The Lovelock gravity in the critical spacetime dimension

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ABSTRACT

It is well known that the vacuum in the Einstein gravity, which is linear in the Riemann curvature, is trivial in the critical $(2+1=3)$ dimension because vacuum solution is flat. It turns out that this is true in general for any odd critical $d = 2n+1$ dimension where n is the degree of homogeneous polynomial in Riemann defining its higher order analogue whose trace is the nth order Lovelock polynomial. This is the "curvature" for nth order pure Lovelock gravity as the trace of its Bianchi derivative gives the corresponding analogue of the Einstein tensor as defined by Dadhich (2010) [1]. Thus the vacuum in the pure Lovelock gravity is always trivial in the odd critical $(2n+1)$ dimension which means it is pure Lovelock flat but it is not Riemann flat unless $n = 1$ and then it describes a field of a global monopole. Further by adding Λ we obtain the Lovelock analogue of the BTZ black hole.

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Since gravity is universal as it links to everything that physically exists including zero mass particles, hence it can only be described by the curvature of spacetime and its dynamics is then entirely determined by the Riemann curvature. The Einstein–Hilbert action which is the trace of the Riemann curvature tensor gives on variation the second rank symmetric Einstein tensor with vanishing divergence. The Einstein tensor provides the second order differential operator, the analogue of $\nabla^2 \phi$, in the equation of motion. There is however an alternative purely geometric way to get to the Einstein tensor by taking the trace of the Bianchi identity satisfied by the Riemann curvature. Inclusion of higher order terms in curvature in the action becomes pertinent to take into account the high energy effects. That is we have to go beyond the linear Einstein–Hilbert term to a polynomial in Riemann and the requirement of the second order quasi-linear equation uniquely identifies the polynomial to the Lovelock polynomial. On the alternative geometric side we have to find an analogue of the Riemann tensor which is a polynomial in Riemann. The Riemann satisfies the Bianchi identity which means vanishing of its Bianchi derivative and the trace of the identity leads to the divergence free Einstein tensor. Now the higher order analogue of Riemann as identified by Dadhich in [1] has non-zero Bianchi derivative and hence does not satisfy the Bianchi identity which is the defining property of Riemann tensor. However the trace of the Bianchi derivative does indeed vanish and that is what is required to get to the divergence

free analogue of the Einstein tensor. The trace of the higher order Riemann analogue is indeed the Lovelock polynomial but for the numerical multiplying factor.

It is well known that for the linear in Riemann Einstein gravity, vacuum is trivial in 3 spacetime dimension as $R_{ab} = 0$ implies $R_{abcd} = 0$. There exists no non-trivial vacuum solution to incorporate dynamics. The vacuum solution is non-trivial only in dimension $\geqslant 4$. To universalize this feature, we should ask whether it is true in general for higher order gravity as well? That is, is vacuum solution trivial in general for the critical $d = 2n+1$ dimension relative to the higher order Riemann analogue where n is the degree of the polynomial? If we denote nth order Riemann analogue by $R^{(n)}_{abcd}$, then $R^{(n)}_{ab} = 0$ implies $R^{(n)}_{abcd} = 0$ for $d = 2n+1$ and the nth order pure Lovelock vacuum will be non-trivial only in $d \geqslant 2(n+1)$ dimension. Even when spacetime is Lovelock flat, it will not be Riemann flat unless $n = 1$.

Our main purpose in this Letter is to establish this universal feature of gravitational field. This means spacetime dimension and the degree of the curvature polynomial, $R^{(n)}_{abcd}$, are intimately related and $d = 2n+1$ is the critical dimension for which the corresponding vacuum is trivial. For the linear and quadratic orders $n = 1, 2$, it is the Einstein and Gauss–Bonnet gravity with critical dimensions $d = 3, 5$ respectively. And vacuum is universally trivial in the critical dimensions. The Lovelock flat is not Riemann flat unless $n = 1$, and the static spacetime in the critical dimension is characterized by $g_{tt} = -1/g_{rr} = const$. Then g_{tt} could be squared out by redefining the time coordinate as constant Newtonian potential is trivial while g_{rr} is the Einstein effect which cannot be absorbed by coordinate transformation and it represents a solid angle deficit for $d > 3$ and has non-zero Riemann curvature [2].

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The Einstein stress tensor so generated is known asymptotically to approximate to that of a global monopole in 4-dimension [3, 4]. It is remarkable that this is true in general for all dimensions > 3. We shall in particular show that the Gauss–Bonnet trivial vacuum spacetime in the critical 5-dimension indeed produces Einstein stresses that describe a 5-dimensional global monopole. Thus Lovelock flat spacetime in the critical dimension $(2n + 1)$ will describe a global monopole in the Einstein gravity.

Following Dadhich [1] with a slight change in notation we define the Lovelock curvature polynomial

$$R^{(n)}_{abcd} = F^{(n)}_{abcd} - \frac{n-1}{n(d-1)(d-2)} F^{(n)}(g_{ac}g_{bd} - g_{ad}g_{bc}),$$

$$F^{(n)}_{abcd} = Q_{ab}{}^{mn} R_{cdmn}, \tag{1}$$

where

$$Q^{ab}{}_{cd} = \delta^{aba_1b_1...a_nb_n}_{cdc_1d_1...c_nd_n} R_{a_1b_1}{}^{c_1d_1} \ldots R_{a_{n-1}b_{n-1}}{}^{c_{n-1}d_{n-1}},$$

$$Q^{abcd}{}_{;d} = 0. \tag{2}$$

The analogue of nth order Einstein tensor is given by

$$G^{(n)}_{ab} = n\left(R^{(n)}_{ab} - \frac{1}{2} R^{(n)} g_{ab} \right) \tag{3}$$

and

$$R^{(n)} = \frac{d-2n}{n(d-2)} F^{(n)}. \tag{4}$$

Note that $R^{(n)} = R^{(n)}_{ab} g^{ab} = 0$ in $2n$ dimension for arbitrary metric g_{ab}. Since $R^{(n)}_{ab}$ is a function of the metric and its first and second derivatives which are all arbitrary, it must vanish in $d = 2n$. That is, $R^{(n)}_{ab} = 0$ identically in $2n$ dimension. On the other hand for the general Lovelock case, the Lagrangian is non-zero for $d = 2n$ but its variation vanishes identically. Here it is much more direct and transparent. Further it turns out that

$$R^{(n)}_{abcd} = \Lambda(g_{ac}g_{bd} - g_{ad}g_{bc}) \tag{5}$$

implies

$$F^{(n)}_{abcd} = \frac{n(d-2)}{d-2n} \Lambda(g_{ac}g_{bd} - g_{ad}g_{bc}), \tag{6}$$

and vice versa. Not only that the corresponding Weyl curvature is also the same for the two. That is

$$\begin{aligned} W^{(n)}_{abcd}\big(R^{(n)}_{abcd}\big) = {}& R^{(n)}_{abcd} - \frac{1}{(d-2)} \\ & \times \big(R^{(n)}_{ac} g_{bd} + R^{(n)}_{bd} g_{ac} - R^{(n)}_{ad} g_{bc} - R^{(n)}_{bc} g_{ad} \big) \\ & + \frac{1}{(d-1)(d-2)} R^{(n)}(g_{ac}g_{bd} - g_{ad}g_{bc}) \\ = {}& W^{(n)}_{abcd}\big(F^{(n)}_{abcd}\big). \end{aligned} \tag{7}$$

The two tensors differ from each other only through their trace.

We shall now explicitly demonstrate for the static spacetime that pure Lovelock vacuum, $G^{(n)}_{ab} = 0$, solution is in fact Lovelock flat, $R^{(n)}_{abcd} = 0$. We write for the static spherically symmetric spacetime,

$$ds^2 = B\, dt^2 - A\, dr^2 - r^2\, d\Omega^2_{(d-2)} \tag{8}$$

where $AB = const. = 1$ due to the null energy condition, $G^{(n)}_{ab} k^a k^b = 0$, $k_a k^a = 0$. Then the vacuum solution with Λ, $G^{(n)}_{ab} = \Lambda g_{ab}$, is given by [5],

$$B = 1/A = 1 - \left(\Lambda r^{2n} + \frac{M}{r^{d-2n-1}} \right)^{1/n}. \tag{9}$$

In the critical dimension $d = 2n + 1$, the pure Lovelock vacuum solution with $\Lambda = 0$ will have $B = 1/A = 1 - K = const.$ which could however be transformed away in g_{tt} but not in g_{rr} for $d > 3$ and hence is Riemann non-flat. However it will have $R^{(n)}_{abcd} = 0$ for any n. It is trivially true for $n = 1$ because it only causes the angle deficit which produces no Riemann curvature and we have verified it for the Gauss–Bonnet case, $n = 2$. This shows that the pure Gauss–Bonnet vacuum is trivial; i.e. $G^{(n)}_{ab} = 0$ implies $R^{(n)}_{abcd} = 0$ in the critical dimension, $d = 2n + 1$ which is 5 in this case. Similarly it could be verified for any n.[1] It is remarkable that in critical dimension spacetime is characterized by the vanishing of the corresponding nth order curvature. Thus for the critical $d = 2n + 1$ dimension, pure Lovelock vacuum is always Lovelock flat but it would not be Riemann flat unless $n = 1$. We now show that it would describe a global monopole in the Einstein gravity.

Since the Lovelock flat spacetime in the critical dimension is not Riemann flat, hence it will generate the Einstein stresses in the equation,

$$G_{ab} = -\kappa T_{ab}. \tag{10}$$

For the critical 5-dimensional Gauss–Bonnet vacuum we have $B = 1/A = 1 - K = const.$ which gives rise to Einstein stresses,

$$G^t_t = G^r_r = 3G^\theta_\theta = -3\frac{K}{r^2}, \qquad G^\theta_\theta = G^\phi_\phi = G^\psi_\psi. \tag{11}$$

To felicitate comparison with the 4-dimensional global monopole solution we have used the same notation as that of Barriola and Vilenkin [3] and write the Lagrangian as,

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi^a \partial^\mu \phi^a - \frac{1}{4}\lambda(\phi^a\phi^a - \eta^2)^2, \tag{12}$$

where ϕ^a is a quadruplet of scalar fields ($a = 1, 2, 3, 4$). The field configuration describing monopole is

$$\phi^a = \eta f(r)\frac{x^a}{r} \tag{13}$$

where x^a are cartesian coordinates with the usual relation to spherical coordinates, and $x^a x^a = r^2$. The energy momentum tensor of monopole then takes the form

$$\begin{aligned} T^t_t &= \frac{1}{2A^2}\eta^2 f'^2 + \frac{3}{2}\frac{\eta^2 f^2}{r^2} + \frac{\lambda}{4}\eta^4(f^2 - 1)^2, \\ T^r_r &= -\frac{1}{2A^2}\eta^2 f'^2 + \frac{3}{2}\frac{\eta^2 f^2}{r^2} + \frac{\lambda}{4}\eta^4(f^2 - 1)^2, \\ T^\theta_\theta &= \frac{1}{2A^2}\eta^2 f'^2 + \frac{1}{2}\frac{\eta^2 f^2}{r^2} + \frac{\lambda}{4}\eta^4(f^2 - 1)^2 \end{aligned} \tag{14}$$

and due to spherical symmetry $T^\theta_\theta = T^\phi_\phi = T^\psi_\psi$. The equation of motion for the field ϕ^a reduces to the following equation for $f(r)$,

$$\frac{f''}{A} + \left[\frac{3}{rA} + \frac{1}{2B}\left(\frac{B}{A}\right)' \right]f' - \frac{3f}{r^2} - \lambda\eta^2 f(f^2 - 1) = 0. \tag{15}$$

As is clear from the above discussion, the critical dimension vacuum spacetime can harbor no core mass for global monopole, and asymptotically $f \approx 1$ and then we have

$$T^t_t = T^r_r = 3T^\theta_\theta = \frac{3\eta^2}{2r^2}. \tag{16}$$

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[1] Since the appearance of the first version on the arXiv, there appeared a paper [6] the very next day establishing this result in general for any n. Thus vacuum for the critical dimension $d = 2n + 1$ is always Lovelock flat.

We can now integrate the Einstein equation to write the metric coefficient in the exterior as given by

$$B = 1/A = 1 - \frac{8\pi G_5}{r} \int T_t^t r^2 \, dr = 1 - 12\pi G_5 \eta^2. \tag{17}$$

The size of the monopole core could be estimated in flat space as $\delta \approx \lambda^{-1/2}\eta^{-1}$. This approximation is believed to hold good as gravity is not expected to substantially alter the structure of the monopole. Our aim is to show that the stresses generated by $B = 1/A = const.$ has the same structure as that of the global monopole and the constant $K = 12\pi G_5 \eta^2$.

We had set out to establish the two remarkable universal features: (a) the universality of vacuum in the critical $d = 2n + 1$ dimension in which spacetime is free of the corresponding curvature $R_{abcd}^{(n)}$; i.e. "vacuum is flat" and (b) this spacetime always describes a global monopole in the Einstein gravity. That's what we have shown. The critical dimension vacuum spacetime could be viewed as due to constant Newtonian potential which geometrically corresponds to solid angle deficit. The remarkable point is that it produces stress structure which agrees with that of a global monopole not only in 4-dimension [3] but also in any dimension $d \geqslant 4$. This is very interesting, why should the stresses always match? It is though understandable that the stresses go as $1/r^2$ because that is what the solid angle deficit could do and so does the prescription of the field ϕ^a. However what is not so obvious is the fact that in 4-dimension all the angular stresses vanish but not in 5-dimension yet the stresses exactly match for the left and right of the equation. What is interesting here is the fact that a global monopole in the critical dimension $(2n + 1)$ in the Einstein gravity is in fact a trivial vacuum solution relative to the Lovelock gravity with vanishing corresponding curvature, $R_{abcd}^{(n)}$. Alternatively we can also view it as a constant potential spacetime which is Lovelock flat in the critical dimension.

If we do not set $\Lambda = 0$ in the solution (9), it would describe the analogue of BTZ black hole [7] in the critical $d = 2n + 1$ dimension. Note that the BTZ black hole is the solution of $G_{ab}^{(n)} = \Lambda g_{ab}$ in the critical dimension, $d = 2n + 1$ and hence it exists only in the critical dimension. Thus BTZ black hole with all its peculiar and remarkable properties exists in all critical dimensions with the corresponding "curvature" $R_{abcd}^{(n)}$ being constant. Though BTZ black hole is well known but what is not so well known is the property that it occurs not only in 3-dimension but in all odd critical dimensions and its spacetime is indeed of constant curvature, $R_{abcd}^{(n)}$. This we believe is a new feature that has got uncovered through our higher order curvature analysis.

The main motivation for this investigation was to explore the universal features of gravity in higher dimensions. Starting from the universality of gravity inside uniform density sphere [8] followed by the thermodynamical universality of pure Lovelock black hole [9], this is yet another new interesting universal feature we have added. The Lovelock gravity is always trivial in the critical dimension $d = 2n + 1$ with the corresponding curvature $R_{abcd}^{(n)}$ vanishing, however it always represents a global monopole for the Einstein gravity. What it means in general is that the Lovelock degree n does not matter for gravity in the critical dimension. In the critical dimension gravity is always universal including the BTZ black hole as well as its global monopole description in the Einstein sector. This is indeed a very remarkable general result.

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