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Equality Conditions for Matrix Kantorovich-Type Inequalities

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Following a recent paper of S. Liu and H. Neudecker (*J. Math. Anal. Appl.* **197**, 1996, 23–26), we present sufficient and necessary conditions (SNECs) under which equalities occur in those corresponding matrix Kantorovich-type inequalities. We also present several relevant inequalities.

1. INTRODUCTION

Marshall and Olkin [10] first presented a matrix version of the Kantorovich inequality involving a positive definite matrix. Baksalary and Puntanen [1] extended it to cover the case of one positive semidefinite matrix, while Mond and Pečarić [11, 12] gave several Kantorovich-type inequalities for the case of one positive definite matrix or for Fan's cases of sums of matrices (see also Fan [2]). Liu [5] gave a related inequality in a special case. More recently, Liu and Neudecker [7] presented further Kantorovich-type inequalities involving one positive semidefinite matrix or sums of such matrices. In this paper all matrices and numbers considered are real. We refer to Magnus and Neudecker [9] for mathematical basics.

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Following Liu and Neudecker [7], we shall further study sufficient and necessary conditions for known and new Kantorovich-type inequalities to become equalities. We shall also present several relevant inequalities.

2. BASIC RESULTS

Let A be an $n \times n$ positive semidefinite matrix with rank $p (p \leq n)$ and with nonzero eigenvalues $M \geq \ldots \geq m > 0$. Let V be an $n \times r$ matrix with rank q such that $\Re(V) \subset \Re(A)$, where $q \leq \min(r, p)$, and $\Re(\cdot)$ denotes the column space of the matrix. Let $+$ indicate the Moore-Penrose inverse. For symmetric matrices B and C, $B \leq C$ means $C - B$ is positive semidefinite.

In the following, from three lemmas we shall derive three basic propositions.

LEMMA 2.1. *If $D > 0$ is a $p \times p$ matrix with eigenvalues $M \geq \ldots \geq m > 0$, then*

$$D^{-1} \leq \frac{M+m}{Mm} I_p - \frac{1}{Mm} D, \tag{1}$$

and

$$D^2 \leq (M+m) D - Mm I_p. \tag{2}$$

See, e.g., Marshall and Olkin [10] *and Liu and Neudecker* [7].

LEMMA 2.2. *If $B \geq 0$, $C \geq 0$, $B^2 \geq C^2$, then $\Re(C^2) \subset \Re(B^2)$.*

See, e.g., Liski and Puntanen [4] or Wang and Chow [17].

LEMMA 2.3. *If $E > 0$, $F \geq 0$, $E^2 \geq F^2$, then $E \geq F$ holds.*

See, e.g., Theorem 2.5.5 in Wang and Chow [17].

PROPOSITION 2.1. *If $A \geq 0$, V is an $n \times r$ matrix with rank q, and $\Re(V) \subset \Re(A)$, we have*

$$VV^+A^+VV^+ \leq \frac{M+m}{Mm} VV^+ - \frac{1}{Mm} VV^+AVV^+; \tag{3}$$

$$VV^+A^2VV^+ \leq (M+m) VV^+AVV^+ - Mm VV^+. \tag{4}$$

Proof. As $A \geq 0$ and $\Re(V) \subset \Re(A)$, we have $A = TDT'$, $AA^+ = TT'$, and $AA^+V = V$, where $D > 0$, $T'T = I_p$, $p = \operatorname{rank}(A)$, matrices D and T

are of order $p \times p$ and $n \times p$, respectively. From (1) and (2) we get

$$A^{+} \leq \frac{M+m}{Mm} AA^{+} - \frac{1}{Mm} A; \tag{5}$$

$$A^{2} \leq (M+m)A - MmAA^{+}, \tag{6}$$

for $A \geq 0$. And then we obtain (3) and (4).

Remark 2.1. Note that (5) and (6) are equivalent. Also (4) can be extended as for any $n \times n$ symmetric matrix C with eigenvalues c_j such that $M \geq c_j \geq m, j = 1, \ldots, n$,

$$C^{2} \leq (M+m)C - MmCC^{+}, \tag{7}$$

where M and m are not necessary positive scalars because $(M - c_j)(m - c_j) \leq 0$ is always true.

PROPOSITION 2.2. *If $A \geq 0$, V is and $n \times r$ matrix with rank q, and $\Re(V) \subset \Re(A)$, then the following five identities hold,*

$$(VV^{+}AVV)^{+} = V(V'AV)^{+} V'; \tag{8}$$

$$(VV^{+}AVV^{+})^{+1/2}(VV^{+}AVV^{+})^{1/2} = VV^{+}; \tag{9}$$

$$(VV^{+}AVV^{+})^{1/4}(VV^{+}AVV^{+})^{+1/2}(VV^{+}AVV^{+})^{1/4} = VV^{+}; \tag{10}$$

$$(VV^{+}AVV^{+})^{1/2}VV^{+} = (VV^{+}AVV^{+})^{1/2}; \tag{11}$$

$$(VV^{+}A^{2}VV^{+})^{1/2}VV^{+} = (VV^{+}A^{2}VV^{+})^{1/2}. \tag{12}$$

Proof. Write $V = SGQ'$ and $VV^{+} = SS'$, where $G > 0$, $S'S = I_q$, $Q'Q = I_q$, $q = \text{rank}(V)$, matrices G, S, and Q are of order $q \times q, n \times q$, and $r \times q$, respectively. Noting that $S'AS > 0$ we have

$$(VV^{+}AVV^{+})^{+} = (SS'ASS')^{+} = S(S'AS)^{-1}S',$$

and

$$V(V'AV)^{+} V' = SGQ'(QGS'ASGQ')^{+} QGS' = S(S'AS)^{-1}S',$$

then (8) holds.

From the following

$$(VV^{+}AVV^{+})^{+1/2} = S(S'AS)^{-1/2}S'; \tag{13}$$

$$(VV^+AVV^+)^\alpha = S(S'AS)^\alpha S', \tag{14}$$

$\alpha = 1/2$, *or* $1/4$;

$$(VV^+A^2VV^+)^{1/2} = S(S'A^2S)^{1/2}S', \tag{15}$$

we get (9) through (12).

Remark 2.2. In (14), α can be any number. Note that $VV^+AVV^+ \geq 0$ but $S'AS > 0$. If $\alpha < 0$, then $-\alpha > 0$ and therefore α for the left-hand-side term has to be replaced with $+(-\alpha)$, where this $+$ indicates the Moore-Penrose inverse (see also (13) above as an example). If $\alpha = 0$, then $(VV^+AVV^+)^0 = SS' = VV^+$. Also (15) and then (12) still holds when A is just a symmetric matrix such that $\Re(V) \subset \Re(A)$.

PROPOSITION 2.3. *If* $B \geq 0$, $C \geq 0$, *and* $B^2 \geq C^2$, *then* $B \geq C$ *holds*.

Proof. Using Lemma 2.2 gives $\Re(C) \subset \Re(B)$. Write $B = RER'$ and $C = RFR'$, where $E > 0$, $F \geq 0$, and $R'R = I_b$ with $b = \text{rank}(B)$. Then $RE^2R' = B^2 \geq C^2 = RF^2R'$, hence $E^2 \geq F^2$. Applying Lemma 2.3 leads to $E \geq F$, and therefore $B \geq C$.

Remark 2.3. Based on Lemma 2.3 another proof of Proposition 2.3 is, due to Professor A.M. Fink's idea, as follows. For any $\epsilon > 0$, we have $(B + \epsilon I)^2 = B^2 + 2\epsilon B + \epsilon^2 I \geq C^2$, then $B + \epsilon I \geq C$, i.e., $B \geq C$.

2. EQUALITY CONDITIONS

We now use Propositions 2.1 and 2.2 to derive sufficient and necessary conditions (SNECs) for several Kantorovich-type inequalities to become equalities.

PROPOSITION 3.1. *The SNECs for* (16) *are* (17) *or* (18):

$$V^+A^+V'^+ \leq \frac{(M+m)^2}{4Mm}(V'AV)^+, \tag{16}$$

$$V'AV = \frac{M+m}{2}V'V, V'A^+V = \frac{M+m}{2Mn}V'V; \tag{17}$$

$$V = 0, \tag{18}$$

where $A \geq 0$ *and* $\Re(V) \subset \Re(A)$.

Proof. By using (3) and (9), and noting that $(VV^+AVV^+)^{+1/2} \geq 0$ and $(VV^+AVV^+)^{1/2} \geq 0$ are symmetric, we have

$$\begin{aligned}
VV^+A^+VV^+ &\leq \frac{M+m}{Mm}VV^+ - \frac{1}{Mm}VV^+AVV^+ \\
&= \frac{(M+m)^2}{4Mm}(VV^+AVV^+)^+ \\
&\quad - \left[\frac{M+m}{2\sqrt{Mm}}(VV^+AVV^+)^{+1/2} - \frac{1}{\sqrt{Mm}}(VV^+AVV^+)^{1/2}\right]^2 \\
&\leq \frac{(M+m)^2}{4Mm}(VV^+AVV^+)^+ . \qquad (19)
\end{aligned}$$

Using (8) and noting that $V^+V(V'AV)^+V'V'^+ = (V'AV)^+$ we see (19) is equivalent to (16). From (19) we find that the SNECs are

$$\text{(i)}\ \frac{M+m}{2\sqrt{Mm}}(VV^+AVV^+)^{+1/2} - \frac{1}{\sqrt{Mm}}(VV^+AVV^+)^{1/2} = 0 \qquad (20)$$

and

$$VV^+A^+VV^+ = \frac{(M+m)^2}{4Mm}(VV^+AVV^+)^+, \qquad (21)$$

or

$$\text{(ii)}\ VV^+A^+VV^+ = \frac{(M+m)^2}{4Mm}(VV^+AVV^+)^+ = 0. \qquad (22)$$

Using (10), $V'VV^+ = V'$, and $VV^+V = V$, we get (17) from (20) and (21). Clearly (22) means that $V'^+V'A^+VV^+ = V'^+V'AVV^+ = 0$. Simply $AV = 0$, or $V = 0$. This is because $\Re(V) \subset \Re(A)$, i.e., $V = AL$, for some matrix L, and then $V'V = L'AV$.

For (16), compare the result (1) in Liu and Neudecker [6].

Remark 3.1. Consider an illustrative example in a simple case for equality conditions. Define the 5×5 diagonal matrix $A = \mathrm{diag}(3,3,2,1,1)$, and the 5×2 matrix $V = (x, y)$ with the 5×1 vector $x = (1/\sqrt{2}, 0, 0, 0, 1/\sqrt{2})'$ and the 5×1 vector $y = (0, 1/\sqrt{2}, 0, 1/\sqrt{2}, 0)'$. In this case $A > 0$ and $V'V = I_2$. A straightforward calculation shows that (17) holds.

Assume that $A \geq 0$ and $\Re(V) \subset \Re(A)$, then a matrix version of the Cauchy–Schwartz inequality is

$$(V'AV)^{+} \leq V^{+}A^{+}V'^{+}, \tag{23}$$

with equality if and only if $\Re(V) = \Re(AV)$.

It can be obtained, by pre- and post-multiplying V^{+} and V'^{+}, respectively, from

$$V(V'AV)^{+}V' \leq A^{+}, \tag{24}$$

for $A \geq 0$ and $R(V) \subset R(A)$.

For (24) with its equality condition $\Re(V) = \Re(A)$, see, e.g., Pukelsheim and Styan [14]. Also (24) can be derived as follows. Given E and F are two symmetric and idempotent matrices, then $EF = F$ implies $E \geq F$; see Liu and Polasek [8]. Using $E = AA^{+}$ and $F = A^{1/2}V(V'AV)^{+}V'A^{1/2}$, where $EF = F$, we get (24).

PROPOSITION 3.2. *The SNECs for* (25) *are* (26) *or* (27) *or* (28),

$$V^{+}AV'^{+} - (V'A^{+}V)^{+} \leq \left(\sqrt{M} - \sqrt{m}\right)^{2}(V'V)^{+}, \tag{25}$$

$$V'AV = \left(M + m - \sqrt{Mm}\right)V'V, V'A^{+}V = \frac{1}{\sqrt{Mm}}V'V; \tag{26}$$

$$V = 0; \tag{27}$$

$$M = m, \tag{28}$$

where $A \geq 0$ *and* $\Re(V) \subset \Re(A)$.

Proof. Using (3) and (9), we have

$$\begin{aligned} &VV^{+}AVV^{+} - (VV^{+}A^{+}VV^{+})^{+} \\ &\quad \leq (M + m)VV^{+} + MmVV^{+}A^{+}VV^{+} - (VV^{+}A^{+}VV^{+})^{+} \\ &\quad = \left(\sqrt{M} - \sqrt{m}\right)^{2}VV^{+} \\ &\qquad - \left[\sqrt{Mm}\,(VV^{+}A^{+}VV^{+})^{1/2} - (VV^{+}A^{+}VV^{+})^{+1/2}\right]^{2} \\ &\quad \leq \left(\sqrt{M} - \sqrt{m}\right)^{2}VV^{+}. \end{aligned} \tag{29}$$

Then (26) follows from using (29) and (10). Also it can be verified that $VV^{+}A^{+}VV^{+} - (VV^{+}A^{+}VV^{+})^{+} = (\sqrt{M} - \sqrt{m})^{2}VV^{+} = 0$ is equivalent to (27) or (28).

For (25), see also Liu and Neudecker [7].

Remark 3.2. Note that A^+ has the nonzero eigenvalues $1/m \geq \dots \geq 1/M > 0$, and a representation of Proposition 3.2 can then be given. The SNECs for (30) are (31) or (27) or (28),

$$V^+A^+V'^+ - (V'AV)^+ \leq \frac{(\sqrt{M} - \sqrt{m})^2}{Mm}(V'V)^+, \tag{30}$$

$$V'AV = \sqrt{Mm}\,V'V, V'A^+V = \frac{M + m - \sqrt{Mm}}{Mm}V'V, \tag{31}$$

where $A \geq 0$ has nonzero eigenvalues $M \geq \dots \geq m > 0$, and $\Re(V) \subset \Re(A)$.

PROPOSITION 3.3. *The SNECs for* (32) *are* (33) *or* (34),

$$V'A^2V \leq \frac{(M+m)^2}{4Mm}V'AVV^+AV, \tag{32}$$

$$V'AV = \frac{2Mm}{M+m}V'V, V'A^2V = MmV'V; \tag{33}$$

$$V = 0, \tag{34}$$

where $A \geq 0$ *and* $\Re(V) \subset \Re(A)$.

Proof. By using (4) and $VV^+AVV^+VV^+ = VV^+AVV^+$, we have

$$\begin{aligned} VV^+A^2VV^+ &\leq (M+m)VV^+AVV^+ - MmVV^+ \\ &= \frac{(M+m)^2}{4Mm}(VV^+AVV^+)^2 \\ &\quad - \left[\frac{M+m}{2\sqrt{Mm}}VV^+AVV^+ - \sqrt{Mm}\,VV^+\right]^2 \\ &\leq \frac{(M+m)^2}{4Mm}(VV^+AVV^+)^2. \end{aligned} \tag{35}$$

Then (32), (33), and (34) follow.

Remark 3.3. Note that from $VV^+ \leq I$, we get for any symmetric matrix C

$$V'CVV^+CV \leq V'C^2V,$$

and equivalently

$$(VV^+CVV^+)^2 \leq VV^+C^2VV^+, \tag{36}$$

both with equalities if and only if $VV^+CV = CV$.

PROPOSITION 3.4. *The SNECs for* (37) *are* (38) *or* (39) *or* (40),

$$V'A^2V - V'AVV^+AV \leq \frac{1}{4}(M-m)^2 V'V, \tag{37}$$

$$V'AV = \frac{M+m}{2}V'V, V'A^2V = \frac{M^2+m^2}{2}V'V; \tag{38}$$

$$V = 0; \tag{39}$$

$$M = m, \tag{40}$$

where $A \geq 0$ *and* $\Re(V) \subset \Re(A)$.

Proof. Using (4) and $VV^+AVV^+VV^+ = VV^+AVV^+$, we have

$$\begin{aligned} VV^+A^2VV^+ &- (VV^+AVV^+)^2 \\ &\leq (M+m)VV^+AVV^+ - MmVV^+ - (VV^+AVV^+)^2 \\ &= \frac{1}{4}(M-m)^2VV^+ - \left[VV^+AVV^+ - \frac{M+m}{2}VV^+\right]^2 \\ &\leq \frac{1}{4}(M-m)^2VV^+. \end{aligned} \tag{41}$$

Then (37) through (40) follow.

For (37) and other equivalent inequalities, see Liu and Neudecker [7].

Remark 3.4. Based on (7) in Remark 2.1, we see from (41) that we can relax A to be a symmetric matrix. For another method to relax A to be symmetric, see Styan [16].

PROPOSITION 3.5. *The SNECs for* (42) *are* (43) *or* (44),

$$(VV^+A^2VV^+)^{1/2} \leq \frac{M+m}{2\sqrt{Mm}}VV^+AVV^+, \tag{42}$$

$$V'AV = \frac{2Mm}{M+m}V'V, V'A^2V = MmV'V; \tag{43}$$

$$V = 0, \tag{44}$$

where $A \geq 0$ *and* $\Re(V) \subset \Re(A)$.

Proof. By (4) and (12), we have

$$
\begin{aligned}
VV^{+}AVV^{+} \\
&\geq \frac{1}{M+m} VV^{+}A^{2}VV^{+} + \frac{Mm}{M+m} VV^{+} \\
&= \frac{2\sqrt{Mm}}{M+m} \left(VV^{+}A^{2}VV^{+} \right)^{1/2} \\
&\quad + \left[\frac{1}{\sqrt{M+m}} \left(VV^{+}A^{2}VV^{+} \right)^{1/2} - \frac{\sqrt{Mm}}{\sqrt{M+m}} VV^{+} \right]^{2} \\
&\geq \frac{2\sqrt{Mm}}{M+m} \left(VV^{+}A^{2}VV^{+} \right)^{12} \qquad (45)
\end{aligned}
$$

Then (42), (43), and (44) hold.

For (42), see Liu and Neudecker [7].

PROPOSITION 3.6. *The SNECs for* (46) *are* (47) *or* (48) *or* (49),

$$\left(VV^{+}A^{2}VV^{+} \right)^{1/2} - VV^{+}AVV^{+} \leq \frac{(M-m)^{2}}{4(M+m)} VV^{+}, \tag{46}$$

$$V'AV = \frac{M^{2}+m^{2}+6Mm}{4(M+m)} V'V, V'A^{2}V = \frac{(M+m)^{2}}{4} V'V; \tag{47}$$

$$V = 0; \tag{48}$$

$$M = m, \tag{49}$$

where $A \geq 0$ *and* $\Re(V) \subset \Re(A)$.

Proof. By (4) and (12), we have

$$
\begin{aligned}
\left(VV^{+}A^{2}VV^{+} \right)^{1/2} - VV^{+}AVV^{+} \\
&\leq \left(VV^{+}A^{2}VV^{+} \right)^{1/2} - \frac{1}{M+m} VV^{+}A^{2}VV^{+} - \frac{Mm}{M+m} VV^{+} \\
&= \frac{(M-m)^{2}}{4(M+m)} VV^{+} \\
&\quad - \left[\frac{1}{\sqrt{M+m}} \left(VV^{+}A^{2}VV^{+} \right)^{1/2} - \frac{\sqrt{M+m}}{2} VV^{+} \right]^{2} \\
&\leq \frac{(M-m)^{2}}{4(M+m)} VV^{+}. \qquad (50)
\end{aligned}
$$

Then (46) through (49) hold.

For (46), see Liu and Neudecker [7].

Remark 3.5. From (7), (12), and (50) we see that (46) can be extended as

$$(M+m)\left[(VV^+C^2VV^+)^{1/2} - VV^+CVV^+\right] \leq \frac{(M-m)^2}{4}VV^+, \tag{51}$$

for any symmetric matrix C such that $\Re(V) \subset \Re(C)$ and $M+m \geq 0$.

4. RELEVANT INEQUALITIES

Applying Proposition 2.3, we can derive some further results.

First from (16) and (19), we get

$$(V^+A^+V'^+)^{1/2} \leq \frac{M+m}{2\sqrt{Mm}}(V'AV)^{+1/2} \tag{52}$$

and

$$(VV^+A^+VV^+)^{1/2} \leq \frac{M+m}{2\sqrt{Mm}}(VV^+AVV^+)^{+1/2}, \tag{53}$$

both with equalities if and only if (17) or (18) holds. Using (23) and its equivalent version, we have

$$(V'AV)^{+1/2} \leq (V^+A^+V'^+)^{1/2}, \tag{54}$$

and

$$(VV^+AVV^+)^{+1/2} \leq (VV^+A^+VV^+)^{1/2}, \tag{55}$$

where $A \geq 0$ and $\Re(V) \subset \Re(A)$. The two equalities occur if and only if $\Re(V) = \Re(AV)$.

Using (35) and the following matrix version of the Cauchy−Schwarz inequality

$$(VV^+AVV^+)^2 \leq VV^+A^2VV^+, \tag{56}$$

we also get respectively Proposition 3.5 and the inequality

$$VV^+AVV^+ \leq (VV^+A^2VV^+)^{1/2}, \tag{57}$$

with equality if and only if $VV^+AV = AV$. Note that A in (56) and (57) can be relaxed to be any symmetric matrix; see also Remark 3.3 and (36) there.

Now, we present the combined matrix inequalities for the Baksalary−Puntanen [1] condition, i.e., for A and V such that $A \geq 0$ and $V'AA^+V$ is idempotent,

$$(V'A^+V)^{+1/2} \leq (V'AV)^{1/2} \leq \frac{M+m}{2\sqrt{Mm}}(V'A^+V)^{+1/2}. \tag{58}$$

Here (58) can be derived from (2.4) and (3.4) in Baksalary and Puntanen [1]. In particular, if $A > 0$ and $V'V$ is idempotent, which is also a special case of $A \geq 0$ and $\Re(V) \subset \Re(A)$, we have

$$V'AV \leq (V'A^2V)^{1/2} \leq \frac{M+m}{2\sqrt{Mm}} V'AV. \tag{59}$$

The first part of (59) follows from (36). The second part is derived from (32), and $V^+ = V'$ which is equivalent to the idempotency of $V'V$.

Keep in mind that (in) equality conditions remain unchanged when we apply Proposition 2.3.

5. CONCLUDING COMMENTS

(i) By using the results of Section 3, we can examine the special cases for the Hadamard product (see, e.g., Horn [3]), and for the upper-left submatrices (see, e.g. Liu [5]) studied in Liu and Neudecker [7].

(ii) Applying the block-method used by Liu [5] and Liu and Neudecker [7], plenty of results for several cases of sums of matrices including Kantorovich and Cauchy−Schwarz inequalities can be easily derived from the results presented in this paper, and for Fan's cases, a special type of the cases of sums of matrices, see Fan [2], Mond and Pečarić [12], and Liu and Neudecker [7]. Also, it is not difficult to give parallel versions of SNECs of equalities for Kantorovich-type inequalities in the cases of sums of matrices.

(iii) Only the case which involves one positive (semi-)definite matrix is considered in Section 3, while the case which involves two such matrices studied in Wang and Shao [18] and Liu and Neudecker [6] can also be treated to give further results.

(iv) A short comment on equality conditions for the matrix version of the Kantorovich inequality can be found in Marshall and Olkin [10]. Studies for a different type of conditions under which Kantorovich inequalities become equalities can be found in Baksalary and Puntanen [1] and Pečarić, Puntanen, and Styan [13]. For considerations in matrix-trace and other relevant cases, see, e.g. Rao [15] and references thereafter.

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