 \) for all \( s \) in \( \Omega \), by Lemma 1. But clearly \( ||F(t)|| \geq \epsilon \), a contradiction. Therefore, \( R(\cdot, s) \) is continuous for every \( s \) in \( \mathcal{S} \).

To avoid excessive repetition of hypotheses, we shall formalize some of the foregoing by making a definition. If \( \mathcal{H} \) is a complex Banach space, by an analytic family of representations of \( G \) on \( \mathcal{H} \), we shall mean a map \( R \) from \( G \times \mathcal{S} \), \( \mathcal{S} \) being an open connected subset of \( \mathbb{C}^n \), to the bounded linear operators on \( \mathcal{H} \) such that

(i) for each \( s \) is \( \mathcal{S} \), \( y \rightarrow R(y, s) \) is a strongly continuous representation of \( G \) on \( \mathcal{H} \)

(ii) for each \( y \) in \( G \), \( s \rightarrow R(y, s) \) is (complex) analytic in \( \mathcal{S} \).

In most of our applications of Theorem 4, \( G \) will be a semi-simple Lie group and \( A \) a set of the form \( G_0P \) where \( G_0 \) is a parabolic subgroup and \( P \) is a finite set of representatives in \( G \) for certain elements of the Weyl group. Generally, we shall construct a uniformly bounded analytic family starting with representations \( R(\cdot, s) \) (\( s \in \mathcal{S} \)) which are unitary and independent of \( s \) on \( G_0 \). Then conditions (2) and (3) of the theorem are easily seen to be equivalent to the assertion that

\[ s \rightarrow R(p, s) \quad (s \in \mathcal{S}) \]

has an analytic continuation into \( \Omega \) for each \( p \) in \( P \). Generally, we shall also be concerned with the case in which \( \mathcal{S} = i\mathbb{R}^n \) and \( \Omega \) is a tube domain invariant under the map \( s \rightarrow -s \). A tube \( \Omega \) will be called symmetric if it contains \( i\mathbb{R}^n \) and is invariant under the map \( s \rightarrow -s \).

**Lemma 4.** Let \( \mathcal{H} \) be a complex Hilbert space and \( \Omega \) a symmetric tube. Suppose \( \{R(\cdot, s) : s \in \mathcal{S}\} \) is an analytic family of representations of \( G \) on \( \mathcal{H} \). If the representations \( R(\cdot, s) \) are all unitary when \( s \in i\mathbb{R}^n \), then

\[ R(y^{-1}, s)^* = R(y, -\bar{s}) \]

for all \((y, s)\) in \( G \times \mathcal{S} \).

**Proof.** To clarify the notation, if \( s \) is the point \((s_1, \ldots, s_n)\) in \( \mathbb{C}^n \), then

\[ -\bar{s} = (-\bar{s}_1, \ldots, -\bar{s}_n). \]
Suppose the representations \( R(\cdot, s) \) are unitary for \( s \) in \( i\mathbb{R}^n \). Then
\[
R(y^{-1}, -\bar{s})^* = R(y, s)
\]
for all \((y, s) \in G \times i\mathbb{R}^n\). By assumption, \( s \mapsto R(y, s) \) is analytic in \( \Omega \) for every \( y \) in \( G \); since \( \Omega \) is symmetric, this implies the fact that \( s \mapsto R(y^{-1}, -\bar{s})^* \) is also analytic in \( \Omega \). Therefore, \( R(y^{-1}, -\bar{s})^* = R(y, s) \) for all \( s \) in \( \Omega \), by Lemma 1.

Lemma 4 states that under the given conditions, \( R(\cdot, -s) \) is necessarily the contragredient of the representation \( R(\cdot, s) \), and this has implications concerning the conditions under which \( R(\cdot, s) \) or a quotient of \( R(\cdot, s) \) is weakly similar to a unitary representation.

If \( M \) and \( N \) are continuous representations of \( G \) on complex Banach spaces \( \mathcal{H} \) and \( \mathcal{K} \), we say that \( N \) is weakly similar to \( M \) if there is a bounded, \( 1-1 \), linear map \( A: \mathcal{H} \to \mathcal{K} \) with dense range such that
\[
AM(y) = N(y)A \tag{2.4}
\]
for all \( y \) in \( G \); when \( A \) maps \( \mathcal{H} \) onto \( \mathcal{K} \), i.e., when \( A \) has a bounded everywhere defined inverse, we say that \( N \) is (strongly) similar to \( M \). Strong similarity is evidently an equivalence relation while weak similarity is in general not transitive or even reflexive.

**Lemma 5.** Let \( R \) be a bounded linear operator with a bounded inverse on a complex Hilbert space \( \mathcal{H} \). Suppose \( A \) is a bounded nonnegative operator on \( \mathcal{H} \) such that
\[
AR = R'A
\]
where \( R' = (R^*)^{-1} \). Let \( B \) be the non-negative square root of \( A \). Then there is a unique unitary operator \( U \) on \( A(\mathcal{H}) \) such that
\[
BR = UB.
\]
Conversely, if \( U \) is unitary on \( \mathcal{H} \) and if \( B \) is a bounded linear map of \( \mathcal{H} \) into \( \mathcal{H} \) such that \( BR = UB \), then
\[
(B^*B)R = R'(B^*B).
\]

**Proof.** Let \( \varphi \) be any vector in \( \mathcal{H} \). Then
\[
(BR\varphi | BR\varphi) = (AR\varphi | R\varphi) = (R'A\varphi | R\varphi) = (A\varphi | \varphi) = (B\varphi | B\varphi).
\]
Thus $BR\varphi = 0$ if and only if $B\varphi = 0$. Hence, there is a single-valued (necessarily unique) linear operator $U$ on $B(\mathcal{H})$ such that

$$U(B\varphi) = BR\varphi$$

for all $\varphi$ in $\mathcal{H}$. But given this, we also have

$$(U(B\varphi) | U(B\varphi)) = (B\varphi | B\varphi)$$

for all $\varphi$; hence $U$ is isometric. Since $R$ is invertible, every vector in $B(\mathcal{H})$ is of the form $BR\varphi$ with $\varphi$ in $\mathcal{H}$. Thus $U$ maps $B(\mathcal{H})$ onto $B(\mathcal{H})$. Therefore, $U$ extends uniquely to a unitary map, again denoted $U$, of $\overline{B(\mathcal{H})}$ onto $\overline{B(\mathcal{H})}$. It follows from the spectral theorem that $A$ and $B$ have the same null space. Because $A$ and $B$ are self-adjoint, this implies $A(\mathcal{H}) = B(\mathcal{H})$.

Conversely, suppose that $U$ is a unitary operator on a Hilbert space $\mathcal{H}$ and that $B$ is a bounded linear map of $\mathcal{H}$ into $\mathcal{H}$ such that $BR = UB$. Then $R^*B^* = B^*U^{-1}$, which implies $B^*U = R^*B^*$. Thus

$$B^*BR = B^*UB - R^*B^*B.$$

The next result relates in an obvious fashion to the problem of constructing complementary series.

**Theorem 5.** Let $\mathcal{H}$ be a complex Hilbert space, $\Omega$ a symmetric tube in $\mathbb{C}^n$, and $\{R(\cdot, s) : s \in \Omega\}$ an analytic family of representations of $G$ on $\mathcal{H}$ which are unitary for $s$ in $i\mathbb{R}^n$. If $s$ is a given point in $\Omega$, then $R(\cdot, s)$ or some quotient of $R(\cdot, s)$ is weakly similar to a unitary representation if and only if there is a bounded non-negative operator $A$ on $\mathcal{H}$ such that

$$AR(y, s) = R(y, -s)A \quad (2.5)$$

for all $y$ in $G$.

**Proof.** Suppose (2.5) holds with $A \geq 0$, and let $B$ denote the non-negative square root of $A$. By Lemma 4, $R(y, -s) = R(y^{-1}, s)^*$ for all $y$ in $G$. Hence, for each $y$ in $G$ there is a unique unitary transformation $U(y, s)$ on $\overline{A(\mathcal{H})}$ such that

$$BR(y, s) = U(y, s)B \quad (2.6)$$

by Lemma 5. Since

$$BR(y_1y_2, s) = BR(y_1, s)R(y_2, s)$$

$$= U(y_1, s)BR(y_2, s)$$

$$= U(y_1, s)U(y_2, s)B$$
it follows that $U(y_1y_2, s) = U(y_1, s) U(y_2, s)$. Thus $y \to U(y, s)$ is a unitary representation of $G$ on $A(\mathcal{H})$ which is strongly continuous by (2.6). From (2.6) it is also clear that $U(\cdot, s)$ is weakly similar to the quotient of $R(\cdot, s)$ defined by the null space of $B$.

Conversely, suppose $U(\cdot, s)$ is a unitary representation of $G$ on a Hilbert space $\mathcal{H}$ which is weakly similar to the quotient of $R(\cdot, s)$ defined by some closed $R(\cdot, s)$ invariant subspace $\mathcal{N}$ of $\mathcal{H}$, via a map $C: \mathcal{H} / \mathcal{N} \to \mathcal{H}$. Let $B$ denote the canonical projection $\mathcal{H} \to \mathcal{H} / \mathcal{N}$ followed by $C$. Then

$$BR(y, s) = U(y, s)B$$

and appealing to Lemma 5, we see that

$$(B*B) R(y, s) = R(y, -s)(B*B).$$

3. Partially Normalized Series

Let $G$ be a complex semisimple Lie group, $G = KAN$ an Iwasawa decomposition of $G$, $M$ the centralizer of $A$ in $K$, and $B = MAN$. For any continuous irreducible finite dimensional unitary representation $\lambda$ of $B$, let $T(\cdot, \lambda)$ denote the unitary representation of $G$ which is induced by $\lambda$, as defined in Section 4 of [3]. Let $M'$ denote the normalizer of $A$ in $K$ and set $\mathfrak{B} = M' / M$. The elements $p$ in $\mathfrak{B}$ act in a natural fashion on the representations $\lambda$ of $B$. For each pair $p, \lambda$, let $A(p, \lambda)$ denote the unitary operator constructed in Sections 6, 7, and 8 of [3] and having the property

$$A(p, \lambda) T(y, \lambda) = T(y, \lambda A(p, \lambda)$$

(3.1)

for all $y$ in $G$.

If for each $\lambda$, $W(\lambda)$ is a unitary operator on the Hilbert space on which $T(\cdot, \lambda)$ acts, we may then define a new family $\{R(\cdot, \lambda)\}$ of unitary representations by setting

$$R(y, \lambda) = W(\lambda) T(y, \lambda) W(\lambda)^{-1}$$

(3.2)

for $y$ in $G$. Our first problem is to construct the operators $W(\lambda)$ so that the unitary equivalence (3.1) of $T(\cdot, \lambda)$ and $T(\cdot, \lambda A(p, \lambda)$ is replaced, in so far as possible, by the identity

$$R(\cdot, \lambda) = R(\cdot, \lambda A(p, \lambda).$$

(3.3)
Ultimately, we require the representations \( R(\cdot, \lambda) \) to have certain analytic properties in addition to the formal property (3.3), and these additional requirements naturally affect our construction of a map \( \lambda \to W(\lambda) \). But for the moment we ignore analytic considerations and focus our attention on the algebraic implications of (3.2) and (3.3).

Observe that (3.3) holds if and only if

\[
W(\rho \lambda)^{-1} W(\lambda) T(y, \lambda) = T(y, \rho \lambda) W(\rho \lambda)^{-1} W(\lambda).
\]

For this it is sufficient that

\[
W(\lambda) = W(\rho \lambda) A(\rho, \lambda)
\] (3.4)

for all \( \rho \) and \( \lambda \). By [3, Theorem 5]

\[
A(\rho \lambda, \lambda) A(\rho, \lambda) = A(\rho, \rho \lambda) A(\rho \lambda, \lambda)
\] (3.5)

for all \( \rho, \lambda \). From (3.5) it is easy to see that (3.4) is valid for all \( \rho \) if it holds for any set of elements generating \( \mathfrak{B} \). More generally, if (3.4) is satisfied by all the elements of a given subset of \( \mathfrak{B} \), it will then hold on the subgroup generated by that set.

Our construction is based on the following observation. The completely straight-forward proof is omitted.

**Lemma 6.** Let \( S \) be an arbitrary subset of \( \mathfrak{B} \) and \( \rho \) an element of \( \mathfrak{B} \) which lies outside the subgroup generated by \( S \). Suppose that for each \( \lambda \) we are given a unitary operator \( Y(\lambda) \) such that

\[
Y(\rho \lambda) = Y(\rho \lambda) A(\rho, \lambda)
\]

for all \( \lambda \) in \( S \). In addition, suppose that \( X(\lambda) \) is a unitary operator such that \( X(q \lambda) = X(\lambda) \) for all \( q \) in \( S \) and

\[
X(\rho \lambda)^{-1} X(\lambda) = Y(\rho \lambda) A(\rho, \lambda) Y(\lambda)^{-1}
\]

Then the operators \( Z(\lambda) = X(\lambda) Y(\lambda) \) have the property that

\[
Z(\rho \lambda) = Z(q \lambda) A(q, \lambda)
\]

for all \( \lambda \) in the set \( \{\rho, S\} \).

At this point one could hope to construct an appropriate map \( \lambda \to W(\lambda) \) inductively making use of Lemma 6. We have only attempted to do this
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for the classical groups. The construction for $SL(n, \mathbb{C})$ is given in [2], and in this paper we treat the other classical groups. There are four separate constructions: one for $SL(n, \mathbb{C})$, another for $Sp(n, \mathbb{C})$, and for the complex orthogonal groups $O(n, \mathbb{C})$ it seems necessary to distinguish between odd and even $n$. Roughly speaking, the differences in the constructions result from or correspond to the differences in the Dynkin diagrams. The extent to which the constructions are similar is mirrored by the extent to which the diagrams are similar. Since ultimately the constructions involve only the angles between the various simple roots, it is reasonable to conjecture that suitable, but technically complicated, modifications of the methods used here would work for any complex semi-simple Lie group with no simple component of type $G_2$.

Before we restrict our attention to special classes of groups, further discussion of the general situation is appropriate.

Let $\mathfrak{g}$ be the Lie algebra of $G$, $\mathfrak{a}$ the Lie algebra of $A$ and $\beta \mapsto H_\beta$ the identification of $\mathfrak{a}^*$ with $\mathfrak{a}$ defined by the Killing form. Give $\mathfrak{a}^*$ the inner product in which $\langle \alpha, \beta \rangle = \alpha(H_\beta)$. A non-zero root $\alpha$ of $\mathfrak{g}$ with respect to $\mathfrak{a}$ is called basic (or simple) if $\exp(\mathfrak{g}_\alpha) \subseteq N$ and $\alpha$ is not the sum of two such roots. The system $(\alpha_1, \ldots, \alpha_r)$ of basic roots is a base for $\mathfrak{a}^*$ over $\mathbb{Z}$, and $\mathfrak{h}$ may be identified with the group of orthogonal transformations on $\mathfrak{a}^*$ which is generated by the basic reflections

$$p(\beta) = \beta - \frac{2}{\langle \alpha_j, \alpha_j \rangle} \alpha_j, \quad \beta \in \mathfrak{a}^*. \quad (3.6)$$

The elements of $\mathfrak{a}$ extend uniquely to complex linear functionals on the Cartan subalgebra $\mathfrak{h} = \mathfrak{a} + i\mathfrak{a}$. Then $\mathfrak{h}^* = \mathfrak{a}^* + i\mathfrak{a}^*$ and the inner product on $\mathfrak{a}^*$ extends by linearity to a nondegenerate bilinear form on $\mathfrak{h}^*$. Let $M^\times$ denote the set of all $\eta$ in $\mathfrak{a}^*$ such that $\eta(H)/2\pi$ is an integer whenever $H \in \mathfrak{a}$ and $\exp(iH)$ is the identity in $G$. Then $\mathfrak{h}^* \times M^\times$ is a group under addition and as such may be identified, as in [3, Section 6] with the group of all continuous homomorphisms of $B$ into the multiplicative group of complex numbers. Under this identification $i\mathfrak{a}^* \times M^\times$ corresponds to the subgroup $A$ of all unitary characters of $B$. The irreducible finite dimensional unitary representations $\lambda$ of $B$ may be identified with characters in $\mathfrak{a}$. The elements $p$ in $\mathfrak{M}$ regarded as transformations on $\mathfrak{a}^*$ extend by linearity to complex orthogonal transformations on $\mathfrak{h}^*$ in such a way that when $\lambda = (\xi, \eta)$ ($\xi \in \mathfrak{h}^*$, $\eta \in M^\times$) then $p\lambda = (p\xi, p\eta)$. Next let $\alpha_j^0 = 2\alpha_j/\langle \alpha_j, \alpha_j \rangle \ (0 \leq j \leq n - 1)$ and $\mathbb{Z}$ denote the group of integers. Then for each $\eta$ in $M^\times$, $\langle \eta, \alpha_j^0 \rangle \in \mathbb{Z}$
Let $U$ be the unitary group of the Hilbert space $H$ on which all of the $T(\cdot, \lambda)$ ($\lambda \in \Lambda$) act. Then by [3, Lemma 27 and Theorem 3] there exists a strongly continuous homomorphism

$$A_j: i\mathbb{R} \times \mathbb{Z} \to U$$

($i\mathbb{R} \times \mathbb{Z}$ being regarded as an additive group) such that for the basic reflection $p_j$ ($0 \leq j \leq n - 1$)

$$A(p_j, \lambda) = A(p_j, \xi, \eta) = A_j(\langle \xi, \alpha_j^0 \rangle, \langle \eta, \alpha_j^0 \rangle).$$

(3.7)

Now suppose that $G$ is a complex classical group of type $B_n$, $C_n$, or $D_n$ with $n \geq 2$, and let $\epsilon_1, \ldots, \epsilon_n$ be the standard orthonormal basis for $\mathbb{R}^n$. Then there is a particularly convenient identification of $\mathfrak{h}^*$ with $\mathbb{C}^n$ and $M^*$ with $\mathbb{Z}^n$ in which the form $\langle \cdot, \cdot \rangle$ is the bilinear extension of the standard inner product on $\mathbb{R}^n$. Moreover, the basic roots $\alpha_0, \ldots, \alpha_{n-1}$ may be indexed in such a way that

$$\alpha_j = \epsilon_{j+1} - \epsilon_j, \quad 1 \leq j \leq n - 1$$

(3.8)

and $\alpha_0$ is $\epsilon_1 + 2\epsilon_2$, or $\epsilon_1 + \epsilon_2$ according as $G$ is of type $B_n$, $C_n$, or $D_n$, [6].

If $\xi = (s_1, \ldots, s_n)$ it follows that

$$\langle \xi, \alpha_j^0 \rangle = s_{j+1} - s_j$$

(3.9)

for $1 \leq j \leq n - 1$. Similarly, if $\eta = (m_1, \ldots, m_n)$ with $m_k$ in $\mathbb{Z}$, then

$$\langle \eta, \alpha_j^0 \rangle - m_{j+1} - m_j, \quad 1 \leq j \leq n - 1.$$ (3.10)

Thus when $\lambda = (\xi, \eta)$ with $\xi = (s_1, \ldots, s_n)$ and $\eta = (m_1, \ldots, m_{n-1})$ ($m_k \in i\mathbb{R}^n, m_k \in \mathbb{Z}^n$) it follows from (3.7), (3.9), and (3.10) that

$$A(p_j, \lambda) = A(p_j, \xi, \eta) = A_j(s_{j+1} - s_j, m_{j+1} - m_j)$$

(3.11)

for $1 \leq j \leq n - 1$. From (3.6), (3.8), and (3.9) we find that $p_j(\xi) = \xi - (s_{j+1} - s_j)(\epsilon_{j+1} - \epsilon_j)$, and from this it follows that

$$p_j(\xi) = \sum_{i < j} s_i s_j + s_{j+1}s_j + s_j s_{j+1} + \sum_{k > j} s_k s_k$$

(3.12)

for $1 \leq j \leq n - 1$. Thus $p_j$ ($1 \leq j \leq n - 1$) is the permutation which interchanges the pair $s_j, s_{j+1}$ and fixes the other coordinates.
Now we are in position to consider the problem of constructing a map \( Y: \Lambda \to \mathcal{U} \) such that
\[
Y(\lambda) = Y(\rho_j \lambda) A(\rho_j, \lambda), \quad 1 \leq j \leq n - 1. \tag{3.13}
\]
Since the roots \( \alpha_1, \ldots, \alpha_{n-1} \) form a system of type \( A_{n-1} \), this is essentially the problem that arises in the case of \( SL(n, \mathbb{C}) \) which is solved in [2]. Although it was not apparent then, the normalizing map \( W: \Lambda \to \mathcal{U} \) for \( SL(n, \mathbb{C}) \) could have been obtained by repeated applications of Lemma 6.

To construct the operators \( Y(\lambda) \) we begin with the simpler problem of finding a map \( Y_{n-1}: \Lambda \to \mathcal{U} \) such that
\[
Y_{n-1}(\rho_{n-1}) = Y_{n-1}(\rho_{n-1} \lambda) A(\rho_{n-1} \lambda, \lambda).
\]
Now this is easily done because \( A_{n-1} \) is a homomorphism of \( i\mathbb{R} \times \mathbb{Z} \) into \( \mathcal{U} \). If \( \lambda = (\xi, \eta) \) with \( \xi = (s_1, \ldots, s_n) \) and \( \eta = (m_1, \ldots, m_n) \), as above, let \( t_j = (s_j, m_j) \) \( (1 \leq j \leq n - 1) \) and set
\[
Y_{n-1}(\lambda) = A_{n-1}(-t_{n-1}). \tag{3.14}
\]
Then by (3.11) and (3.12)
\[
Y_{n-1}(\rho_{n-1} \lambda) A(\rho_{n-1} \lambda, \lambda) = A_{n-1}(-t_n) A_{n-1}(t_n - t_{n-1}) = Y_{n-1}(\lambda).
\]
Next, we try to find \( X_{n-2}: \Lambda \to \mathcal{U} \) such that
\[
\begin{align*}
(a) \quad X_{n-2}(\rho_{n-1} \lambda) &= X_{n-2}(\lambda), \quad \text{and} \\
(b) \quad X_{n-2}(\rho_{n-2} \lambda)^{-1} X_{n-2}(\lambda) &= Y_{n-1}(\rho_{n-2} \lambda) A(\rho_{n-2} \lambda, \lambda) Y_{n-1}(\lambda)^{-1}.
\end{align*}
\]
By (3.11), (3.12), and (3.14)
\[
Y_{n-1}(\rho_{n-2} \lambda) A(\rho_{n-2} \lambda, \lambda) Y_{n-1}(\lambda)^{-1} = A_{n-1}(-t_{n-2}) A_{n-2}(t_{n-1} - t_{n-2}) A_{n-1}(t_{n-1}) = A_{n-1}(-t_{n-2}) A_{n-2}(-t_{n-2}) A_{n-2}(t_{n-1}) A_{n-1}(t_{n-1}).
\]
Setting
\[
X_{n-2}(\lambda) = A_{n-1}(-t_{n-2}) A_{n-2}(-t_{n-2}) \tag{3.15}
\]
we then have (a) and the equation
\[
Y_{n-1}(\rho_{n-2} \lambda) A(\rho_{n-2} \lambda, \lambda) Y_{n-1}(\lambda)^{-1} = X_{n-2}(\lambda) X_{n-2}(\rho_{n-2} \lambda)^{-1}.
\]
If it were true that
\[ X_{n-2}(\lambda) X_{n-2}(\rho_{n-2})^{-1} = X_{n-2}(\rho_{n-2})^{-1} X_{n-2}(\lambda) \]
we would also have (b), and Lemma 6 would tell us that the operators
\[ Y_{n-2}(\lambda) = X_{n-2}(\lambda) Y_{n-1}(\lambda), \quad \lambda \in A \quad (3.16) \]
have the property that
\[ Y_{n-2}(\lambda) = Y_{n-2}(\rho \lambda) A(\rho \lambda, \lambda) \]
for \( n - 2 \leq j \leq n - 1 \). In the next section, we shall prove that for fixed \( k, 1 \leq k \leq n - 1 \), the operators
\[ X_k(\lambda) = A_{n-1}(-t_k) \cdots A_k(-t_k) \quad (3.17) \]
form a commutative family in \( \lambda = (t_1, \ldots, t_n), t_j \) in \( \mathbb{R} \times \mathbb{Z} \). Assuming this for now, we set
\[ Y_k(\lambda) = X_k(\lambda) \cdots X_{n-1}(\lambda), \quad 1 \leq k \leq n - 1. \quad (3.18) \]

**Lemma 7.** For fixed \( k \) with \( 1 \leq k \leq n - 1 \)
\[ Y_k(\lambda) = Y_k(\rho \lambda) A(\rho \lambda, \lambda) \]
for all \( j \) such that \( k \leq j \leq n - 1 \).

**Proof.** We already know this for \( k = n - 1 \). With our assumption that
\[ X_j(\lambda_1) X_j(\lambda_2) = X_j(\lambda_2) X_j(\lambda_1), \quad 1 \leq j \leq n - 1 \quad (3.19) \]
for all \( \lambda_1 \) and \( \lambda_2 \), the argument above proves the lemma for the case \( k = n - 2 \). Suppose \( n \geq 4 \) and that \( k \leq n - 3 \), then
\[ Y_k(\lambda) = X_k(\lambda) Y_{k+1}(\lambda) \]
and we may assume by induction that
\[ Y_{k+1}(\lambda) = Y_{k+1}(\rho \lambda) A(\rho \lambda, \lambda) \]
for \( k + 1 \leq j \leq n - 1 \). It follows from (3.12) and (3.17) that
\[ X_k(\rho \lambda) = X_k(\lambda), \quad k + 1 \leq j \leq n - 1. \quad (3.20) \]
Moreover, if $\lambda = (t_1, ..., t_n)$, then

$$Y_{k+1}(p_\lambda) A(p_\lambda, \lambda) Y_{k+1}(\lambda)^{-1} = X_{k+1}(p_\lambda) Y_{k+2}(p_\lambda) A_k(-t_k) A_k(t_{k+1}) Y_{k+2}(\lambda)^{-1} X_{k+1}(\lambda)^{-1}. $$

Since $p_k$ commutes with $p_j$ for $k + 1 \leq j \leq n - 1$, it follows from [3, Theorem 4], that $A_k(t)$ commutes with $A_j(t')$ whenever $k + 1 \leq j \leq n - 1$. Therefore, $A_k(t)$ commutes with $X_j(\lambda)$ for $k + 1 \leq j \leq n - 1$; hence $A_k(t)$ commutes with

$$Y_{k+2}(p_\lambda) = Y_{k+2}(\lambda)$$

and with $Y_{k+2}(\lambda)^{-1}$ as well. Thus

$$Y_{k+1}(p_\lambda) A(p_\lambda, \lambda) Y_{k+1}(\lambda)^{-1} = X_{k+1}(p_\lambda) A_k(-t_k) A_k(t_{k+1}) X_{k+1}(\lambda)^{-1}. $$

Now $X_{k+1}(p_\lambda) A_k(-t_k) = X_k(\lambda)$, and $A_k(t_{k+1}) X_{k+1}(\lambda)^{-1} = X_k(p_\lambda)^{-1}$. These relations and (3.19) imply that

$$X_k(p_\lambda)^{-1} X_k(\lambda) = Y_{k+1}(p_\lambda) A(p_\lambda, \lambda) Y_{k+1}(\lambda)^{-1}. \quad (3.21)$$

In view of (3.20), (3.21), and our inductive assumption, we may now apply Lemma 6 to complete the proof.

At this point we define the partially normalized principal series for $G$ be setting

$$R_k(y, \lambda) = Y_k(\lambda) T(y, \lambda) Y_k(\lambda)^{-1}, \quad y \in G \quad (3.22)$$

for $1 \leq k \leq n - 1$. We also introduce some additional notation. If $\lambda = (t_1, ..., t_n)$ with $t_j$ in $i\mathbb{R} \times \mathbb{Z}$ and $1 \leq k \leq n - 1$, let

$$t_j^{(k)} = \begin{cases} t_j & \text{if } j < k \\ 0 & \text{if } k \leq j \leq n - 1 \\ \sum_{i \geq k} t_i & \text{if } j = n \end{cases}$$

and set

$$\lambda^{(k)} = (t_1^{(k)}, ..., t_n^{(k)}). \quad (3.23)$$

As in [3] let $V$ denote the analytic subgroup of $G$ which corresponds to the subalgebra

$$v = \sum_{\alpha < 0} g_\alpha.$$
of \( g \), and let \( H \) be the closed minimal parabolic subgroup \( MAV \). If \( n \geq 3 \) and \( 1 \leq k \leq n - 2 \), let \( G^{(k)} \) be the parabolic subgroup of \( G \) which is generated by \( H \) and the basic reflections \( p_j \) with \( k \leq j \leq n - 2 \).

**Theorem 6.** Let \( G \) be a complex classical group of type \( B_n, C_n, \) or \( D_n \) with \( n \geq 2 \). Suppose \( 1 \leq k \leq n - 1 \) and let \( \mathfrak{W}^{(k)} \) be the subgroup of \( \mathfrak{W} \) which is generated by the basic reflections \( p_k, \ldots, p_{n-1} \). Then

\[
R_k(\cdot, \rho \lambda) = R_k(\cdot, \lambda)
\]

for all \( \rho \) in \( \mathfrak{W}^{(k)} \). If \( n \geq 3 \) and \( 1 \leq k \leq n - 2 \), then

\[
R_k(y, \lambda) = R_k(y, \lambda^{(k)})
\]

for all \( y \) in \( G^{(k)} \).

In the proof we shall use some lemmas.

**Lemma 8.** Let \( 1 \leq k \leq n - 1 \), \( q_k = p_{n-1} \cdots p_k \), and \( \lambda = (t_1, \ldots, t_n) \). If

\[
\lambda_k = (0, \ldots, 0, t_k, 0, \ldots, 0)
\]

with \( t_k \) in the \( k \)th place, then

\[
X_k(\lambda) = A(q_k, \lambda_k).
\]

**Proof.** Use (3.5), (3.11), and (3.12) to express \( A(q_k, \lambda_k) \) as a product of basic intertwining operators, and compare the result with (3.17).

Next we note that from the definition [3] of the representations \( T(\cdot, \lambda) \), it follows readily that when \( \lambda = (t_1, \ldots, t_n) \) then \( T(p_k, \lambda) \) is a function of \( t_{k+1} - t_k \) only; hence we may write

\[
T(p_n, \lambda) = T(p_k, t_{k+1} - t_k), \quad 1 \leq k \leq n - 1.
\]

If \( t \) is any given element of \( i\mathbb{R} \times \mathbb{Z} \), we may always choose \( \lambda \) so that \( t = t_{k+1} - t_k \). Thus it makes sense to consider the basic operators \( T(p_k, t) \) for arbitrary \( t \) in \( i\mathbb{R} \times \mathbb{Z} \).

The next result is essentially Lemma 16 of [2].

**Lemma 9.** Let \( 1 \leq j, k \leq n - 1 \). Then

\[
A_j(t) T(p_j, t) = T(p_j, -t) A_j(t)
\]

\[
A_j(t) T(p_k, t') = T(p_k, t') A_j(t) \quad \text{if } |k - j| > 1
\]

\[
A_j(t) T(p_k, t') = T(p_k, t + t') A_j(t) \quad \text{if } |k - j| = 1.
\]
Proof. By (3.1) and (3.12)

\[ A_j(t_{j+1} - t_j) T(p_j, t_{j+1} - t_j) = T(p_j, t_j - t_{j+1}) A_j(t_{j+1} - t_j) \]

and replacing \( t_{j+1} - t_j \) by \( t \), we obtain (3.29)

If \( |k - j| > 1 \) then \( T(p_k, \lambda) = T(p_k, p_j \lambda) \)

and this together with (3.1) implies (3.30).

In proving (3.31), we suppose that \( k = j + 1 \), the argument being similar in the case \( j = k + 1 \). If \( t \) and \( t' \) are given elements of \( iR \times Z \), we can always choose \( \lambda = (t_1, \ldots, t_n) \) so that \( t = t_k - t_j \) and \( t' = t_{k+1} - t_k \). Then

\[ T(p_k, p_j \lambda) = T(p_k, t_{k+1} - t_j) \]

and since \( t_{k+1} - t_j = t + t' \), we obtain (3.31) from (3.1).

Proof of Theorem 6. Suppose 1 \( \leq k \leq j \leq n - 1 \). Then by Lemma 7

\[ Y_k(p_j \lambda)^{-1} Y_k(\lambda) = A(p_j, \lambda) \]

so by (3.1)

\[ Y_k(p_j \lambda)^{-1} Y_k(\lambda) T(y, \lambda) = T(y, p_j \lambda) Y_k(p_j \lambda)^{-1} Y_k(\lambda). \]

Therefore

\[ Y_k(\lambda) T(y, \lambda) Y_k(\lambda)^{-1} = Y_k(p_j \lambda) T(y, p_j \lambda) Y_k(p_j \lambda)^{-1} \]

for all \( y \) in \( G \); by (3.22) this is the statement that (3.24) holds for the element \( p_j (k \leq j \leq n - 1) \). It follows immediately that (3.24) is valid for all \( p \) in \( \Phi^{(k)} \).

Suppose that \( n \geq 3 \) and that \( 1 \leq k \leq n - 2 \). Then since

\[ R_k(y_1 y_2, \lambda) = R_k(y_1, \lambda) R_k(y_2, \lambda) \]

for all \( y_1, y_2 \) in \( G \), the validity of (3.25) for \( y_1, y_2 \) in \( G^{(k)} \) implies its validity for \( y_1 y_2 \). Because \( H = CV \) where \( C = MA \), it therefore suffices to prove (3.25) for the cases \( y \in V, y \in C \), and \( y = p_j, k \leq j \leq n - 2 \).

For \( y \) in \( V \), \( T(y, \lambda) \) is independent of \( \lambda \). In fact, \( y \rightarrow T(y, \lambda) \) \((y \in V)\) is just the right regular representation of \( V \), [3]. From this and (3.1) it follows that the operators \( A_j(t) \) commute with the operators \( T(y, \lambda) \) for
y in $V$. Hence the same is true for the operators $X_j(\lambda)$ and the operator $Y_k(\lambda)$. Therefore $R_k(y, \lambda) = T(y, \lambda) = T(y, \lambda(\epsilon^k)) - R_k(y, \lambda(\epsilon^k))$ for all $y$ in $V$.

Now suppose $y = c \in C$ and let $\epsilon$ denote the identity character, i.e., $\epsilon = (0, \ldots, 0)$. Then it also follows from the definition of $T(\cdot, \lambda)$ that

$$T(c, \lambda) = \lambda(c) T(c, \epsilon). \quad (3.32)$$

From this and (3.1) we find that

$$A(p, \lambda) T(c, \epsilon) = \frac{(p\lambda)(\epsilon)}{\lambda(\epsilon)} T(c, \epsilon) A(p, \lambda).$$

Multiplying this equation by $\lambda'(c)$, $\lambda'$ being an arbitrary unitary character of $B$, we obtain the relation

$$A(p, \lambda) T(c, \lambda') = \frac{(p\lambda)(\epsilon)}{\lambda(\epsilon)} T(c, \lambda') A(p, \lambda). \quad (3.33)$$

Suppose $\lambda = (t_1, \ldots, t_n)$ and define $\lambda_j$ ($k \leq j \leq n - 1$) and $q_j$ as in Lemma 8. In (3.33) replace $p$ by $q_j$, $\lambda$ by $\lambda_j$, and $\lambda'$ by $\lambda$. Then this special case of (3.33) may be written in the form

$$X_j(\lambda) T(c, \lambda) \rightarrow \frac{(q\lambda_j)(\epsilon)}{\lambda_j(\epsilon)} T(c, \lambda) X_j(\lambda) \quad (3.34)$$

by Lemma 8. Since $q_j$ is the permutation of $\lambda$ that fixes $t_1, \ldots, t_{j-1}$ and transforms $t_j, \ldots, t_n$ into $t_{j+1}, \ldots, t_n$, $t_j$, it follows in particular that

$$c \rightarrow (q_j\lambda_j)(\epsilon)/\lambda_j(\epsilon)$$

is the character defined by the $n$-tuple

$$(0, \ldots, 0, -t_j, 0, \ldots, 0, t_j)$$

with $-t_j$ in the $j$th place. From this and (3.34), the definition (3.18) of $Y_k(\lambda)$, and (3.32), we find that

$$Y_k(\lambda) T(c, \lambda) Y_k(\lambda)^{-1} = \lambda^{(\epsilon^k)}(c) T(c, \epsilon).$$

Thus $R_k(c, \lambda) = T(c, \lambda^{(\epsilon^k)})$ for all pairs $c, \lambda$. Because $(\lambda^{(\epsilon^k)})^{(\epsilon^k)} = \lambda^{(\epsilon^k)}$, it follows that $R_k(c, \lambda^{(\epsilon^k)}) = T(c, \lambda^{(\epsilon^k)})$ as well, and therefore (3.25) holds on $C$. 

Finally, consider a basic reflection $p_j$ with $k \leq j \leq n - 2$, and suppose $\lambda = (t_1, \ldots, t_n)$. By (3.28), $T(p_j, \lambda) = T(p_j, t_{j+1} - t_j)$. Now from this, (3.30), and (3.17) it follows that

$$X_i(\lambda) T(p_j, \lambda) = T(p_j, \lambda) X_i(\lambda)$$

for $i > j + 1$. In addition, by (3.31) and (3.30), we see that

$$X_{i+1}(\lambda) T(p_j, t_{j+1} - t_j) = T(p_j, -t_j) X_{i+1}(\lambda).$$

Next, using all the statements of Lemma 9, we find that

$$X_j(\lambda) T(p_j, -t_j) = T(p_j, \epsilon) X_j(\lambda).$$

Another application of Lemma 9 shows that

$$X_i(\lambda) T(p_j, \epsilon) = T(p_j, \epsilon) X_i(\lambda)$$

for $k \leq i < j$. Combining the relations given in (3.35) through (3.38), we see that

$$Y_k(\lambda) T(p_j, \lambda) = T(p_j, \epsilon) Y_k(\lambda).$$

Hence, $R_k(p_j, \lambda) = T(p_j, \epsilon)$ for all $\lambda$, and this evidently implies

$$R_k(p_j, \lambda) = R_k(p_j, \lambda^k)$$

so that (3.25) is true for the basic reflections $p_j$ with $k \leq j \leq n - 2$. By our remarks at the beginning of the proof of (3.25), this completes the proof of the theorem.

4. **Lemmas on Commuting Families**

The next logical step in our program of successive normalizations of the principal series in the construction of a map $X_0: \Lambda \to U$ such that

$$X_0(p_j, \lambda) = X_0(\lambda)$$

for $1 \leq j \leq n - 1$ and

$$X_0(p_0, \lambda)^{-1} X_0(\lambda) = Y_1(p_0, \lambda) A(p_0, \lambda) Y_1(\lambda)^{-1}.$$
Unfortunately, there seems to be no way of constructing such an $X_0$ as a product of the basic intertwining operators $A_j(t)$ as in (3.17). Of course (3.17) was suggested by the form of the various products

$$Y_{n-1}(p_{n-2}, \lambda) A(p_{n-2}, \lambda) Y_{n-1}(\lambda)^{-1}, \ldots, Y_{k+1}(p_k, \lambda) A(p_k, \lambda) Y_{k+1}(\lambda)^{-1}.$$ 

In the present case, the operators

$$Y_1(p_0, \lambda) A(p_0, \lambda) Y_1(\lambda)^{-1}$$

have a rather different form when written as a product of the basic intertwining operators. Suppose, for example, that $G = \text{Sp}(n, \mathbb{C})$ and define $C: i\mathbb{R} \times \mathbb{Z} \to \mathbb{U}$ by the equation

$$C(t) = A_{n-1}(t) \cdots A_1(t) A_0(t) A_1(t) \cdots A_{n-1}(t).$$

Then it turns out that for $\lambda = (t_1, \ldots, t_n)$ with $t_j$ in $i\mathbb{R} \times \mathbb{Z}$

$$Y_1(p_0, \lambda) A(p_0, \lambda) Y_1(\lambda)^{-1} = C(t).$$

Nevertheless, our problem has a solution which is based on the special algebraic and analytic properties of the map $C$. The situation is similar but slightly different when $G$ is of type $B_n$ or $D_n$. However, in each case the appropriate map $C$ has the properties described in the following definition.

Let $\mathcal{H}$ be a complex Hilbert space and $\mathcal{U}((\mathcal{H})$ the unitary group on $\mathcal{H}$. Then a map $F: i\mathbb{R} \times \mathbb{Z} \to \mathcal{U}(\mathcal{H})$ is admissible if

(i) $F(t) F(t') = F(t') F(t)$ for all $t, t'$ in $i\mathbb{R} \times \mathbb{Z}$;

(ii) for each triple $(ic, m_1, m_2)$ in $i\mathbb{R} \times \mathbb{Z} \times \mathbb{Z}$, the operator-valued function $s \to F(s + ic, m_1) F(-s, m_2)$, $\text{Re}(s) = 0$ has an analytic continuation into the strip $-1 < \text{Re}(s) < 1$;

(iii) there is a positive integer $N$ and a constant $K_\epsilon$ ($0 < \epsilon < 1$) such that

$$\|F(a + ib + ic, m_1) F(-a - ib, m_2)\| \leq K_\epsilon \left| 1 + |b| + |c| + |m_1| + |m_2| \right|^N$$

for all $a + ib$ with $-1 + \epsilon \leq a \leq 1 - \epsilon$ and $-\infty < b < \infty$.

Thus $F$ is admissible if $\{F(s, m): s \in i\mathbb{R}, m \in \mathbb{Z}\}$ is a commutative family of unitary operators with a relative analytic continuation, in the sense of (ii), which is of polynomial growth in the strip $-1 < \text{Re}(s) < 1$, as defined in (iii).
The next result is the key to our construction of the desired operators \( X_\phi(\lambda) \).

**Lemma 10.** If \( F: iR \times Z \to U(\mathcal{H}) \) is admissible and \( F(-t) = F(t)^{-1} \) for all \( t \) in \( iR \times Z \), then there is an admissible map \( D: iR \times Z \to U(\mathcal{H}) \) such that \( D(t) \) commutes with every operator that commutes with all \( F(t) \) and

\[
D(-t)^{-1} D(t) = F(t) \quad (4.1)
\]

for all \( t \) in \( iR \times Z \).

**Proof.** Suppose \( F \) is admissible and that \( F(-t) = F(t)^{-1} \) (\( t \in iR \times Z \)). If \( F \) were a homomorphism, we could simply set \( D(s, m) = F(s/2, m/2) \) for the case in which \( m \) is even. But in general \( F \) is not a homomorphism and another approach is necessary. If \( F \) had an analytic continuation into the strip, and if it were possible to define a branch of \( s \to \log F(s, m) \) in the strip with the property that

\[
\log F(-s, -m) = -\log F(s, m)
\]

we could set \( D(s, m) = \exp[\frac{1}{2} \log F(s, m)] \). Although in general \( F \) does not have such an analytic continuation (and indeed it does not in our main application) we can proceed by suitably combining these two ideas.

First we consider the case in which \( m \) is an even integer, and for this case we set

\[
D(s, m) = E(s, m) F(s/2, m/2), \quad \text{Re}(s) = 0
\]

where \( E(s, m) \) is to be determined. Then denoting \( (s, m) \) by \( t \), we have

\[
D(-t)^{-1} D(t) = F(t/2) E(-t)^{-1} E(t) F(t/2).
\]

Assuming the \( E \)'s commute with the \( F \)'s, we find that \( E \) should satisfy the equation

\[
E(-t)^{-1} E(t) = F(-t/2)^2 F(t).
\]

Now let \( t = \frac{1}{2}(s, m) \) (\( s \in iR \)) and set

\[
K(s_1, s_2, m) = F(t_1) F(-t_2) F(-t_1 + t_2).
\]  

(4.2)

Then, by our assumptions, for fixed \( s \) and \( m \), \( s_2 \to K(s_1, s_2, m) \) has an analytic continuation into the strip \(-2 < \text{Re}(s_2) < 2 \). Similarly, for
fixed $s_2$ and $m$, $s_1 \rightarrow K(s_1, s_2, m)$ has an analytic continuation into the strip $-2 < \text{Re}(s_1) < 2$. Because these functions satisfy growth conditions similar to (iii), it follows from a straightforward variant of [2, Lemma 21] that $K$ has a joint analytic continuation into the tube

$$|\text{Re}(s_1)| + |\text{Re}(s_2)| < 2$$

which is of polynomial growth in $s_1$ and $s_2$.

Next we define $J$ by setting

$$J(s, m) = K(-s, s, m)$$

for $-1 < \text{Re}(s) < 1$ and $m$ an even integer. By (4.2)

$$K(-s, s, m) = F\left(\frac{-s}{2}, \frac{-m}{2}\right)^2 F(s, m)$$

when $\text{Re}(s) = 0$. Thus $J(\cdot, m)$ is an analytic continuation of

$$s \rightarrow F\left(\frac{-s}{2}, \frac{-m}{2}\right)^2 F(s, m) \quad (\text{Re}(s) = 0)$$

into the strip $-1 < \text{Re}(s) < 1$. Since

$$J(s_1, m_1) J(s_2, m_2) = J(s_2, m_2) J(s_1, m_1)$$

and

$$J(s, m) J(-s, -m) = J(-s, -m) J(s, m) = I$$

whenever $\text{Re}(s_1) = \text{Re}(s_2) = \text{Re}(s) = 0$, these relations also hold by analyticity throughout the strip. If $J'(\cdot, m)$ denotes the (complex) derivative of $J(\cdot, m)$, it follows from (4.5) that

$$J'(s, m) J(-s, -m) = J'(-s, -m) J(s, m).$$

On the other hand, from the definition of $J'(s, m)$ as a limit and (4.4), it follows that the values of $J'$ commute with the values of $J$. Therefore, we have the relation

$$J'(s, m) J(-s, -m) = J(s, m) J'(-s, -m).$$

Now let

$$L(s, m) = \int_0^1 s J'(ts, m) J(-ts, -m) dt$$
for $-1 < \text{Re}(s) < 1$. Then

$$L(-s, -m) = - \int_0^1 s J'(-ts, -m) J(ts, m) \, dt$$

$$= - \int_0^1 s J'(ts, m) J(-ts, -m) \, dt$$

by (4.6); hence

$$L(-s, -m) = -L(s, m). \quad (4.7)$$

Since $\frac{\partial}{\partial s}[J(s, m) \exp(-L(s, m))] = 0$ and $L(o, m) = 0$, it follows that

$$J(s, m) = J(o, m) \exp(L(s, m)). \quad (4.8)$$

Let $E_m = J(o, m)$ when $m \geq 0$, $E_m = I$ when $m < 0$, and define $E$ for $-1 < \text{Re}(s) < 1$ by the equation

$$E(s, m) = E_m \exp(\frac{1}{2}L(s, m)). \quad (4.9)$$

Then it is easy to see from (4.7) and (4.8) that

$$E(-s, -m)^{-1} E(s, m) = J(s, m) \quad (4.10)$$

when $-1 < \text{Re}(s) < 1$.

Next suppose that $B$ is a bounded linear operator that commutes with all $F(s, m)$ ($\text{Re}(s) = 0$). Then when $\text{Re}(s) = 0$

$$BJ(s, m) = J(s, m)B \quad (4.11)$$

and by analyticity (4.11) also holds for all $s$ in the strip $1 < \text{Re}(s) < 1$. Therefore, $B$ commutes with all $J'(s, m)$ and hence with all $L(s, m)$. This implies that $B$ commutes with each $E(s, m)$. In particular, $F(s/2, m/2)$ commutes with $E(s, m)$, and setting

$$D(s, m) = E(s, m)F\left(\frac{s}{2}, \frac{m}{2}\right), \quad \text{Re}(s) = 0 \quad (4.12)$$

we find that

$$D(-s, -m)^{-1} D(s, m) = F\left(\frac{s}{2}, \frac{m}{2}\right)^2 E(-s, -m)^{-1} E(s, m).$$

From (4.10) and the fact that

$$J(s, m) = F\left(\frac{-s}{2}, \frac{-m}{2}\right)^2 F(s, m)$$
when $\text{Re}(s) = 0$, we conclude that

$$D(-s, -m)^{-1} D(s, m) = F(s, m), \quad \text{Re}(s) = 0. \quad (4.13)$$

This proves (4.1) for the case in which $t = (s, m)$, $\text{Re}(s) = 0$, and $m$ is even. Note also that $D(s, m)$ commutes with every bounded operator $B$ which commutes with all $F(s, m)$.

Next we consider the growth of $E$. By (4.14),

$$E(s, m)^2 = E_m^2 \exp(L(s, m)),$$

and by (4.8) this implies

$$\| E(s, m) \|^2 = \| J(s, m) \|.$$  

By (4.3) and our observations about the growth of $K(s_1, s_2, m)$, there is a positive integer $N$ and a constant $K_\epsilon (0 < \epsilon < 1)$ such that

$$\| J(s, m) \| \leq K_\epsilon (1 + |\text{Im}(s)| + |m|)^N$$

whenever $-1 + \epsilon \leq \text{Re}(s) \leq 1 - \epsilon$. Taking square roots we obtain the same type of bound for $\| E(s, m) \|$.

The construction of $D(s, m)$ for the case in which $m$ is odd is slightly different but based on the same idea, and we shall only sketch the method.

For odd $m$ and $\text{Re}(s) = 0$, take

$$J(s, m) = F \left( \frac{-s}{2}, \frac{-m-1}{2} \right) F \left( \frac{-s}{2}, \frac{1-m}{2} \right) F(s, m). \quad (4.15)$$

Then $s \rightarrow J(s, m)$ ($\text{Re}(s) = 0$) has an analytic continuation into the strip which is of polynomial growth, and $J(s, m)^{-1} = J(-s, -m)$. Define $L(s, m)$ as before and set

$$E(s, m) = E_m \exp(\frac{1}{2}L(s, m)) \quad (4.16)$$

where $E_m = J(0, m)$ when $m > 0$ and $E_m = I$ when $m < 0$. Finally, set

$$D(s, m) = E(s, m) F \left( \frac{s}{2}, \frac{m-1}{2} \right). \quad (4.17)$$

Then, by essentially the same argument that was given earlier

$$D(-s, -m)^{-1} D(s, m) = F(s, m)$$
and $D(s, m)$ commutes with every operator that commutes with all of the $F$'s. In addition, $s \rightarrow E(s, m)$ is of polynomial growth in the strip $-1 < \text{Re}(s) < 1$.

Now suppose that $m_1$ and $m_2$ are arbitrary integers and that $c$ is real. Then

$$D(s + ic, m_1) D(-s, m_2)$$

$$= E(s + ic, m_1) E(-s, m_2) F \left( \frac{s + ic}{2}, m_1' \right) F \left( \frac{-s}{2}, m_2' \right)$$

where $m_1'$ and $m_2'$ are determined by (4.12) or (4.17) according to the parity of $m_1$, $m_2$. As functions of $s$, $E(s + ic, m_1)$ and $E(-s, m_2)$ are analytic and of polynomial growth in the strip, and by assumption

$$s \rightarrow F \left( \frac{s + ic}{2}, m_1' \right) F \left( \frac{-s}{2}, m_2' \right), \quad \text{Re}(s) = 0$$

has an analytic continuation into the strip which is of polynomial growth. It follows that $D$ is an admissible map of $i\mathbb{R} \times \mathbb{Z}$ into $\mathcal{U}(\mathcal{H})$, and this completes the proof of the lemma.

As mentioned in the proof of Lemma 10, homomorphisms of $i\mathbb{R} \times \mathbb{Z}$ into $\mathcal{U}(\mathcal{H})$ provide trivial examples of admissible maps. We now wish to discuss a non-trivial example of fundamental importance.

For this purpose, let $A(s, m) = A_1(s, m)$ where $A_1(s, m)$ is the basic intertwining operator (3.7) which arises in the case $G = SL(2, \mathbb{C})$. Then $A(s, m)$ is a suitably regularized fractional integral on $L^2(\mathbb{C})$, and $A$ is a strongly continuous homomorphism of $i\mathbb{R} \times \mathbb{Z}$ into $\mathcal{U}(L^2(\mathbb{C}))$, [2] or [3]. To give a precise characterization of $A$, we define $B: i\mathbb{R} \times \mathbb{Z} \rightarrow \mathcal{U}(L^2(\mathbb{C}))$ by

$$B(s, m) f(z) = |z|^{-s} [z]^{-m} f(z)$$

(4.18)

where $[z] = z/|z|$ when $z \neq 0$. Then $B$ is unitarily equivalent to $A$ via the Fourier transform, [3]. If $AB$ and $BA$ are defined in the obvious fashion, i.e.

$$(AB)(s, m) = A(s, m) B(s, m)$$

(4.19)

then $AB$ and $BA$ are both admissible. This would be obvious if $AB$ were a homomorphism of $i\mathbb{R} \times \mathbb{Z}$ into $\mathcal{U}(L^2(\mathbb{C}))$, but it is not even true
that $AB = BA$. Because $B(s, m)A(s, m) = (A(-s, -m)B(-s, -m))^{-1}$, and
\[
B(s + ic, m_1)A(s + ic, m_1)B(-s, m_2)A(-s, m_2)
= (A(-\bar{s}, -m_2)B(-\bar{s}, -m_2)A(s - ic, -m_1)B(s - ic, -m_1))^*
\]
it is easy to see that $BA$ is admissible if and only if $AB$ is admissible. Although there are several proofs of this fact, the most perspicuous involves a concrete spectral decomposition of the family \{$A(t)B(t): t \in i\mathbb{R} \times \mathbb{Z}$\} in terms of the "Mellin transform".

For this purpose we combine the unitary map $f(z) \mapsto |z|^\frac{1}{2}f(z)$ of $L^2(\mathbb{C})$ onto $L^2(\mathbb{C}^*, |z|^{-2}dz)$, $\mathbb{C}^*$ being the multiplicative group of complex numbers, with the Fourier-Mellin transform of $L^2(\mathbb{C}^*, |z|^{-2}dz)$ onto $L^2(i\mathbb{R} \times \mathbb{Z})$. In this way we obtain a map $\mathcal{M}: L^2(\mathbb{C}) \rightarrow L^2(i\mathbb{R} \times \mathbb{Z})$ with the property that for some constant $c$
\[
(f | g) = c(\mathcal{M}f | \mathcal{M}g)
\]
for all $f$ and $g$ in $L^2(\mathbb{C})$. Setting $\zeta(z) = |z|^{ip}[z]^k$ when $\zeta$ is the element $(ip, k)$ of $i\mathbb{R} \times \mathbb{Z}$ and $\mathcal{M}f = \tilde{f}$, we have the specific formula
\[
\tilde{f}(\zeta) = \int f(z) \zeta(z) |z|^{-1}dz
\]
for all $f$ in $L^2(\mathbb{C})$ such that
\[
\int |f(z)| |z|^{-1}dz < \infty.
\]

**Lemma 11.** For each $t$ in $i\mathbb{R} \times \mathbb{Z}$ set $\bar{A}(t) = \mathcal{M}A(t)\mathcal{M}^{-1}$ and $\bar{B}(t) = \mathcal{M}B(t)\mathcal{M}^{-1}$. In addition, let
\[
\omega(\zeta) = 2\pi(i)^k 2^{-ip} \frac{\Gamma\left(\frac{|k| + 1 - ip}{2}\right)}{\Gamma\left(\frac{|k| + 1 + ip}{2}\right)} \tag{4.21}
\]
when $\zeta = (ip, k)$. Then for an arbitrary $g$ in $L^2(i\mathbb{R} \times \mathbb{Z})$
\[
(\bar{A}(t)g)(\zeta) = \frac{\omega(\zeta + t)}{\omega(\zeta)} g(\zeta + t) \tag{4.22}
\]
and
\[
(\bar{B}(t)g)(\zeta) = g(\zeta - t). \tag{4.23}
\]
Proof. Since various results essentially equivalent to this are in the literature (see e.g. [7]), we shall only sketch the idea of the proof. If \( \mathcal{F} \) denotes the usual Fourier transform on \( L^2(\mathbb{C}) \), then

\[
B(t) = \mathcal{F} A(t) \mathcal{F}^{-1}.
\]

On the other hand, let \( \mathbb{F} = M \mathcal{F} M^{-1} \). Then because

\[
w(\zeta) = \lim_{\epsilon \to 0^+} \int e^{i \epsilon \Re z - i |z|} \xi(z) \frac{dz}{|z|}
\]

it follows that

\[
(\mathbb{F}g)(\zeta) = w(-\zeta) g(-\zeta), \quad g \in L^2(i\mathbb{R} \times \mathbb{Z}).
\]

Given (4.24) and (4.25), one obtains (4.22) and (4.23) by a simple computation.

**Lemma 12.** \( AB \) is an admissible map of \( i\mathbb{R} \times \mathbb{Z} \) into \( \mathcal{U}(L^2(\mathbb{C})) \).

Proof. \( MA(t) B(t) M^{-1} = A(t) B(t) \), and from (4.22) and (4.23) it follows that

\[
(\mathcal{A}(t) \mathcal{B}(t) g)(\zeta) = \frac{w(\zeta + t)}{w(\zeta)} g(\zeta), \quad g \in L^2(i\mathbb{R} \times \mathbb{Z}).
\]

Thus, \( A(t) B(t) \) is unitarily equivalent to multiplication by

\[
\zeta \mapsto w(\zeta + t)/w(\zeta)
\]

on \( L^2(i\mathbb{R} \times \mathbb{Z}) \). Hence \( \{A(t) B(t) : t \in i\mathbb{R} \times \mathbb{Z}\} \) is a commutative family of unitary operators. To show that \( AB \) has a relative analytic continuation of polynomial growth, it is enough to show that this is true of \( AB \).

For this purpose, let \( t = (s, m) \) and

\[
w_0(t) = w_0(s, m) = \frac{\Gamma\left(\frac{|m| + 1 - s}{2}\right)}{\Gamma\left(\frac{|m| + 1 + s}{2}\right)}, \quad \Re(s) < 1.
\]

Then \( s \to w_0(s, m) \) is analytic in the half-plane \( \Re(s) < 1 \). To obtain an estimate for \( |w_0(t)| \), we first note that

\[
w_0(s, m) = \frac{\Gamma\left(\frac{|m| + 3 - s}{2}\right)}{\Gamma\left(\frac{|m| + 3 + s}{2}\right)} \frac{\Gamma\left(\frac{|m| + 1 + s}{2}\right)}{\Gamma\left(\frac{|m| + 1 - s}{2}\right)}
\]
when \( \text{Re}(s) < 1 \). Now it was shown in [2, Lemma 17] that there exists a constant \( K \) such that for all \( c \geq 1 \)

\[
\left| \frac{\Gamma(c - x - iy)}{\Gamma(c + x + iy)} \right| \leq K |c + iy|^{-2x}
\]

in the strip \(-\frac{1}{2} < x < \frac{1}{2}\). If \( 0 < \epsilon < 1 \) and \( s = a + ib \) with \(-1 + \epsilon < a < 1 - \epsilon \), it follows that

\[
|w_0(s, m)| \leq K \left| \frac{m + 1 + s}{|m| + 1 - s} \right| \left| \frac{|m| + 3 + ib}{2} \right|^{-a}
\]

\[\leq \frac{4K}{\epsilon} \left| m + 3 + ib \right|^{-a}\]

If \( \zeta \) is the element \((ip, k)\) of \( i\mathbb{R} \times \mathbb{Z} \) and \( t = (s, m) \) with \( s = a + ib \) and \( a < 1 \), then by (4.21) and (4.25)

\[
\frac{w_0(\zeta + t)}{w(\zeta)} = i^{m_2 - s} \frac{w_0(\zeta + t)}{w_0(\zeta)}.
\]  

Since \( |w_0(\zeta)| = 1 \), it follows that

\[
\left| \frac{w(\zeta + t)}{w(\zeta)} \right| \leq \frac{8K}{\epsilon} |3 + |m + k| + ib + p|^{-a}
\]  

in the strip \(-1 + \epsilon < a < 1 - \epsilon \). Because

\[
\sup_{\zeta} \left| \frac{w(\zeta + t)}{w(\zeta)} \right| < \infty
\]

when \( 0 \leq a \leq 1 - \epsilon \), it follows that the operator-valued function \( s \to A(s, m) B(s, m) \) has an analytic continuation into the strip

\( 0 \leq \text{Re}(s) < 1 \);

however, it does not have an analytic continuation into \(-1 < \text{Re}(s) \leq 0\) for

\[
\sup_{\zeta} \left| \frac{w(\zeta + t)}{w(\zeta)} \right| = \infty
\]

when \(-1 < a < 0\). Now fix a real number \( c \) and integers \( m_1 \) and \( m_2 \).
Set $t_1 = (s + ic, m_1)$ and $t_2 = (-s, m_2)$ where $-1 < \Re(s) < 1$. Then by (4.26)

$$\frac{w(\zeta + t_1) \cdot w(\zeta + t_2)}{w(\zeta)} = \frac{w_0(\zeta + t_1)}{w_0(\zeta)} \cdot \frac{w_0(\zeta + t_2)}{w_0(\zeta)}$$

which for fixed $\zeta$ is analytic as a function of $s$ in the strip $-1 < \Re(s) < 1$. By (4.27)

$$\left| \frac{w(\zeta + t_1) \cdot w(\zeta + t_2)}{w(\zeta)} \right| \leq K \epsilon \left| 3 + |m_1 + k| + i(b + c + p) \right|^{-\alpha} \left| 3 + |m_2 + k| + i(p - b) \right|^\alpha$$

when $s = a + ib$ and $-1 + \epsilon \leq a \leq 1 - \epsilon$. By elementary estimates

$$\left| \frac{3 + |m_1 + k| + i(b + c + p)}{3 + |m_2 + k| + i(p - b)} \right| \leq 2(1 + |m_1| + |m_2| + |b| + |c|)$$

for all $p$ and $k$. Hence, when $-1 + \epsilon \leq a \leq 1 - \epsilon$

$$\left| \frac{w(\zeta + t_1) \cdot w(\zeta + t_2)}{w(\zeta)} \right| \leq K \epsilon 2^{-\alpha} \left( 1 + |m_1| + |m_2| + |b| + |c| \right)^{1-\epsilon}$$

for all $p$ and $k$. Thus the operator valued function

$$s \to \bar{A}(s + ic, m_1) \hat{B}(s + ic, m_1) \bar{A}(-s, m_2) \hat{B}(-s, m_2)$$

has an analytic continuation which is of polynomial growth in the strip $-1 < \Re(s) < 1$.

**Lemma 13.** $ABA$ and $BAB$ are homomorphisms of $i\mathbb{R} \times \mathbb{Z}$ into $L^2(\mathbb{C})$ and hence admissible.

**Proof.** By (4.22) and (4.24)

$$\bar{A}(t) \hat{B}(t) \bar{A}(t) g(\zeta) = \left( \frac{w(\zeta + t)}{w(\zeta)} \right)^a g(\zeta + t) \quad (4.28)$$

and by (4.23) and (4.24)

$$\hat{B}(t) \bar{A}(t) \hat{B}(t) g(\zeta) = \frac{w(\zeta)}{w(\zeta - t)} g(\zeta - t) \quad (4.29)$$
for $g$ in $L^2(i\mathbb{R} \times \mathbb{Z})$. It follows from (4.28) that $AB\overline{A}$ is a homomorphism and similarly from (4.29) that $BA\overline{B}$ is a homomorphism.

Next we use Lemma 12 in proving the following result.

**Lemma 14.** If $1 \leq j \leq n - 2$ and $A_j$ is the homomorphism of $i\mathbb{R} \times \mathbb{Z}$ into $U$ which is characterized by (3.7), then $A_jA_{j+1}$ and $A_{j+1}A_j$ are admissible.

**Proof.** As noted earlier, in a similar situation, it is enough to show that $A_jA_{j+1}$ is admissible. Suppose $1 \leq j \leq n - 2$ and let $t \in i\mathbb{R} \times \mathbb{Z}$. Then it follows from Theorem 4 of [3] and its proof that $A_j(t) A_{j+1}(t)$ is unitarily equivalent to

$$
(A(t) B(t) \otimes I \otimes B(t)) \otimes I_j
$$

where $A(t)$ and $B(t)$ are the operators on $L^2(\mathbb{C})$ that are characterized by Lemma 11, $I$ is the identity operator on $L^2(\mathbb{C})$, and $I_j$ is the identity operator on a certain Hilbert space $\mathcal{H}_j$ whose exact description is unimportant for the present purpose. The admissibility of $A_jA_{j+1}$ is therefore an easy consequence of Lemma 12.

Now we wish to obtain a generalization of Lemma 14 dealing with products involving more than two factors. In doing this, we use the following result.

**Lemma 15.** Let $X$ and $Z$ be maps and $Y$ a homomorphism of $i\mathbb{R} \times \mathbb{Z}$ into $U(\mathcal{H})$. Suppose $XY$ and $YZ$ are admissible and that $X(t)$ commutes with $Z(t')$ for all $t, t' \in i\mathbb{R} \times \mathbb{Z}$. Then $XYZ$ is admissible.

**Proof.** We first prove that $\{X(t) Y(t) Z(t): t \in i\mathbb{R} \times \mathbb{Z}\}$ is a commutative family. Let $t$ and $t'$ be arbitrary elements of $i\mathbb{R} \times \mathbb{Z}$. Then

$$
X(t) Y(t) Z(t) \cdot X(t') Y(t') Z(t')
= X(t) Y(t) \cdot X(t') Y(t') \cdot Y(-t') Y(-t) \cdot Y(t) Z(t) \cdot Y(t') Z(t')
$$

(4.30)

because $X(t')$ commutes with $Z(t)$ and $Y$ is a homomorphism. Since $XY$ and $YZ$ are admissible

$$
X(t) Y(t) \cdot X(t') Y(t') = X(t') Y(t') \cdot X(t) Y(t).
$$

and

$$
Y(t) Z(t) \cdot Y(t') Z(t') = Y(t') Z(t') \cdot Y(t) Z(t).
$$
Therefore, we have

\[ X(t) Y(t) Z(t) \cdot X(t') Y(t') Z(t') \]

\[ = X(t') Y(t') \cdot X(t) Y(t) \cdot Y(-t') Y(-t) \cdot Y(t') Z(t') \cdot Y(t) Z(t) \]

\[ = X(t') Y(t') X(t) Z(t) Y(t) Z(t) \]

\[ = X(t') Y(t') Z(t') \cdot X(t) Y(t) Z(t). \]

Next, we take \( t = (s + ic, m) \) and \( t' = (-s, m') \) in (4.30). Then \( Y(-t') Y(-t) = Y(-t' - t) \) is independent of \( s \), and the maps

\[ s \rightarrow X(s + ic, m) Y(s + ic, m) X(-s, m') Y(-s, m'), \quad \text{Re}(s) = 0 \]

\[ s \rightarrow Y(s + ic, m) Z(s + ic, m) Y(-s, m') Y(-s, m'), \quad \text{Re}(s) = 0 \]

have analytic continuations which are of polynomial growth in the strip \(-1 < \text{Re}(s) < 1\). Thus, it follows from (4.30) that

\[ s \rightarrow X(s + ic, m) Y(s + ic, m) Z(s + ic, m) X(-s, m') Y(-s, m') Z(-s, m') \]

has an analytic continuation of polynomial growth in the strip \(-1 < \text{Re}(s) < 1\).

Next, we generalize Lemma 14 by proving that the product of any consecutive string of \( A_i \)'s \( (1 < j < n - 1) \) is admissible.

**Lemma 16.** If \( 1 \leq j < k \leq n - 1 \), then the products \( A_j \cdots A_k \) and \( A_k \cdots A_j \) are admissible.

**Proof.** Because the \( A_i \) with \( j \leq i \leq k \) are homomorphisms, it is enough to prove the admissibility of the first product in which the indices go in increasing order. By Lemma 14, the result is true when \( k - j = 1 \). Suppose \( k - j > 1 \). Then the product in question involves at least 3 factors and has the form

\[ A_j A_{j+1} A_{j+2} \cdots A_k. \]

Let \( X = A_j \), \( Y = A_{j+1} \), and \( Z = A_{j+2} \cdots A_k \). Then \( X(t) Z(t') = Z(t') X(t) \), \( XY \) is admissible, and by induction \( YZ \) is admissible. Hence \( XYZ \) is admissible by Lemma 15.

As a corollary we obtain the fact that the particular products

\[ J_k = A_{n-1} A_{n-2} \cdots A_k, \quad 1 \leq k \leq n - 1 \quad (4.31) \]
are admissible. This observation allows us to verify the as yet unproved assumption made in Section 3. If \( \lambda = (t_1, \ldots, t_n) \) with \( t_j \) in \( i\mathbb{R} \times \mathbb{Z} \), it follows from (3.17) that
\[
X_k(\lambda) = J_k(-t_k), \quad 1 \leq k \leq n - 1. \tag{4.32}
\]
Hence, for each \( k \), \( \{X_k(\lambda) : \lambda \in \Lambda\} \) is indeed a commutative family.

Now let
\[
C_b = A_{n-1} \cdots A_1 A_0^2 A_1 \cdots A_{n-1} \tag{4.33}
\]
when \( G \) is of type \( B_n \)
\[
C_c = A_{n-1} \cdots A_1 A_0 A_1 \cdots A_{n-1} \tag{4.34}
\]
when \( G \) is of type \( C_n \), and
\[
C_d = A_{n-1} \cdots A_2 A_1 A_0 A_2 \cdots A_{n-1} \tag{4.35}
\]
when \( G \) is of type \( D_n \) \((n \geq 3)\).

The rest of this section is devoted to the proof of the theorem that follows. Once the theorem is established, we will use it and Lemma 10 to construct maps \( X_0 : \Lambda \to \mathcal{U} \), in each of the 3 cases, such that
\[
X_j(\lambda) = X_0(\lambda) \quad \text{for} \quad 1 \leq j \leq n - 1, \quad \text{and}
\]
\[
X_0(p_0\lambda)^{-1} X_0(\lambda) = Y_1(p_0) A(p_0, \lambda) Y_1(\lambda)^{-1}.
\]
This construction will be given in the next section where we shall complete our normalization of the principal series.

**Theorem 7.** The maps \( C_b \), \( C_c \), and \( C_d \) are all admissible. Suppose \( \lambda = (t_1, \ldots, t_n) \) with \( t_j \) in \( i\mathbb{R} \times \mathbb{Z} \) and define \( Y_1 \) by (3.18). If \( G \) is of type \( B_n \), then
\[
Y_1(p_0) A(p_0, \lambda) Y_1(\lambda)^{-1} = C_b(t_1).
\]
If \( G \) is of type \( C_n \), then
\[
Y_1(p_0) A(p_0, \lambda) Y_1(\lambda)^{-1} = C_c(t).
\]
On the other hand, if \( G \) is of type \( D_n \) \((n \geq 3)\) then
\[
Y_1(p_0) A(p_0, \lambda) Y_1(\lambda)^{-1} = C_d(t_1) C_d(t_2).
\]
In the proof, we use one general result on the construction of admissible
maps and a number of special results designed to treat the cases in which $G$ is of type $B_n$, $C_n$, or $D_n$. The general result will be proved next.

**Lemma 17.** Suppose $X$ and $Z$ are maps of $i\mathbb{R} \times \mathbb{Z}$ into $\mathcal{U}(\mathcal{H})$ such that $X(t) Z(t') = Z(t') X(t)$ and $X(t)^{-1} = X(-t)$ for all $t, t'$ in $i\mathbb{R} \times \mathbb{Z}$. In addition, suppose that $Y$ is a homomorphism of $i\mathbb{R} \times \mathbb{Z}$ into $\mathcal{U}(\mathcal{H})$ such that $XY$ and $YZY$ are both admissible. Then $XYZYX$ is also admissible.

**Proof.** Let $t_1$ and $t_2$ be arbitrary elements of $i\mathbb{R} \times \mathbb{Z}$, and write $X_i$, $Y_i$, $Z_i$ for $X(t_i)$, $Y(t_i)$, and $Z(t_i)$ ($i = 1, 2$). Since $(XY)^{-1}(t) = Y(-t) X(-t)$ for all $t$ in $i\mathbb{R} \times \mathbb{Z}$, it follows from our assumptions that $Y_1 X_1$ commutes with $X_2 Y_2$. Therefore

$$X_1 Y_1 Z_1 Y_1 X_1 \cdot X_2 Y_2 Z_2 Y_2 X_2 \cdot X_1 Y_1 Z_1 Y_1 X_1 \cdot X_2 Y_2 Z_2 Y_2 X_2$$

$$= X_1 Y_1 Z_1 \cdot X_2 Y_2 \cdot Y_1 X_1 \cdot Z_2 Y_2 X_2$$

$$- X_1 Y_1 \cdot X_2 Y_2 \cdot Y_2^{-1} Z_1 Y_2 Y_1 Z_2 Y_1^{-1} \cdot Y_1 X_1 \cdot Y_2 X_2$$

The product $Y_2^{-1} Z_1 Y_2 Y_1 Z_2 Y_1^{-1}$ which appears in the middle of this expression may be written in the form

$$Y_2^{-1} Y_1^{-1} \cdot Y_1 Z_1 Y_1 \cdot Y_2 Z_2 Y_2 \cdot Y_2^{-1} Y_1^{-1}.$$ 

Thus we have

$$X_1 Y_1 Z_1 Y_1 X_1 \cdot X_2 Y_2 Z_2 Y_2 X_2$$

$$= X_1 Y_1 \cdot X_2 Y_2 \cdot Y_2^{-1} Y_1^{-1} \cdot Y_1 Z_1 Y_1 \cdot Y_2 Z_2 Y_2 \cdot Y_2^{-1} Y_1^{-1} \cdot Y_1 X_1 \cdot Y_2 X_2.$$ 

(4.36)

Because $Y(s, m) X(s, m) = (X(\hat{s}, -m) Y(\hat{s}, -m))^*$ ($\text{Re}(s) = 0$) and $XY$ is admissible, it follows that $YX$ is also admissible. Now take $t_1 = (s + ic, m_1)$, $c$ real, and $t_2 = (-s, -m_2)$. Then

$$Y_2^{-1} Y_1^{-1} = Y(-ic, -m_1 - m_2)$$

and the products $X_1 Y_1 \cdot X_2 Y_2$, $Y_1 Z_1 Y_1 \cdot Y_2 Z_2 Y_2$, and $Y_1 X_1 \cdot Y_2 X_2$ regarded as functions of $s$ have analytic continuations into the strip $-1 < \text{Re}(s) < 1$ which are of polynomial growth. Thus it follows from (4.36) that $X_1 Y_1 Z_1 Y_1 X_1 \cdot X_2 Y_2 Z_2 Y_2 X_2$ regarded as a function of $s$ also has an analytic continuation into the strip $-1 < \text{Re}(s) < 1$ which is of polynomial growth. In addition, we may rewrite the right side of (4.36)
by interchanging the subscripts 1 and 2 because $X_1Y_1$ commutes with $X_2Y_2$ etc. Having done this and combining terms, we see that

$$X_1Y_1Z_1X_1 \cdot X_2Y_2Z_2Y_2X_2$$

$$= X_2Y_2X_1Z_2Y_1Z_1X_2X_1$$

$$= X_2Y_2Z_2 \cdot X_1Y_1 \cdot Y_2X_2 \cdot Z_1Y_1X_1$$

$$= X_2Y_2Z_2 \cdot Y_2X_2 \cdot X_1Y_1Z_1Y_1X_1$$

for arbitrary $t_1$ and $t_2$ in $i\mathbb{R} \times \mathbb{Z}$. Therefore, $XYZYX$ is admissible.

**Lemma 18.** If $G$ is of type $C_n$, then $A_1A_0A_1$ is admissible.

**Proof.** Define $B^+: i\mathbb{R} \times \mathbb{Z} \to U(L^2(\mathbb{C}))$ by

$$B^+(s, m) f(z) = |z + 1|^{-s} [z + 1]^{-m} f(z). \quad (4.36)$$

Then, as shown in the proof of Theorem 4 in [3], $A_1(t) A_0(t) A_1(t)$ is unitarily equivalent to

$$B^+(t) A(t) B(t) A(t) B^+(t) \otimes M(t) \otimes I$$

where $A(t)$ and $B(t)$ are the operators on $L^2(\mathbb{C})$ that are characterized by Lemma 11, $M$ is a homomorphism of $i\mathbb{R} \times \mathbb{Z}$ into the unitary group on a certain Hilbert space, and $I$ is the identity operator on another space. The exact description of $M$ and the other two spaces is of no importance for our present purpose. For it follows that $A_1A_0A_1$ is admissible if and only if $B^+ABA^B$ is admissible. By Lemma 13 $ABA$ is admissible. In addition, $B^+A$ is admissible. To see this, let $T$ be the translation operator defined on $L^2(\mathbb{C})$ by

$$(Tf)(z) = f(z + 1)$$

Then by (4.18) and (4.36) $B^+(t) = TB(t)T^{-1}$ for all $t$ in $i\mathbb{R} \times \mathbb{Z}$. Since $TA(t)T^{-1} = A(t)$, it follows that

$$B^+(t) A(t) = TB(t) A(t) T^{-1}.$$

Therefore, $B^+A$ is admissible by Lemma 12. Since $B^+A$ and $ABA$ are admissible, and $B^+$ is a homomorphism, with $B^+(t) B(t') = B(t') B^+(t)$ the admissibility of $B^+ABA^B$ is a consequence of Lemma 17.
Lemma 19. If \( G \) is of type \( C_n \), then \( A_k \cdots A_1 A_0 A_1 \cdots A_k \) is admissible for each \( k \) such that \( 1 \leq k \leq n - 1 \).

Proof. We may assume \( n \geq 3 \). Then \( A_2 A_1 \) is admissible by Lemma 16, and \( A_1 A_0 A_1 \) is admissible by Lemma 18. Since \( A_2 \) is a homomorphism and \( A_2(t) A_0(t') = A_0(t') A_2(t) \) it follows from Lemma 17, that \( A_2 A_1 A_0 A_1 A_2 \) is admissible. We proceed by induction. Suppose \( 2 \leq k \leq n - 2 \) and that we have shown

\[
A_j \cdots A_2 A_1 A_0 A_1 A_2 \cdots A_i
\]

is admissible for \( 2 \leq j \leq k \). Let \( X = A_{k+1} \), \( Y = A_k \) and

\[
Z = A_{k-1} \cdots A_1 A_0 A_1 \cdots A_{k-1}.
\]

Then \( X(t) Z(t') = Z(t') X(t) \) because \( A_{k+1}(t) \) commutes with all \( A_j(t') \) when \( 0 \leq j \leq k - 1 \). By Lemma 16, \( XY \) is admissible, and \( YZY \) is admissible by our inductive assumption. Therefore

\[
A_{k+1} A_k \cdots A_2 A_1 A_0 A_1 A_2 \cdots A_k A_{k+1}
\]

is admissible by Lemma 17.

Lemma 20. If \( G \) is of type \( B_n \), then

\[
t \to A_1(t) A_0(2t) A_1(t), \quad t \in i\mathbb{R} \times \mathbb{Z}
\]

is admissible.

Proof. The proof is similar to that of Lemma 18. In fact, the argument given there applies to the present situation (basically because \( B_2 \cong C_2 \)) and could be used to show that \( A_0 A_1 A_0 \) is admissible. But that is not what we wish to prove.

Using the proof of Theorem 4 in [3], in particular, equations (7.23), (7.24), (7.29), and (7.32), and the fact that \( \alpha_0 \) is shorter than \( \alpha_1 \) when \( G \) is of type \( B_n \) (cf. (3.8)), we obtain the following: There is a unitary transformation \( U \) such that for \( t = (s, m) \) in \( i\mathbb{R} \times \mathbb{Z} \)

\[
UA_0(t) U^{-1} = 4^s B^+(t) A(t) B^+(t) \otimes M_0(t) \otimes I \quad (4.37)
\]

and

\[
UA_1(t) U^{-1} = (-1)^m 4^{-s} B^+(-2t) B(t) \otimes M_1(t) \otimes I \quad (4.38)
\]
where $A(t)$, $B(t)$, and $B^+(t)$ are the unitary operators defined earlier (cf. the proof of Lemma 18) and $M_0, M_1$ are homomorphisms of $iR \times Z$ into the unitary group on a certain Hilbert space, $M_0(t) M_1(t') = M_1(t') M_0(t)$ and $I$ is the identity operator on another space. Setting $M = M_1 M_0^2 M_1$, we conclude from (4.37) and (4.38) that

\[ UA_1(t) A_0(2t) A_1(t) U^{-1} = B(t) A(2t) B(t) \otimes M(t) \otimes I. \quad (4.39) \]

Because $M$ is a homomorphism, it follows from (4.39) that it is enough to show that $BA^2B$ is admissible. Now $BA^2B = BA \cdot AB$ and $AB, BA$ are admissible by Lemma 12. Since $B(t) A(t) = (A(-t) B(-t))^{-1}$, it follows that $A(t) B(t)$ commutes with $B(t') A(t')$ for all $t$ and $t'$. Hence, $BA^2B$ is admissible.

**Lemma 21.** If $G$ is of type $B_n$, then $A_1 A_0^2 A_1 \cdots A_k$ is admissible for each $k$ with $1 \leq k \leq n - 1$.

**Proof.** This follows by induction from Lemma 17 and Lemma 20 in the same way that Lemma 19 follows from Lemma 17 and Lemma 18.

**Lemma 22.** If $G$ is of type $D_n$ ($n \geq 3$) then $A_2 A_0$ and $A_2 A_0 A_1 A_2$ are both admissible.

**Proof.** By (3.8) $\alpha_0$ is orthogonal to $\alpha_1$. Hence, $A_1(t) A_0(t') = A_0(t') A_1(t)$ by Theorem 4 of [3]. On the other hand

\[ \langle \alpha_0, \alpha_2 \rangle = \langle \alpha_1, \alpha_2 \rangle = -1 \]

by (3.8). Using this and Theorem 4 of [3], we see that the proof of Lemma 14 applies to the present situation and hence that $A_0 A_2$ and $A_2 A_0$ are admissible. In addition we know that $A_1 A_2$ and $A_2 A_1$ are admissible. Moreover, $A_2(t) A_1(t) A_1(t') A_2(t') = A_1(t') A_2(t') A_2(t) A_1(t)$ and $A_0(t) A_2(t) A_2(t') A_0(t') = A_0(t') A_2(t') A_0(t) A_2(t)$. If $t$ and $t'$ are arbitrary elements of $iR \times Z$, it follows that

\[ A_2(t) A_1(t) A_0(t) A_2(t) \cdot A_2(t') A_1(t') A_0(t') A_2(t') \]

\[ = A_2(t) A_1(t) \cdot A_0(t) A_2(t) \cdot A_2(t') A_0(t') \cdot A_1(t') A_2(t') \]

\[ = A_2(t) A_1(t) \cdot A_2(t') A_0(t') \cdot A_0(t) A_2(t) \cdot A_1(t') A_2(t'). \]
Now using the fact that $A_1$ and $A_0$ are commuting homomorphisms, we find that

$$A_2(t) \ A_1(t) \ A_0(t) \ A_2(t) \cdot A_2(t') \ A_1(t') \ A_0(t') \ A_2(t')$$

$$= A_2(t) \ A_1(t) \cdot A_2(t') \ A_1(t') \cdot A_1(-t' - t) \ A_0(t' + t) \cdot A_1(t) \ A_2(t)$$

$$\cdot A_1(t') \ A_2(t').$$

(4.40)

If $t = (s + ic, m)$ with $c$ real and $t' = (-s, m')$, it follows from (4.40) that

$$A_2(t) \ A_1(t) \ A_0(t) \ A_2(t) \cdot A_2(t') \ A_1(t') \ A_0(t') \ A_2(t')$$

regarded as a function of $s$ has an analytic continuation into the strip which is of polynomial growth. It also follows from (4.40) and our observations made above that

$$A_2(t) \ A_1(t) \ A_0(t) \ A_2(t) \cdot A_2(t') \ A_1(t') \ A_0(t') \ A_2(t')$$

$$= A_2(t') \ A_1(t') \cdot A_2(t) \ A_0(t) \cdot A_0(t' + t) \ A_1(t') \ A_0(t') \ A_1(t) \ A_2(t)$$

$$= A_2(t') \ A_1(t') \cdot A_2(t) \ A_0(t) \cdot A_0(t') \ A_2(t') \ A_1(t) \ A_0(t) \ A_2(t)$$

which completes the proof that $A_2 A_1 A_0 A_2$ is admissible.

**Lemma 23.** If $G$ is of type $D_n$ $(n \geq 3)$ then $A_k \cdots A_2 A_1 A_0 A_2 \cdots A_k$ is admissible for each $k$ such that $2 \leq k \leq n - 1$.

**Proof.** This follows by induction from Lemma 22 and Lemma 17 by an elementary argument similar to the one given in the proof of Lemma 19.

**Proof of Theorem 7.** The maps $C_b$, $C_c$, and $C_d$ defined by (4.33), (4.34), and (4.35) are admissible by Lemmas 21, 19, and 23.

Suppose $G$ is of type $B_n$. Then by (3.7) and (3.8)

$$A(\rho_0, \lambda) = A_0(2t_1)$$

(4.41)

when $\lambda = (t_1, \ldots, t_n)$ with $t_j$ in $i\mathbb{R} \times \mathbb{Z}$. Moreover, it follows from (3.6) that

$$\rho_0(t_1, t_2, \ldots, t_n) = (-t_1, t_2, \ldots, t_n).$$

(4.42)

By (3.8) $\alpha_0$ is orthogonal to $\alpha_k$ for $2 \leq k \leq n - 1$; hence, $A_0(t)$ commutes
with $A_k(t')$ for arbitrary $t, t'$ when $2 \leq k \leq n - 1$, by [3, Theorem 4]. Therefore, for the same values of $k$, $A_o(t)$ commutes with the operators $J_k(t')$ defined by (4.31). Thus, if $\lambda = (t_1, \ldots, t_n)$, it follows from (4.32) that $A_o(2t_1) = A(p_0, \lambda)$ commutes with $X_k(\lambda)$ for $2 \leq k \leq n - 1$. Moreover, $X_k(p_0\lambda) = X_k(\lambda)$ for $2 \leq k \leq n - 1$ by (4.32) and (4.42). Since $Y_1 = X_1 \cdots X_{n-1}$ it follows that

$$Y_1(p_0\lambda) A(p_0, \lambda) Y_1(\lambda)^{-1} = X_1(p_0\lambda) A(p_0, \lambda) X_1(\lambda)^{-1}$$

$$= J_1(t_1) A_o(2t_1) J_1(-t_1)^{-1}$$

$$= C_o(t_1)$$

(4.43)

when $\lambda = (t_1, \ldots, t_n)$.

Now suppose $G$ is of type $C_n$. Then by (3.7) and (3.8)

$$A(p_0, \lambda) = A_o(t_1)$$

(4.44)

for $\lambda = (t_1, \ldots, t_n)$. From (3.6) and (3.8) we find that

$$p_0(t_1, t_2, \ldots, t_n) = (-t_1, t_2, \ldots, t_n)$$

(4.45)

By an argument almost identical to that given above, $A(p_0, \lambda)$ commutes with $X_k(\lambda)$ and $X_k(p_0\lambda) = X_k(\lambda)$ for $2 \leq k \leq n - 1$. Therefore

$$Y_1(p_0\lambda) A(p_0, \lambda) Y_1(\lambda)^{-1} = X_1(p_0\lambda) A(p_0, \lambda) X_1(\lambda)^{-1}$$

$$= J_1(t_1) A_o(t_1) J_1(-t_1)^{-1}$$

$$= C_o(t_1)$$

(4.46)

when $\lambda = (t_1, \ldots, t_n)$.

Next suppose $G$ is of type $D_n (n \geq 3)$. Then by (3.7) and (3.8)

$$A(p_0, \lambda) = A_o(t_1 + t_2), \quad \lambda = (t_1, t_2, \ldots, t_n)$$

(4.47)

By (3.6) and (3.8)

$$p_0(t_1, t_2, t_3, \ldots, t_n) = (-t_2, -t_1, t_3, \ldots, t_n)$$

(4.48)

By (3.8) $\alpha_0$ is orthogonal to $\alpha_1$ and to $\alpha_k$ for $3 \leq k \leq n - 1$. Therefore, $A_o(t)$ commutes with $A_i(t')$ and with the operators $J_k(t'), 3 \leq k \leq n - 1$. Thus, if $\lambda = (t_1, \ldots, t_n)$, it follows that $A_o(t_1 + t_2) = A(p_0, \lambda)$ commutes with $X_k(\lambda)$ when $3 \leq k \leq n - 1$. By (4.48) $X_k(p_0\lambda) = X_k(\lambda)$ for
3 \leq k \leq n - 1. Hence

\[ Y_1(p_0, \lambda) A(p_0, \lambda) Y_1(\lambda)^{-1} = X_1(p_0, \lambda) X_2(p_0, \lambda) A(p_0, \lambda) X_1(\lambda)^{-1} \]

Suppose \( n = 3 \). Then (4.49) reduces to the equation

\[ Y_1(p_0, \lambda) A(p_0, \lambda) Y_1(\lambda)^{-1} = J_1(t_2) J_2(t_2) A_0(t_1) A_0(t_2) J_2(-t_2)^{-1} J_1(-t_1)^{-1}. \]

(4.49)

and commuting the two middle terms (cf. Lemma 22), we obtain the desired equations

\[ Y_1(p_0, \lambda) A(p_0, \lambda) Y_1(\lambda)^{-1} = C_d(t_2) C_d(t_1) \]

Now suppose \( n > 3 \). Then

\[ J_2(t_1) A_0(t_1) = A_{n-1}(t_1) \cdots A_0(t_1) \cdot A_0(t_2) \cdot A_0(t_3) \cdot A_0(t_4) \cdot A_0(t_5) \]

and hence \( J_2 \) is admissible by Lemma 15. Thus \( J_2(t_1) A_0(t_2) \) commutes with

\[ (J_2(-t_2) A_0(-t_2))^{-1} = A_0(t_2) J_2(-t_2)^{-1} = A_0(t_2) A_0(t_2) A_0(t_2) \cdots A_0(t_2). \]

Therefore, (4.49) may be written in the form

\[ Y_1(p_0, \lambda) A(p_0, \lambda) Y_1(\lambda)^{-1} = J_1(t_2) A_0(t_2) J_2(-t_2)^{-1} \cdot J_2(t_1) A_0(t_1) J_1(-t_1)^{-1} = C_d(t_2) \cdot C_d(t_1) \]

Because \( C_d(t_2) \) commutes with \( C_d(t_1) \), this completes the proof of Theorem 7.

5. The Final Normalization

Let \( C \) denote any one of the maps \( C_b, C_c, C_d \) defined by (4.33), (4.34), (4.35). From the fact that \( C \) is a product of homomorphisms of \( iR \times Z \) into \( U \) and the form of these equations, it is apparent that
C(t)\(^{-1}\) = C(\(-t\)) for all t. Let D be the admissible map corresponding to C which is constructed in Lemma 10. We recall that one of the basic properties of D is the relation

\[ D(-t)^{-1} D(t) = C(t), \quad t \in \mathbb{R} \times \mathbb{Z}. \]  

When it becomes necessary to indicate that C is C, , C, , or C, , we shall denote the corresponding D by D, , D, , or D, .

If \( \lambda = (t_1, \ldots, t_n) \) with \( t_j = (s_j, m_j) \), let \( N = N(\lambda) \) denote the number of odd integers in the sequence \( m_1, m_2, \ldots, m_n \), set \( N' = N'(\lambda) = N \) if \( N \) is even, \( N' = N - 1 \) if \( N \) is odd, and let

\[ X_0(\lambda) = D(0, N') D(s_1, m_1) D(s_2, m_2) \cdots D(s_n, m_n) \]  

In addition, let \( \lambda^{(0)} \) be the trivial character

\[ \lambda^{(0)} = \epsilon \]  

if \( N(\lambda) \) is even, and

\[ \lambda^{(0)} = ((0, 0), \ldots, (0, 1)) \]  

if \( N(\lambda) \) is odd. Finally, let \( G^{(0)} \) denote the maximal parabolic subgroup of G which is generated by the minimal parabolic subgroup \( H = MAV \) and the basic reflections \( p_j \) with \( 0 \leq j \leq n - 2 \). If \( n \geq 3 \), \( G^{(0)} \) is generated by \( p_0 \) and \( G^{(1)} \) (cf. Theorem 6, Section 3).

We may now state the main result to be proved in this section.

**Theorem 8.** Let G be a complex classical group of type \( B_n \), \( C_n \), or \( D_n \). When G is of type \( D_n \), assume that \( n \geq 3 \) and otherwise that \( n \geq 2 \). Define \( W: \Lambda \to \mathfrak{U} \) by

\[ W(\lambda) = X_0(\lambda) Y_1(\lambda) \]  

where \( X_0 \) is given by (5.2) and \( Y_1 \) by (3.18). Then the normalized principal series defined by

\[ R(y, \lambda) = W(\lambda) T(y, \lambda) W(\lambda)^{-1}, \quad y \in G \]  

are such that

\[ R(\cdot, p\lambda) = R(\cdot, \lambda) \]  

for all p in the Weyl group \( \mathfrak{W} \), and

\[ R(y, \lambda) = R(y, \lambda^{(0)}) \]  

for all y in the maximal parabolic subgroup \( G^{(0)} \).
Proof of (5.7). It is clear that \( N'(p\lambda) = N'(\lambda) \) for all \( p \) in \( \mathcal{M} \). If \( \lambda = (t_1, \ldots, t_n) \) with \( t_j = (s_j, m_j) \) in \( i\mathbb{R} \times \mathbb{Z} \), then because \( D \) is admissible \( D(t_j) \) commutes with \( D(t_k) \) for all pairs \( t_j, t_k \). Hence, it follows from (3.12) and (5.2) that

\[
X(p\lambda) = X(\lambda) \tag{5.9}
\]

for \( 1 \leq j \leq n - 1 \). If \( G \) is of type \( B_n \) or \( C_n \), then

\[
X(p_0\lambda)^{-1} X(\lambda) = D(-t_1)^{-1} D(t_2)^{-1} \cdots D(t_n)^{-1} D(t_1) D(t_2) \cdots D(t_n) = D(-t_1)^{-1} D(t_1) = C(t_1)
\]

and by Theorem 7

\[
C(t_1) = Y_1(p_0\lambda) A(p_0, \lambda) Y_1(\lambda)^{-1}.
\]

If \( G \) is of type \( D_n \), then

\[
X(p_0\lambda)^{-1} X(\lambda) = D(-t_1)^{-1} D(-t_2)^{-1} D(t_3)^{-1} \cdots D(t_n)^{-1} \cdot D(t_1) D(t_2) \cdots D(t_n) = D(-t_1)^{-1} D(t_1) D(-t_2)^{-1} D(t_2) = C(t_1) C(t_2)
\]

and by Theorem 7

\[
C(t_1) C(t_2) = Y_1(p_0\lambda) A(p_0, \lambda) Y_1(\lambda)^{-1}.
\]

Thus we have

\[
X_0(p_0\lambda)^{-1} X_0(\lambda) = Y_1(p_0\lambda) A(p_0, \lambda) Y_1(\lambda)^{-1} \tag{5.10}
\]

in all three cases. As observed earlier in Section 3 (cf. (3.4), Lemma 6, and Theorem 6) equations (5.9) and (5.10) imply (5.7).

For the proof of (5.8) we shall need some preliminary lemmas.

From the definition [3] of the principal series \( T(\cdot, \lambda) (\lambda \in \Lambda) \) it is easy to see that when \( \lambda = (t_1, \ldots, t_n) \), then \( T(p_0, \lambda) \) is naturally a function of \( 2t_1, t_2, \) or \( t_1 + t_2 \) according as \( G \) is of type \( B_n, C_n, \) or \( D_n \), these being the same functions of \( \lambda \) that arise when \( A(p_0, \lambda) \) is expressed in terms of \( A_0 \). Thus, as in (3.28), we shall write \( T(p_0, 2t_1), T(p_0, t_1) \), or \( T(p_0, t_1 + t_2) \) for \( T(p_0, \lambda) \) according as \( G \) is of type \( B_n, C_n, \) or \( D_n \).
Lemma 24. If $G$ is of type $B_n$, then
\begin{align}
A_0(2t) T(p_0, 2t) &= T(p_0, -2t) A_0(2t) \quad (5.11) \\
A_0(2t) T(p_1, t') &= T(p_1, 2t + t') A_0(2t) \quad (5.12) \\
A_0(2t) T(p_j, t') &= T(p_j, t') A_0(2t), \quad 2 \leq j \leq n - 1 \quad (5.13) \\
A_1(t) T(p_0, 2t') &= T(p_0, 2t + 2t') A_1(t) \quad (5.14)
\end{align}
for all $t, t'$ in $i\mathbb{R} \times \mathbb{Z}$. If $G$ is of type $C_n$, then
\begin{align}
A_0(t) T(p_0, t) &= T(p_0, -t) A_0(t) \quad (5.15) \\
A_0(t) T(p_1, t') &= T(p_1, 2t + t') A_0(t) \quad (5.16) \\
A_0(t) T(p_j, t') &= T(p_j, t') A_0(t), \quad 2 \leq j \leq n - 1 \quad (5.17) \\
A_1(t) T(p_0, t') &= T(p_0, t + t') A_1(t) \quad (5.18)
\end{align}
for all $t, t'$ in $i\mathbb{R} \times \mathbb{Z}$. If $G$ is of type $D_n$, then
\begin{align}
A_0(t) T(p_0, t) &= T(p_0, -t) A_0(t) \quad (5.19) \\
A_0(t) T(p_2, t') &= T(p_2, t + t') A_0(t) \quad (5.20) \\
A_0(t) T(p_j, t') &= T(p_j, t') A_0(t), \quad j \neq 0, \ j \neq 2 \quad (5.21) \\
A_2(t) T(p_0, t') &= T(p_0, t + t') A_2(t) \quad (5.22)
\end{align}
for all $t, t'$ in $i\mathbb{R} \times \mathbb{Z}$.

Proof. In principle the proof is identical with that of Lemma 9. Equations (5.11) through (5.13) follow from the basic intertwining relations (3.1), (4.41), and (4.42) by straightforward computations. On the other hand, (5.14) follows from (3.1), (3.11) and (3.12). Equations (5.15) through (5.17) follow from (3.1), (4.44), and (4.45); (5.18) is a consequence of (3.1), (3.11), and (3.12). Equations (5.19) through (5.21) are implied by (3.1), (4.47), and (4.48). The final equation (5.20) results from (3.1), (3.11) and (3.12).

Lemma 25. For every $\lambda$ in $\Lambda$
\[ Y_1(\lambda) T(p_0, \lambda) Y_1(\lambda)^{-1} = T(p_0, \epsilon). \]

Proof. If $G$ is of type $B_n$ or $C_n$, then
\[ A_k(t) T(p_0, \lambda) = T(p_0, \lambda) A_k(t), \quad 2 \leq k \leq n - 1 \quad (5.23) \]
for all $t$ and $\lambda$. For if $\lambda = (t_1,\ldots,t_n)$ and $k \geq 2$, then by (3.1)
\[ A_k(t_{k+1} - t_k) T(p_0, 2t_1) = T(p_0, 2t_1) A_k(t_{k+1} - t_k) \]
when $G$ is of type $B_n$, and
\[ A_k(t_{k+1} - t_k) T(p_0, t_1) = T(p_0, t_1) A_k(t_{k+1} - t_k) \]
when $G$ is of type $C_n$. If $G$ is of type $D_n$, then
\[ A_k(t) T(p_0, t) = T(p_0, t) A_k(t) \]
for all $t$ and $\lambda$ when $k = 1$ or $3 < k < n - 1$. For if $\lambda = (t_1,\ldots,t_n)$ and $k = 1$ or $k \geq 3$, then
\[ h_{k+1} - t_k) T(p_0, 2h) = T(p_0, 2h) A_k(h_{k+1} - t_k) \]
by (3.1).

Suppose $X = (t_1,\ldots,t_n)$. Then $Y_1 = X_1X_2\cdots X_{n-1}$ where $X_k(\lambda) = A_{n-1}(-t_k) \cdots A_k(-t_k)$. If $G$ is of type $B_n$ or $C_n$, it follows from (5.23) that
\[ Y_1(\lambda) T(p_0, \lambda) Y_1(\lambda)^{-1} = X_1(\lambda) T(p_0, \lambda) X_1(\lambda)^{-1} \]
If $G$ is of type $B_n$, we have
\[ A_1(-t_1) T(p_0, 2t_1) A_1(t_1) = T(p_0, \epsilon) \]
by (5.14). On the other hand, if $G$ is of type $C_n$, then
\[ A_1(-t_1) T(p_0, t_1) A_1(t_1) = T(p_0, \epsilon) \]
by (5.18). Thus in either case
\[ X_1(\lambda) T(p_0, \lambda) X_1(\lambda)^{-1} = A_{n-1}(-t_1) \cdots A_2(-t_1) T(p_0, \epsilon) A_2(t_1) \cdots A_{n-1}(t_1) \]
and the right side of this equation reduces to $T(p_0, \epsilon)$ by (5.23).

Now suppose $G$ is of type $D_n$. Then
\[ Y_1(\lambda) T(p_0, \lambda) Y_1(\lambda)^{-1} = X_1(\lambda) X_2(\lambda) T(p_0, \lambda) X_2(\lambda)^{-1} X_1(\lambda)^{-1} \]
by (5.24). By (5.22)
\[ A_2(-t_2) T(p_0, t_1 + t_2) A_2(t_2) = T(p_0, t_1) \]
so that $X_x(\lambda) \ T(p_0, \lambda) \ X_x(\lambda)^{-1} = T(p_0, t_1)$ by (5.24). Thus

$$Y_1(\lambda) \ T(p_0, \lambda) \ Y_1(\lambda)^{-1} = X_1(\lambda) \ T(p_0, t_1) \ X_1(\lambda)^{-1}.$$ 

By (5.24), $A_1(-t_1) \ T(p_0, t_1) \ A_1(t_1) = T(p_0, t_1)$, and by (5.22),

$$A_x(-t_1) \ T(p_0, t_1) \ A_x(t_1) = T(p_0, \epsilon).$$ 

Therefore

$$X_1(\lambda) \ T(p_0, t_1) \ X_1(\lambda)^{-1} = A_{n-1}(-t_1) \cdots A_0(-t) \cdot T(p_0, \epsilon) \cdot A_0(t) \cdots A_{n-1}(t)$$

and this reduces to $T(p_0, \epsilon)$ by (5.24). Thus $Y_1(\lambda) \ T(p_0, \lambda) \ Y_1(\lambda)^{-1} = T(p_0, \epsilon)$ in each of the three cases.

**Lemma 26.** If $\lambda$ is an arbitrary character and $y$ an arbitrary element of $G^{(0)}$, then

$$Y_1(p_0\lambda) \ A(p_0, \lambda) \ Y_1(\lambda)^{-1} \cdot T(y, \lambda^{(1)})$$

$$= T(y, (p_0\lambda)^{(1)}) \cdot Y_1(p_0\lambda) \ A(p_0, \lambda) \ Y_1(\lambda)^{-1} \quad (5.25)$$

**Proof.** As in the proof of Theorem 6 it is sufficient to check that (5.25) holds on $V$, on $C = MA$, and for the basic reflections $p_j$ with $0 \leq j \leq n - 2$. When $y \in V$, $T(y, \lambda)$ is independent of $\lambda$ and commutes with all of the $A_j(t)$, $0 \leq j \leq n - 1$; hence (5.25) holds on $V$.

Now consider the case in which $y$ is an element of $C$ or $n \geq 3$ and $y$ is one of the basic reflections $p_j$ with $1 \leq j \leq n - 2$. Then by the proof of Theorem 6

$$Y_1(\lambda) \ T(y, \lambda) \ Y_1(\lambda)^{-1} = T(y, \lambda^{(1)}). \quad (5.26)$$

This also holds when $y = p_0$ by Lemma 25. Hence

$$T(y, \lambda) = Y_1(\lambda)^{-1} \ T(y, \lambda^{(1)}) \ Y_1(\lambda)$$

and conjugating by $A(p_0, \lambda)$, we find that

$$T(y, p_0\lambda) = A(p_0, \lambda) \ Y_1(\lambda)^{-1} \ T(y, \lambda^{(1)}) \ Y_1(\lambda) \ A(p_0, \lambda)^{-1}.$$ 

Now using (5.26), we see that

$$T(y, (p_0\lambda)^{(1)})$$

$$= Y(p_0\lambda) \ A(p_0, \lambda) \ Y_1(\lambda)^{-1} \cdot T(y, \lambda^{(1)}) \cdot Y_1(\lambda) \ A(p_0, \lambda)^{-1} \ Y_1(p_0\lambda)^{-1}$$

which is equivalent to (5.25).
Lemma 27. For every character $\lambda$

$$X_0(\lambda) T(p_j, \epsilon) = T(p_j, \epsilon) X_0(\lambda)$$

(5.27)

for $0 \leq j \leq n - 2$.

Proof. First suppose that $\lambda = (t, 0, \ldots, 0)$. Then it follows from Theorem 7 that

$$Y_1(p_0, \lambda) A(p_0, \lambda) Y_1(\lambda)^{-1} = C(t).$$

In addition

$$T(p_j, \lambda^{(1)}) = T(p_j, (p_0 \lambda)^{(1)}) = T(p_j, \epsilon)$$

for $0 \leq j \leq n - 2$. Hence, $C(t)$ commutes with $T(p_j, \epsilon)$ ($0 \leq j \leq n - 2$) by Lemma 26. By Lemma 10, the operator $D(t)$, which satisfies (5.1), commutes with $T(p_j, \epsilon)$. Thus, if $\lambda$ is any character, it follows from (5.2) that $X_0(\lambda)$ commutes with each of the operators $T(p_j, \epsilon)$ ($0 \leq j \leq n - 2$).

Lemma 28. Suppose $t = (s, m)$ lies in $i\mathbb{R} \times \mathbb{Z}$, and let $\chi(\cdot, t) = \chi(\cdot, s, m)$ be the character of $C$ which is given by

$$\chi(\cdot, t) = (0, \ldots, 0, t)$$

(5.28)

Let $c$ be an arbitrary element of $C$. Then

$$D(s, m) T(c, \epsilon) D(s, m)^{-1} = \chi(c, -s, -m) T(c, \epsilon)$$

(5.29)

if $m$ is even, and

$$D(s, m) T(c, \epsilon) D(s, m)^{-1} = \chi(c, -s, 1 - m)$$

(5.30)

if $m$ is odd.

Proof. Take $\lambda = (t, 0, \ldots, 0)$ and $\gamma = \epsilon$ in (5.25). Then, by Theorem 7, (5.25) reduces to the relation

$$C(t) T(c, \lambda^{(1)}) C(t)^{-1} = T(c, (p_0 \lambda)^{(1)}).$$

By (3.32), we can write this in the form

$$\lambda^{(1)}(c) C(t) T(c, \epsilon) C(t)^{-1} = (p_0 \lambda)^{(1)}(c) T(c, \epsilon).$$
By (3.23), \( \lambda^{(1)} = (0, \ldots, 0, t) \), and by (4.42), (4.45), and (4.48), \( (p_0 \lambda)^{(1)} = (0, \ldots, 0, -t) \). Thus, we have

\[
C(t) T(c, \varepsilon) C(t)^{-1} = \chi(c, -2t) T(c, \varepsilon). \tag{5.31}
\]

Now suppose \( m \) is even. Then it follows from (5.31) that

\[
C(-t/2)^2 C(t) T(c, \varepsilon) C(-t) C(t/2)^2 = \chi(c, -2t) \chi(c, t)^2 T(c, \varepsilon)
\]

Thus, \( T(c, \varepsilon) \) commutes with the operator \( J(t) \) defined by (4.3). From this it follows that \( T(c, \varepsilon) \) commutes with the operator \( E(t) \) defined by (4.9). By (4.12), \( D(t) = E(t) C(t/2) \). Hence

\[
D(t) T(c, \varepsilon) D(t)^{-1} = E(t) C(t/2) T(c, \varepsilon) C(-t/2) E(t)^{-1}
\]

which proves (5.29).

Next, suppose \( m \) is odd. Then it follows from (5.31) that

\[
C\left(\frac{-s}{2}, \frac{-m-1}{2}\right) C\left(\frac{-s}{2}, \frac{1-m}{2}\right) C(s, m) T(c, \varepsilon)
\]

\[
= \chi(c, -2s, -2m) \chi(c, s, m - 1) \chi(c, s, m + 1) \cdot T(c, \varepsilon)
\]

\[
\cdot C\left(\frac{-s}{2}, \frac{-m-1}{2}\right) C\left(\frac{-s}{2}, \frac{1-m}{2}\right) C(s, m)
\]

Thus, \( T(c, \varepsilon) \) commutes with the operator \( J(t) \) defined by (4.15) with \( F(t) = C(t) \). It follows that \( T(c, \varepsilon) \) commutes with the operator \( E(t) \) given by (4.16). By (4.17) \( D(s, m) = E(s, m) C(s/2, (m - 1)/2) \). Hence, (5.30) is implied, as above, by (5.31).

**Proof of (5.8).** As in the proof of Theorem 6 and also in the proof of Lemma 26, it suffices to check that (5.8) holds on \( V \), on \( C = MA \), and for the basic reflections \( p_j \) with \( 0 \leq j \leq n - 1 \). Suppose \( v \in V \). Then \( T(v, \lambda) \) is just right translation by \( v \) on \( L^2(V) \); hence \( T(v, \lambda) \) is independent of \( \lambda \) and commutes with each of the operators \( A_j(t), 0 \leq j \leq n - 1 \).
Therefore, $T(v, \lambda)$ commutes with $Y_1(\lambda)$, and by (5.25) and Theorem 7, it also commutes with $C(t)$ for every $t$ in $i\mathbb{R} \times \mathbb{Z}$. By Lemma 10, $T(v, \lambda)$ commutes with $D(t)$ and hence also with $X_0(\lambda)$ by (5.2). Thus, $T(v, \lambda)$ commutes with the normalizing operator $W(\lambda)$ given by (5.5). It follows that $R(v, \lambda) = R(v, \epsilon) = R(v, \lambda^{(1)})$ for all $v$ in $V$.

By Lemma 25 and (5.26)

$$Y_1(\lambda) T(p_j, \lambda) Y_1(\lambda)^{-1} = T(p_j, \epsilon)$$

for $0 \leq j \leq n - 2$. On the other hand

$$X_0(\lambda) T(p_j, \epsilon) X_0(\lambda)^{-1} = T(p_j, \epsilon), \quad 0 \leq j \leq n - 1$$

by Lemma 27. Hence, it follows from (5.5) and (5.6) that

$$R(p_j, \lambda) = R(p_j, \epsilon) = R(p_j, \lambda^{(1)})$$

for $0 \leq j \leq n - 2$.

Next suppose that $c$ is an element of the Cartan subgroup $C = MA$. Then

$$Y_1(\lambda) T(c, \lambda) Y_1(\lambda)^{-1} = T(c, \lambda^{(1)})$$

by (5.26). Suppose $\lambda = (t_1, \ldots, t_n)$ with $t_j$ in $i\mathbb{R} \times \mathbb{Z}$. Then by (3.23)

$$\lambda^{(1)} = \left(0, \ldots, 0, \sum_j t_j\right)$$

so that $\lambda^{(1)} = \chi(\cdot, \sum_j t_j)$ by (5.28). Thus, we have the relation

$$Y_1(\lambda) T(c, \lambda) Y_1(\lambda)^{-1} = \chi(\cdot, \sum_j t_j) T(c, \epsilon)$$

(5.34)

Now suppose $t_k = (s_k, m_k)$. If $m_k$ is even it follows from (5.29) that

$$D(t_k) Y_1(\lambda) T(c, \lambda) Y_1(\lambda)^{-1} D(t_k)^{-1} = \chi(\cdot, \sum_j t_j) \chi(c, -t_k) T(c, \epsilon).$$

On the other hand, if $m_k$ is odd, we have

$$D(t_k) Y_1(\lambda) T(c, \lambda) Y_1(\lambda)^{-1} D(t_k)^{-1} = \chi(\cdot, \sum_j t_j) \chi(c, -s, 1 - m_k) T(c, \epsilon)$$

by (5.30). Thus, if we set

$$F(\lambda) = D(t_1) D(t_2) \cdots D(t_n)$$
it follows that
\[ F(\lambda) \ Y_1(\lambda) \ T(c, \lambda) \ Y_1(\lambda)^{-1} F(\lambda)^{-1} = \chi(c, 0, N(\lambda)) \ T(c, \epsilon) \]

where \( N(\lambda) \) is the number of odd integers in the sequence \( m_1, m_2, \ldots, m_n \). Recall that \( N'(\lambda) = N(\lambda) \) if \( N(\lambda) \) is even and that \( N'(\lambda) = N(\lambda) - 1 \) if \( N(\lambda) \) is odd. Hence
\[
D(0, N'(\lambda)) F(\lambda) \ Y_1(\lambda) \ T(c, \lambda) \ Y_1(\lambda)^{-1} F(\lambda)^{-1} D(0, N'(\lambda))^{-1} = \chi(c, 0, N(\lambda)) \chi(c, 0, -N'(\lambda)) \ T(c, \epsilon)
\]

by (5.29). Since
\[
\chi(c, 0, N(\lambda)) \chi(c, 0, -N'(\lambda)) = \chi(c, 0, N(\lambda) - N'(\lambda))
\]
and \( X_0(\lambda) = D(0, N'(\lambda))F(\lambda) \), it follows from the above and (5.4) that
\[
W(\lambda) \ T(c, \lambda) \ W(\lambda)^{-1} = T(c, \lambda^{(0)})
\]
for all \( c \) in \( C \) and all character \( \lambda \). This completes the proof of (5.8) and hence the proof of Theorem 8.

6. Some Results on Several Complex Variables

In this section, we prove some extensions of the results in [2, Section 8] which will be used in the analytic continuation of the normalized principal series.

**Theorem 9.** Let \( \gamma_1, \ldots, \gamma_m \) be real vectors spanning \( \mathbb{R}^n \) and \( \mathcal{F} \) the tube in \( \mathbb{C}^n \) whose base is the convex set generated by all vectors of the form \( u\gamma_j (-1 < u < 1) \). Let \( F \) be defined and continuous on \( i\mathbb{R}^n \). Suppose that for each \( s \) in \( i\mathbb{R}^n \) and each \( \gamma_j \), the function
\[
w \to F(s + w\gamma_j)
\]
initially defined for \( \text{Re}(w) = 0 \), has an analytic continuation into the strip
\[-1 < \text{Re}(w) < 1 \quad \text{and} \quad |F(s + w\gamma_j)| \leq 1\]
there. Then \( F \) extends uniquely to an analytic function in \( \mathcal{F} \) which is necessarily bounded by 1.
Proof. Let \( \epsilon_1, \ldots, \epsilon_m \) denote the standard basis for \( \mathbb{C}^m \) and \( T \) the linear map of \( \mathbb{C}^m \) onto \( \mathbb{C}^n \) such that \( T\epsilon_j = \gamma_j \) for \( 1 \leq j \leq m \). For \( y \) in \( \mathbb{R}^n \), set \( g(iy) = F(iTy) \). Then \( w \mapsto g(iy + we_j) \) has an analytic continuation into the strip \(-1 < \text{Re} w < 1\), and \( |g(iy + we_j)| \leq 1 \) there. It follows from Lemma 21 of [2] that \( g \) has an analytic continuation into the tube \( T \) consisting of all points \( z = (z_1, \ldots, z_m) \) such that

\[
|\text{Re} z_1| + |\text{Re} z_2| + \cdots + |\text{Re} z_m| < 1
\]

and that \( |g(z)| \leq 1 \) in \( T \). Since \( F = T(F) \) and \( g(iy) = g(iy') \) whenever \( y, y' \in \mathbb{R}^n \) and \( Ty = Ty' \), it follows from Lemma 22 of [2] that \( F \) has an analytic continuation with the stated properties.

In our application we need a slightly more general result dealing with functions of polynomial growth.

Suppose \( \mathcal{F} \) is a cone in \( \mathbb{C}^n \) whose base (base \( (\mathcal{F}) \)) is an open subset of \( \mathbb{R}^n \), and let \( F \) be a function defined on \( \mathcal{F} \). We say that \( F \) is of polynomial growth in \( \mathcal{F} \) if there is an integer \( N \geq 0 \) and for each compact subset \( E \) of base \( (\mathcal{F}) \) a constant \( K_E \) such that

\[
|F(s_1, \ldots, s_n)| = K_E \left(1 + \sum_j |\text{Im}(s_j)|\right)^N
\]

for all \((s_1, \ldots, s_n)\) in the tube over \( E \). To simplify the notation, we shall adopt the convention that when \( s \) is the point \((s_1, \ldots, s_n)\) of \( \mathbb{C}^n \), then

\[
|\text{Im}(s)| = \sum_j |\text{Im}(s_j)|.
\]

Corollary. With \( \gamma_1, \ldots, \gamma_m \) and \( \mathcal{F} \) as before, let \( F \) be defined and continuous on \( i\mathbb{R}^n \). Suppose that for each \( s \) in \( i\mathbb{R}^n \) and each \( \gamma_j \), the function

\[
w \mapsto F(s + w\gamma_j)
\]

initially defined for \( \text{Re}(w) = 0 \), has an analytic continuation into the strip \(-1 < \text{Re}(w) < 1\). Assume there is an integer \( N \geq 0 \) and for each \( \epsilon \) with \( 0 < \epsilon < 1 \) a constant \( K_\epsilon > 0 \) such that for every \( j \)

\[
|F(s + w\gamma_j)| \leq K_\epsilon (1 + |\text{Im}(s + w\gamma_j)|)^N
\]

for all \( w \) in the strip \(-1 + \epsilon \leq \text{Re}(w) \leq 1 - \epsilon \). Then \( F \) extends uniquely to a function, complex-analytic and of polynomial growth in \( \mathcal{F} \).

Proof. Let \( 0 < \epsilon < 1 \), \( d = 1 - \epsilon \), and set

\[
F_d(s) = F(ds), \quad s \in i\mathbb{R}^n.
\]
Then for each $j$, $w 	o F_d(s + w\gamma_j)$ has an analytic continuation such that

$$|F_d(s + w\gamma_j)| \leq K_s(1 + |\text{Im}(s + w\gamma_j)|)^N$$

for $-1 < \text{Re}(w) < 1$. Suppose $\gamma_j = (\gamma_{j1}, \ldots, \gamma_{jm})$ and set

$$M = 1 + \max_k \left(\sum_j |\gamma_{jk}|\right).$$

Define $H$ for any complex $s_1, \ldots, s_n$ by

$$H(s_1, \ldots, s_n) = 2^n(M + s_1)(M + s_2) \cdots (M + s_n).$$

Then it is easy to check that

$$1 + |\text{Im}(s_k)| < 2 |M + s_k| < 4M(1 + |\text{Im}(s_k)|)$$

and hence that

$$1 + |\text{Im}(s)| < |H(s)| < (4M)^n(1 + |\text{Im}(s)|)^n$$

for all $s$ in $F$. Thus the function

$$G_d(s) = \frac{F_d(s)}{K_s(H(s))^N}$$

is defined and continuous on $i\mathbb{R}^n$; moreover, for each $s$ in $i\mathbb{R}^n$ and each $j$, $w \to G_d(s + w\gamma_j)$ is analytic and

$$|G_d(s + w\gamma_j)| \leq 1$$

in the strip $-1 < \text{Re}(w) < 1$. Since

$$F_d(s) = K_s G_d(s)(H(s))^N$$

it follows from the theorem that $F_d$ extends to a function analytic in $F$ such that

$$|F_d(s)| \leq K_s(4M)^n(1 + |\text{Im}(s)|)^nN$$

for all $s$ in $F$. If $0 < d_1 < d_2 < 1$ it follows from (6.3) and analyticity that

$$F_{d_1}(s) = F_{d_2}\left(\frac{d_1}{d_2} s\right)$$

(6.5)
for all $s$ in $\mathcal{F}$. If $s_0 \in \mathcal{F}$ there is a neighborhood $U$ of $s_0$ and $d_1$ such that $s/d_1 \in \mathcal{F}$ for all $s$ in $U$. If $d_2$ also has this property and $d_1 < d_2$, then

$$F_{d_1}\left(\frac{s}{d_1}\right) = F_{d_2}\left(\frac{d_1}{d_2}\left(\frac{s}{d_1}\right)\right) = F_{d_2}\left(\frac{s}{d_2}\right)$$

for all $s$ in $U$ by (6.5). Thus we may extend $F$ to an analytic function in $\mathcal{F}$ by setting

$$F(s) = F_{d}\left(\frac{s}{d}\right)$$

whenever $s \in \mathcal{F}, 0 < d < 1$, and $s/d \in \mathcal{F}$.

Now suppose $E$ is a compact subset of base ($\mathcal{F}$). Then there is a positive $\epsilon < 1$ such that $E$ is contained in the convex set spanned by all points of the form $u_\gamma j$ with $-d < u < d$, $d = 1 - \epsilon$. For $s$ in this convex set, we have

$$F(s) = F_{d}\left(\frac{s}{d}\right)$$

and setting $K_E = d^{-1}K_n(4M)^n$ and $N' = nN$, we see from (6.4) that

$$|F(s)| \leq K_E(1 + |\text{Im}(s)|)^{N'}$$

for all $s$ in the tube over $E$.

7. Analytic Continuation of the Normalized Principal Series

In this section, we construct an analytic continuation of the normalized principal series.

For this purpose some additional notation will be useful. If $\lambda$ is the quasi-character of $B = MAN$ which corresponds to the $n$-tuple $(t_1, \ldots, t_n)$ with $t_j = (s_j, m_j)$ in $\mathbb{C} \times \mathbb{Z}$ (cf. Section 3), we shall write $\lambda = (t_1, \ldots, t_n)$ or

$$\lambda = (s, m)$$

where $s = (s_1, \ldots, s_n)$ and $m = (m_1, \ldots, m_n)$ depending on which is more convenient.

We shall specify an (open) symmetric tube $\Omega$ in $\mathbb{C}^n$ and show that the maps

$$s \to R(y, s, m), \quad y \in G, \quad m \in \mathbb{Z}^n$$
which are initially defined on \( i\mathbb{R}^n \) have analytic continuations into \( \Omega \) with properties analogous to those established in [2] for \( SL(n, \mathbb{C}) \). The tube given in our construction for \( SL(n, \mathbb{C}) \) may be characterized as the tube in \( \mathbb{C}^{n-1} \) whose base is the interior of the convex hull of the set of all roots. In general it is not as large as it should be. Although the proper size of the tube is not entirely clear, our construction could be modified in the case \( n \geq 4 \), to yield an analytic continuation into the larger tube whose base is the convex set generated by all points of the form

\[
\sum_{i=1}^{j} x_i \beta_i , \quad -1 < x_i < 1
\]

where \( \beta_1, ..., \beta_j \) vary over all possible systems of mutually orthogonal roots. Alternatively, if \( \{\alpha_1, ..., \alpha_{n-1}\} \) is a system of basic roots, the base of the tube just described is the interior of the convex set generated by all points of the form

\[
\rho(\alpha_k + \alpha_{k+2} + \cdots + \alpha_{n-1})
\]

where \( 1 \leq k \leq n - 1 \), \( n - k - 1 \) is even, and \( \rho \) varies over the Weyl group.

In the present situation, in which \( G \) is of type \( B_n, C_n, \) or \( D_n \), \( \Omega \) will denote the tube in \( \mathbb{C}^n \) whose base is the interior of the convex set generated by all points of the form

\[
\rho \sum_{j} \alpha_{k+2j}, \quad 0 \leq j \leq \frac{1}{2}(n - k - 1)
\]

where \( 1 \leq k \leq n - 1 \), \( n - k - 1 \) is even, \( \alpha_1, ..., \alpha_{n-1} \) are given by (3.9), and \( \rho \) varies over \( \mathbb{W} \). It is easy to see from (3.12), (4.42), (4.45), and (4.48) that

\[
\{ \rho \alpha_{n-1}; \rho \in \mathbb{W} \} = \{ \pm (\epsilon_k \pm \epsilon_j); 1 \leq j < k \leq n \}.
\]

Since this set contains a basis for \( \mathbb{R}^n \), e.g., \( \{\epsilon_1 + \epsilon_2, \alpha_1, ..., \alpha_{n-1}\} \), it follows that \( \Omega \) is an open tube in \( \mathbb{C}^n \); moreover, \( \Omega \) is invariant under the map \( s \mapsto -s \), and hence symmetric as defined in Section 2.

It should be noted that in contrast to the situation described above \( \Omega \) in general cannot be characterized as the tube whose base is the convex set generated by all points of the form

\[
\sum_{i=1}^{j} x_i \beta_i , \quad -1 < x_i < 1
\]
where $\beta_1, \ldots, \beta_j$ vary over all possible systems of mutually orthogonal roots. This is already the case when $G$ is of type $C_2$. For then $\alpha_0 = 2\epsilon_1$ and base $(\Omega)$ is the interior of the "square" with vertices $\epsilon_1 + \epsilon_2$, $\epsilon_2 - \epsilon_1$, $-\epsilon_1 - \epsilon_2$, and $\epsilon_1 - \epsilon_2$.

Next, let $\Lambda^*$ denote the set of all quasi-characters $\lambda = (s, m)$ for which $s \in \Omega$.\(^1\)

Our main result may now be formulated as follows.

**Theorem 10.** Let $G$ be a complex classical group of type $B_n$, $C_n$, or $D_n$. When $G$ is of type $D_n$, assume $n \geq 3$ and otherwise that $n \geq 2$. If $y \in G$ and $\lambda \in \Lambda$, let $R(y, \lambda)$ denote the operator of the normalized principal series given by (5.6). Then for each $y$ in $G$ and $m$ in $\mathbb{Z}^n$, the operator valued function

$$s \mapsto R(y, s, m)$$

initially defined on $i\mathbb{R}^n$ can be extended to $\Omega$ in such a way that

1. it is complex analytic in $\Omega$
2. for each $\lambda$ in $\Lambda^*$

$$y \mapsto R(y, \lambda)$$

is a continuous uniformly bounded representation of $G$; moreover, there is an integer $N \geq 0$ and for each compact subset $E$ of base $(\Omega)$ a constant $K_E$ such that

$$\sup_{y \in G} \| R(y, s, m) \| \leq K_E (1 + |\text{Im}(s)|)^N (1 + |m|)^N$$

(7.1)

for all $s$ in the tube over $E$ where $|\text{Im}(s)|$ is given by (6.2), and $|m| = |m_1| + \cdots + |m_n|$.

3. $R(\cdot, s, m)$ is completely irreducible for all $s$ in the complement of some negligible subset of $\Omega$ and all $m$ in $\mathbb{Z}^n$.
4. $R(p\lambda) = R(\cdot, \lambda)$ for all $p$ in $\mathbb{W}$ and all $\lambda$ in $\Lambda^*$.
5. $R(y^{-1}, s, m)^* = R(y, -s, m)$ for all $y$ in $G$ and all $(s, m)$ in $\Lambda^*$.

The following lemma is crucial.

**Lemma 30.** Let $m_0, m_1, m_2$ be integers and $c$ a real number. Suppose $1 \leq k \leq n - 1$. Then the map

$$w \mapsto A_k(w, m_0) T(p_k, ic + 2w, m_1) A_k(-w, m_2), \quad \text{Re}(w) = 0$$

\(^1\) The normalized complementary series given by $\Omega$ and theorem 12 should be compared to Theorem 9 of [10].
has an analytic continuation into the strip $-1 < \text{Re}(w) < 1$; moreover, if $w = u + iv$ where $1 - \epsilon < u < 1$ and $-\infty < v < \infty$, then

$$
\| A_k(w, m_0) T(p_k, ic + 2w, m_1) A_k(-w, m_2) \| \\
\leq K_\epsilon (1 + |m_1| + |m_2| + |v| + |\epsilon|).
$$

(7.2)

**Remark.** With minor changes this could be formulated more abstractly and proved for an arbitrary complex semi-simple Lie group.

**Proof of Lemma 30.** From the remarks preceding Lemma 32 of [3], the lemma itself, and Lemma 25 of [3], it is easy to see that it is enough to prove the result for $SL(2, \mathbb{C})$. Although this was done in [2], we shall sketch another and better proof here.

Let $T(p) = T(p, \epsilon)$. Then it follows from the definition of the principal series for $SL(2, \mathbb{C})$ that for any $t$ in $i\mathbb{R} \times \mathbb{Z}$

$$
T(p, t) = B(t) T(p)
$$

(7.3)

where $B(t)$ is given by (4.18). It follows from the analogue of (3.29) that

$$
A(t) B(t) T(p) = B(-t) T(p) A(t)
$$

and hence that

$$
T(p) A(t) T(p)^{-1} = B(t) A(t) B(t)
$$

(7.4)

Therefore, by (7.3) and (7.4), we have

$$
A(t_0) T(p, t_1) A(t_2)
$$

$$
= A(t_0) B(t_1) T(p) A(t_2) T(p)^{-1} \cdot T(p)
$$

$$
= A(t_0) B(t_1 + t_2) A(t_2) B(t_2) \cdot T(p)
$$

$$
= A(t_0 - t_1 - t_2) \cdot A(t_1 + t_2) B(t_1 + t_2) A(t_2) B(t_2) \cdot T(p)
$$

for all $t_0, t_1, t_2$ in $i\mathbb{R} \times \mathbb{Z}$. Now take $t_0 = (w, m_0)$, $t_1 = (ic + 2w, m_1)$, and $t_2 = (-w, m_2)$. Then, if $U = A(-ic, m_0 - m_1 - m_2)$, it follows that

$$
A(w, m_0) T(p, ic + 2w, m_1) A(-w, m_2)
$$

$$
= U \cdot A(ic + w, m_1 + m_2) B(ic + w, m_1 + m_2) A(-w, m_2) B(-w, m_2) \cdot T(p)
$$

(7.5)

whenever $\text{Re}(w) = 0$. 

By Lemma 12, $AB$ is an admissible map of $i\mathbb{R} \times \mathbb{Z}$ into $\mathcal{U}(L^2(\mathbb{C}))$. Hence, since $U$ and $T(p)$ are unitary, it follows from (7.5) that

$$w \to A(w, m_0) T(p, ic + 2w, m_1) A(-w, m_0)$$

has an analytic continuation into the strip $-1 < \Re(w) < 1$. The bound given in (7.2) follows from (7.5) and the estimates obtained in the proof of Lemma 12.

The result just established may be extended in the following way.

**Lemma 31.** Let $(s, m) \in i\mathbb{R}^n \times \mathbb{Z}^n$, and suppose

$$\gamma = \alpha_k + \alpha_{k+2} + \cdots + \alpha_{n-1}$$

(7.6)

where $1 \leq k \leq n - 1$ and $n - k - 1$ is even. Let $R_k(\gamma, s, m)$ be the member of the partially normalized series for $G$ given by (3.22). Then the map

$$w \to R_k(p_{n-1}, s + w\gamma, m)$$

has an analytic continuation into the strip $-1 < \Re(w) < 1$; moreover, there is an integer $N \geq 0$ and for each positive $\epsilon < 1$ a constant $K_\epsilon$ such that

$$\| R_k(p_{n-1}, s + w\gamma, m) \| \leq K_\epsilon (1 + |\Im(s + w\gamma)| + |m|)^N$$

(7.7)

for all $w$ in the strip $-1 + \epsilon \leq \Re(w) \leq 1 - \epsilon$.

**Proof.** First suppose that $k = n - 1$. Then $\gamma = \alpha_{n-1}$, $w\gamma = -w\epsilon_{n-1} + w\epsilon_n$, and

$$s + w\gamma = \sum_{j<n-1} s_j \epsilon_j + (s_{n-1} - w) \epsilon_{n-1} + (s_n + w) \epsilon_n.$$

Hence, it follows from (3.14), (3.22), and (3.28) that

$$R_{n-1}(p_{n-1}, s + w\alpha_{n-1}, m)$$

$$= A_{n-1}(-s_{n-1} + w, -m_{n-1}) T(p_{n-1}, s_n - s_{n-1} | 2w, m_n \quad m_{n-1})$$

$$\cdot A_{n-1}(s_{n-1} - w, m_{n-1}).$$

(7.8)

From this and Lemma 30 it follows that $w \to R_{n-1}(p_{n-1}, s + w\alpha_{n-1}, m)$ has an analytic continuation into the strip $-1 < \Re w < 1$. By (7.2)

$$\| R_{n-1}(p_{n-1}, s + w\alpha_{n-1}, m) \| \leq K_\epsilon (1 + |m_n - m_{n-1}| + |m_{n-1}| + |\Im(w - s_{n-1})| + |s_n + s_{n-1}|)$$
whenever $-1 + \epsilon \leq \Re(w) \leq 1 - \epsilon$. Since

$$|s_n + s_{n-1}| \leq |\Im(s_n + w)| + |\Im(s_{n-1} - w)|,$$

it follows that

$$\|R_{n-1}(p_{n-1}, s + \omega \alpha_{n-1}, m)\| \leq 2K_1(1 + |\Im(s + \omega \alpha_{n-1})| + |m|)$$

for all $w$ in the strip $-1 + \epsilon \leq \Re(w) \leq 1 - \epsilon$.

Now assume $n \geq 4$ and that $k = n - 3$. Let $y \in G$ and suppose $\lambda$ is a unitary character. Then $Y_{n-3} = X_{n-3} X_{n-2} X_{n-1}$, and it follows from (3.22) that

$$R_{n-3}(y, \lambda) = X_{n-3}(\lambda) X_{n-2}(\lambda) R_{n-1}(y, \lambda) X_{n-2}(\lambda)^{-1} X_{n-3}(\lambda)^{-1}.$$  (7.9)

By (4.31) and (4.32)

$$X_j(s, m) = J_j(-s_j, -m_j) = A_{n-1}(-s_j, -m_j) \cdots A_j(-s_j, -m_j)$$

for $1 \leq j \leq n - 1$. Since

$$w \gamma = -\omega \epsilon_{n-3} + \omega \epsilon_{n-2} - \omega \epsilon_{n-1} + \omega \epsilon_n$$

it follows that

$$X_{n-3}(s + w \gamma, m) X_{n-3}(s + w \gamma, m)$$

$$= J_{n-3}(-s_{n-3} + w, -m_{n-3}) J_{n-2}(-s_{n-2} - w, -m_{n-2})$$

$$= J_{n-3}(-s_{n-3} + w, -m_{n-3}) J_{n-2}(-s_{n-2} - w, -m_{n-2})$$

$$A_{n-3}(-s_{n-2} + w, m_{n-2}).$$  (7.10)

To simplify the notation, set

$$Z_{n-3}(s + w \gamma, m)$$

$$= J_{n-3}(-s_{n-3} + w, -m_{n-3}) J_{n-3}(-s_{n-2} - w, -m_{n-2}).$$  (7.11)

Now recall that $A_{n-3}(t)$ commutes with $A_{n-1}(t')$ for all $t, t'$ in $i\mathbb{R} \times \mathbb{Z}$. In addition, $A_{n-3}(t)$ commutes with $T(p_{n-1}, t')$ for all $t, t'$ by (3.30). Thus it follows from (7.9), (7.10), (7.11), and (7.8) that

$$R_{n-3}(p_{n-1}, s + w \gamma, m)$$

$$= Z_{n-3}(s + w \gamma, m) R_{n-1}(p_{n-1}, s + w \gamma, m) Z_{n-3}(s + w \gamma, m)^{-1}. $$  (7.12)

By Lemma 16 and computations similar to those used above for the case
$k = n - 1$, $w \to Z_{n-3}(s + w\gamma, m)$ and its reciprocal have analytic continuations into the strip $-1 < \Re(w) < 1$ with bounds of the form

$$K_r(1 + |\Im(s + w\gamma)| + |m|^N)$$

in each substrip: $-1 + \epsilon \leq \Re(w) \leq 1 - \epsilon$. We also have

$$R_{n-1}(p, s + w\gamma, m) = R_{n-1}(p, s + w\varphi_{n-1}, m).$$

The proof of the lemma for the case $n \geq 4$, $k = n - 3$ follows from these observations, (7.12), and what we have already shown regarding the map $w \to R_{n-1}(p_{n-1}, s + w\varphi_{n-1}, m)$.

Next we consider the final case:

$$n \geq 6, \quad k = n - 2l - 1, \quad 5 \leq 2l + 1 \leq n - 1.$$ 

The argument here is similar to that used for the preceding case but somewhat more complicated. Let

$$V_k = (X_k X_{k+1}) \cdots (X_{n-3} X_{n-2})$$

$$= \prod_{j=0}^{l-1} X_{k+2j} X_{k+2j+1}$$

(7.13)

Then $Y_k = V_k X_{n-1}$, and

$$R_k(y, \lambda) = V_k(\lambda) R_{n-1}(y, \lambda) V_k(\lambda)^{-1}. \quad (7.14)$$

For $\lambda = (t_1, \ldots, t_n)$ with $t_j$ in $i\mathbb{R} \times \mathbb{Z}$ set

$$Z_{k+2j}(\lambda) = J_{k+2j}(-t_{k+2j}) J_{k+2j}(-t_{k+2j+1}).$$

(7.15)

Then $X_{k+2j}(\lambda) X_{k+2j+1}(\lambda) = Z_{k+2j}(\lambda) A_{k+2j}(t_{k+2j})$ and hence

$$V_k(\lambda) = \prod_{j=0}^{l-1} Z_{k+2j}(\lambda) A_{k+2j}(t_{k+2j}).$$

Since $A_{k+2j}(t)$ commutes with $J_i(t')$ whenever $i \geq k + 2j + 2$, it follows that

$$V_k(\lambda) = \left(\prod_{j=0}^{l-1} Z_{k+2j}(\lambda)\right) \left(\prod_{j=0}^{l-1} A_{k+2j}(t_k + 2j)\right).$$

(7.16)

In addition, each $A_{k+2j}(t_k + 2j)$ commutes with $X_{n-1}(\lambda)$ and $T(p_{n-1}, \lambda)$. 


Now let
\[ W_k(\lambda) = \prod_{j=0}^{l-1} Z_{k+2j}(\lambda). \]  
(7.17)

Then it follows from (7.14), (7.16), and the remarks above that
\[ R_k(p_{n-1}, \lambda) = W_k(\lambda) R_{n-1}(p_{n-1}, \lambda) W_k(\lambda)^{-1}. \]  
(7.18)

Since \( \gamma = \sum_{j=0}^{l} -\epsilon_{k+2j} + \epsilon_{k+2j+1} \), we have
\[ Z_{k+2j}(s + w\gamma, m) = J_{k+2j}(-s_{k+2j} + w, -m_{k+2j}) J_{k+2j}(-s_{k+2j+1} - w, -m_{k+2j+1}) \]
by (7.15). By Lemma 16, \( J_{k+2j} \) and \( J_{k+2j}^{-1} \) are admissible. From this it follows that \( w \rightarrow Z_{k+2j}(s + w\gamma, m) \) and its reciprocal have analytic continuations into the strip \(-1 < \Re(w) < 1\) with bounds of the form
\[ K_e(1 + |\Im(s + w\gamma)| + |m|)^N \]
in each of the strips \(-1 + \epsilon < \Re(w) < 1 - \epsilon\). By (7.17) the same is true of the function \( w \rightarrow W_k(s + w\gamma, m) \), and its reciprocal. We also have
\[ R_{n-1}(p_{n-1}, s + w\gamma, m) = R_{n-1}(p_{n-1}, s + w\alpha_{n-1}, m). \]

Since \( w \rightarrow R_{n-1}(p_{n-1}, s + w\alpha_{n-1}, m) \) has an appropriate analytic continuation, it follows from (7.18) that
\[ w \rightarrow R_k(p_{n-1}, s + w\gamma, m) \]
has analytic continuation with the desired properties.

Next we need an additional observation that could have been made in connection with Lemma 10.

**Lemma 32.** Let \( D \) be the admissible map given by the construction of Lemma 10 which satisfies (5.1). Then \( D^{-1} \) is also admissible.

**Proof.** The only difficulty here is that generally \( D(s, m)^{-1} \neq D(-s, -m) \). However, when \( m \) is an even integer
\[ D(s, m) = E_m \exp(\frac{1}{2}L(s, m)) C \left( \frac{s}{2}, -\frac{m}{2} \right) \]
by (4.9) and (4.12). When \( m \) is an odd integer

\[
D(s, m) = E_m \exp(\frac{1}{2} L(s, m)) C \left( \frac{s}{2}, \frac{m-1}{2} \right)
\]

by (4.16) and (4.17). In both cases \( E_m \) is a unitary operator independent of \( s \), and the factors in the product defining \( D \) commute with one another. In the first case

\[
D(s, m)^{-1} = E_m^{-1} \exp(\frac{1}{2} L(-s, -m)) C \left( \frac{-s}{2}, \frac{-m}{2} \right)
\]

and in the second

\[
D(s, m)^{-1} = E_m^{-1} \exp(\frac{1}{2} L(-s, -m)) C \left( \frac{-s}{2}, \frac{1-m}{2} \right).
\]

The admissibility of \( D^{-1} \) follows from these formulas and commutativity.

The next result in the main step in the proof of Theorem 10.

**Lemma 33.** Let \( (s, m) \in i\mathbb{R} \times \mathbb{Z}^n \) and suppose that

\[
\gamma = \rho (\alpha_k + \alpha_{k+2} + \cdots + \alpha_{n-1})
\]

where \( 1 < k < n-1, n-k-1 \) is even, and \( \rho \in \mathbb{W} \). Then

\[
\omega \rightarrow R(p_{n-1}, s + w\gamma, m)
\]

has an analytic continuation into the strip \(-1 < \Re \omega < 1\) with a bound of the form (7.7).

**Proof.** By Theorem 8

\[
R(p_{n-1}, s + w\gamma, m) = R(p_{n-1}, p^{-1}s + wp^{-1}\gamma, p^{-1}m)
\]

when \( \Re(\omega) = 0 \). Thus it is enough to consider the case in which \( \gamma \) is given by (7.19) with \( p = 1 \). By (5.5) and (5.6)

\[
R(y, \lambda) = X_0(\lambda) R_1(y, \lambda) X_0(\lambda)^{-1}
\]

(7.20)

for any \( y \) in \( G \) and any unitary character \( \lambda \), \( R_1(\cdot, \lambda) \) being the member of the partially normalized series given by the equation

\[
R_1(y, \lambda) = Y_1(\lambda) T(y, \lambda) Y_1(\lambda)^{-1}.
\]
Let \( Y_k' = \prod_{j<k} X_j \). Then \( Y_1 = Y_k' Y_k \) and we can write

\[
R_1(y, \lambda) = Y_k'(\lambda) R_k(y, \lambda) Y_k'(\lambda)^{-1}.
\]

(7.21)

Let \( l = \frac{1}{2}(n - k - 1) \). Then since

\[
s + w\gamma = \sum_{j<k} s_j \epsilon_j + \sum_{0 < j \leq l} -w\epsilon_{k+2j} + w\epsilon_{k+2j+1}
\]

it follows from (7.21) and (3.17) that

\[
R_1(p_{n-1}, s + w\gamma, m) = Y'(s, m) R_k(p_{n-1}, s + w\gamma, m) Y'(s, m)^{-1}.
\]

Since \( Y'(s, m) \) is a fixed unitary operator, it now follows from Lemma 31, that

\[
w \rightarrow R_1(p_{n-1}, s + w\gamma, m)
\]

has an analytic continuation into the strip \(-1 < \text{Re} \, w < 1\) with a bound of the form (7.7). This being the case, it is enough, by (7.20), to show that \( w \rightarrow X_0(s + w\gamma, m) \) and \( w \rightarrow X_0(s + w\gamma, m)^{-1} \) have analytic continuations with the desired properties.

Since \( D \) and \( D^{-1} \) are admissible, the map

\[
w \rightarrow D(s_{k+2j} \rightarrow w, m_{k+2j}), D(s_{k+2j+1} + w, m_{k+2j+1})
\]

and its reciprocal have analytic continuations with bounds of the form (7.7). Thus, if we set

\[
Q(s, m) = \prod_{i=0}^{l} D(s_{k+2j}, m_{k+2j}), D(s_{k+2j+1}, m_{k+2j+1})
\]

it follows that

\[
w \rightarrow Q(s + w\gamma, m)
\]

and its reciprocal have analytic continuations into the strip \(-1 < \text{Re} \, w < 1\) with bounds of the form (7.7). Next, let

\[
P(s, m) = \prod_{j<k} D(s_j, m_j).
\]

Then, by (5.2), we have

\[
X_0(s + w\gamma, m) = D(0, N') P(s, m) Q(s + w\gamma, m)
\]
so that \( w \to X_0(s + w\gamma, m) \) and its reciprocal have analytic continuations with the desired properties.

**Proof of Theorem 10.** Since

\[
1 + |\text{Im}(s)| + |m| \leq (1 + |m|)(1 + |\text{Im}(s)|)
\]

it follows from Lemma 33, that there is an integer \( N > 0 \) and a constant \( K_\epsilon (0 < \epsilon < 1) \) such that

\[
||R(p_{n-1}, s + w\gamma, m)|| \leq K_\epsilon(1 + |m|)^N(1 + |\text{Im}(s)|)^N
\]

for all \( w \) in the strip \(-1 + \epsilon \leq \text{Re}(w) \leq 1 - \epsilon\). Hence, it follows from the corollary to Theorem 9 that

\[
s \to R(p_{n-1}, s, m), \quad s \in i\mathbb{R}^n
\]

extends uniquely to a function analytic in \( \Omega \); moreover, the bounds obtained in the proof of the corollary show that there is a larger integer \( N > 0 \) and for each compact subset \( E \) of base \( \Omega \), a constant \( K_E \) such that

\[
||R(p_{n-1}, s, m)|| \leq K_E(1 + |\text{Im}(s)|)^N(1 + |m|)^N
\]

(7.22)

for all \( s \) in the tube over \( E \).

Now recall that \( G^{(0)} \) is the parabolic subgroup of \( G \) which is generated by the minimal parabolic subgroup \( M_{AV} \) and the basic reflections \( p_j \) with \( 0 \leq j \leq n - 2 \). If \( x \) and \( y \) are any elements of \( G^{(0)} \) and \( s \in i\mathbb{R}^n \), it follows from (5.3), (5.4), and (5.8) that

\[
R(xp_{n-1}y, s, m) = R(x, 0, m) R(p_{n-1}, s, m) R(y, 0, m).
\]

Thus, for each element \( a \) of \( G^{(0)}p_{n-1}G^{(0)} \), the function \( s \to R(a, s, m) \) has an analytic continuation into \( \Omega \), and if \( a = xp_{n-1}y \), then

\[
||R(a, s, m)|| = ||R(p_{n-1}, s, m)||
\]

(7.23)

because \( R(x, 0, m) \) and \( R(y, 0, m) \) are unitary. It is an elementary consequence of Bruhat's lemma [8] that

\[
G = (G^{(0)}p_{n-1}G^{(0)})^k
\]

for some integer \( k > 0 \). Having made these observations, we may now apply Theorem 4 to obtain statements (1) and (2) of the present theorem. The inequality (7.1) is a consequence of (2.2), (7.22), and (7.23).
From the known facts about the principal series and Theorem 3 we obtain the fact that for a given \( m \) in \( \mathbb{Z}^n \), \( R(\cdot, s, m) \) is completely irreducible for all \( s \) in the complement of some negligible subset of \( \mathcal{O} \). Since \( \mathbb{Z}^n \) is countable and a countable union of negligible sets is again negligible (cf. Lemma 1), this implies statement (3) of the theorem.

Statement (4) follows from analyticity and equation (5.7) of Theorem 8, and (5) is a direct consequence of Lemma 4. This completes the proof of Theorem 10.

Our next goal is to identify the representations \( R(\cdot, \lambda) (\lambda \in \mathbb{A}^*) \) which are unitary or similar to unitary representations. At the same time, we shall determine when two of the representations \( R(\cdot, \lambda_1) \) and \( R(\cdot, \lambda_2) \) are similar. Although the results are analogous to those we proved earlier for \( SL(n, \mathbb{C}) \), the proofs are necessarily different. The difference results from the fact that, unlike the situation for \( SL(n, \mathbb{C}) \), the restrictions of the \( R(\cdot, \lambda) \) to \( G^{(0)} \) are generally reducible. Fortunately, the results do not depend on irreducibility and follow from elementary properties of the characters of the representations \( R(\cdot, \lambda) (\lambda \in \mathbb{A}^*) \).

On the Cartan subgroup \( C \) of \( G \) the characters \( \psi(\cdot, \lambda) \) of the principal series \( T(\cdot, \lambda) \) are given by

\[
\psi(c, \lambda) = \sum_{\rho \in \mathcal{B}} \frac{\rho(c) \lambda(c)}{D(c)} 
\]  

where \( D \) is an appropriate “discriminant” ([9, p. 228]) which depends on the type of \( G \). Let \( G' \) denote the set of regular elements of \( G \). Then, since (7.23) makes sense for any \( \lambda \) unitary or not, we can regard \( \psi \) as a function on \( G' \times \mathbb{C}^n \times \mathbb{Z}^n \) to \( \mathbb{C} \). It is convenient to make the additional convention that \( \psi(x, s, m) = 0 \) when \( x \notin G' \). Then it is easy to check that \( \psi \) has the following properties.

**Lemma 34.** For fixed \( x \) and \( m \), \( s \to \psi(x, s, m) \) is analytic on \( \mathbb{C}^n \). If \( \lambda_1 \) and \( \lambda_2 \) are any quasi-characters of \( B \), then

\[
\psi(x, \lambda_1) = \psi(x, \lambda_2) \quad \text{a.e. in } x
\]

if and only if \( \lambda_1 = \rho \lambda_2 \) for some \( \rho \) in \( \mathbb{W} \). In addition

\[
\psi(x^{-1}, s, m) = \psi(x, -\overline{s}, m) 
\]  

for all \( x \) in \( G \).

Next we show that the \( \psi(\cdot, \lambda) \) are the characters of the representations \( R(\cdot, \lambda) (\lambda \in \mathbb{A}^*) \).
Theorem 11. Let $G$ be a complex classical group of type $B_n$, $C_n$ ($n \geq 2$) or $D_n$ ($n \geq 3$). Suppose $f$ is a bounded Baire function of compact support on $G$. Then for each $s$ in $\Omega$ and $m$ in $\mathbb{Z}^n$ the operator

$$ R(f, s, m) = \int_G f(x) R(x, s, m) \, dx $$

is of Hilbert-Schmidt class, and the map

$$ s \rightarrow R(f, s, m) $$

is analytic in $\Omega$. If $f$ is sufficiently smooth, e.g., the convolution of two bounded Baire functions of compact support, then $R(f, s, m)$ is of trace-class, and

$$ s \rightarrow \text{tr} \ R(f, s, m) $$

is analytic in $\Omega$; moreover

$$ \text{tr} \ R(f, s, m) = \int_G f(x) \psi(x, s, m) \, dx \quad (7.25) $$

where $\psi$ is given by (7.23).

Proof. By Theorem 10(3), $R(\cdot, s, m)$ is completely irreducible for all $s$ in the complement of some negligible subset of $\Omega$. (Actually we only need to know that one of the $R(\cdot, \lambda), \lambda \in \Lambda$ is irreducible). This being the case, all of the statements of theorem, with the exception of (7.25), are immediate consequences of Theorem 10, Theorem 3, and the corollary to Theorem 3.

To prove (7.25) we recall that $R(y, \lambda) = W(\lambda) T(y, \lambda) W(\lambda)^{-1}$ for $\lambda \in \Lambda$. Hence, $R(f, \lambda) = W(\lambda) T(f, \lambda) W(\lambda)^{-1}$, and $\text{tr} \ R(f, \lambda) = \text{tr} \ T(f, \lambda)$. But for any sufficiently smooth $f$ of compact support

$$ \text{tr} \ T(f, \lambda) = \int_G f(x) \psi(x, \lambda) \, dx. $$

In addition, it is easy to see that

$$ s \rightarrow \int_G f(x) \psi(x, s, m) \, dx $$

is analytic in $\mathbb{C}^n$; since $s \rightarrow \text{tr} \ R(f, s, m)$ is analytic in $\Omega$ and these two functions agree on $\mathbb{R}^n$, it follows that $\text{tr} \ R(f, s, m)$ is given by (7.25) for all $s$ in $\Omega$. 
Theorem 12. The representations $R(\cdot, \lambda_1)$ and $R(\cdot, \lambda_2)$ with $\lambda_1$ and $\lambda_2$ in $\mathcal{A}^*$ are similar if and only if $\lambda_1 = p\lambda_2$ for some $p$ in $\mathcal{M}$, and when this is the case $R(\cdot, \lambda_1) = R(\cdot, \lambda_2)$. The representation $R(\cdot, s, m)$ \((s \in \Omega)\) is similar to a unitary representation if and only if $(-s, m) = p(s, m)$ for some $p$ in $\mathcal{M}$, and then it is already unitary.

Proof. Suppose $A$ is a bounded linear operator with a bounded inverse such that $R(y, \lambda_1) = AR(y, \lambda_2)A^{-1}$ for all $y$ in $G$. Suppose $f$ is the convolution of two bounded Baire functions of compact support. Then $R(f, \lambda_1) = AR(f, \lambda_2)A^{-1}$ and $\text{tr} R(f, \lambda_1) = \text{tr} R(f, \lambda_2)$. Hence

$$\int_{G} f(x) \psi(x, \lambda_1) \, dx = \int_{G} f(x) \psi(x, \lambda_2) \, dx$$

by Theorem 11. Since this is true for all such $f$, it follows that

$$\psi(x, \lambda_1) = \psi(x, \lambda_2)$$

a.e. in $x$. Thus, by Lemma 34, $\lambda_1 = p\lambda_2$ for some $p$ in $\mathcal{M}$; under these circumstances

$$R(\cdot, \lambda_1) = R(\cdot, p\lambda_2) = R(\cdot, \lambda_2)$$

by Theorem 10.

Now suppose $s \in \Omega$ and that $A$ is a bounded linear operator with a bounded inverse such that the representation $S(\cdot, s, m)$ given by

$$S(y, s, m) = AR(y, s, m)A^{-1}, \quad y \in G$$

is unitary. Again let $f$ be the convolution of two bounded Baire functions of compact support. Then

$$S(f, s, m) = AR(f, s, m)A^{-1}$$

so that

$$\text{tr} S(f, s, m) = \int_{G} f(x) \psi(x, s, m) \, dx.$$ 

Since $S$ is unitary, $S(f, s, m)^* = S(f^*, s, m)$ where $f^*(x) = f(x^{-1})$. From this it follows that

$$\int_{G} f(x) \psi(x, s, m) \, dx = \int_{G} f(x^{-1}) \psi(x, s, m) \, dx = \int_{G} f(x) \psi(x^{-1}, s, m) \, dx.$$
This being true for all such \( f \), we conclude that
\[
\psi(x, s, m) = \overline{\psi(x^{-1}, s, m)}
\]
a.e. in \( x \). By Lemma 34
\[
\overline{\psi(x^{-1}, s, m)} = \psi(x, -\bar{s}, m).
\]
Hence, \( \psi(x, s, m) = \psi(x, -\bar{s}, m) \) a.e. in \( x \). Using Lemma 34 again, we now see that
\[
(-\bar{s}, m) = p(s, m)
\]
for some \( p \) in \( \mathfrak{W} \). When this is the case, it follows from Theorem 10(4) that \( R(\cdot, s, m) = R(\cdot, -\bar{s}, m) \); thus \( R(\cdot, s, m) \) is a unitary representation because
\[
R(y, s, m) = R(y^{-1}, s, m)^*
\]
for all \( y \) in \( \mathcal{G} \) by Theorem 10(5).

The unitary representations \( R(\cdot, s, m) \) with \( s \) in \( \Omega \) and \( \text{Re}(s) \neq 0 \) will be referred to as members of the *normalized complementary series*. They are determined by the conjugacy classes of elements of order 2 in \( \mathfrak{W} \).

Specifically, suppose that \( p \in \mathfrak{W} \), \( p^2 = 1 \), and \( p \neq 1 \). Let \( E_p^- = \{ \sigma \in \mathbb{R}^n : p\sigma = -\sigma \} \) and \( E_p^+ = \{ \tau \in \mathbb{R}^n : p\tau = \tau \} \). Then \( \mathbb{R}^n \) is the orthogonal direct sum of \( E_p^- \) and \( E_p^+ \). In addition, \( \dim(E_p^-) \geq 1 \), and \( R(\cdot, \sigma + i\tau, m) \) is a member of the normalized complementary series whenever
\[
0 \neq \sigma \in E_p^- \cap \text{base } (\Omega)
\]
\[
\tau \in E_p^+
\]
\[
m \in E_p^+ \cap \mathbb{Z}^n.
\]  
(7.26)

With \( \sigma \), \( \tau \), and \( m \) as above
\[
R(\cdot, \sigma + i\tau, m) = R(\cdot, q\sigma + iq\tau, qm)
\]
for every \( q \) in \( \mathfrak{W} \); since
\[
0 \neq q\sigma \in E_{q\sigma}^- \cap \text{base } (\Omega)
\]
\[
q\tau \in E_{q\tau}^+
\]
\[
qm \in E_{qm}^+ \cap \mathbb{Z}^n
\]
the conjugates of \( p \) will lead, as above, to the same complementary series as \( p \). Every member of the normalized complementary series arises in this fashion. For if \( \text{Re}(s) \neq 0, p \in \mathbb{M} \), and \( ps = -\bar{s} \), then it can be shown that \( -\bar{s} = qs \) for some \( q \) in \( \mathbb{M} \) of order 2.

**Theorem 13.** Suppose \( p \in \mathbb{M}, p^2 = 1, \) and \( p \neq 1 \). Then except for \( (\sigma, \tau) \) in a real \( n \)-dimensional set of Lebesgue measure 0, the associated normalized complementary series \( R(\cdot, \sigma + i\tau, m) \) (\( \sigma, \tau, \) and \( m \) as in (7.26)) are irreducible.

**Proof.** Let \( V_p = E_p^- + iE_p^+ \). Then \( V_p \) is a real form of \( \mathbb{C}^n \). By Theorem 10, the representations \( R(\cdot, s, m) \) (\( s \in \Omega, m \in \mathbb{Z}^n \)) are completely irreducible for all \( s \) in the complement of some negligible set \( S \). Hence, \( S \cap V_p \) is a real \( n \)-dimensional set of Lebesgue measure 0.

**References**