Some Inequalities for the Rational Power of a Nonnegative Definite Matrix

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ABSTRACT

In this note the author gives a simple proof of the following fact: Let \( r \) and \( s \) be two positive rational numbers such that \( r \leq s \) and let \( A \) and \( B \) be two \( n \times n \) nonnegative definite Hermitian matrices such that \( A^r \succeq B^r \). Then \( A^s \succeq B^s \).

1. INTRODUCTION AND STATEMENT OF THE THEOREMS

We denote by \( F \) the field \( R \) of real numbers, the field \( C \) of complex numbers, or the skew field \( H \) of real quaternions. If \( X \) is a matrix with elements in \( F \), we denote by \( X^* \) its conjugate transpose. In all three cases of \( F \), an \( n \times n \) matrix \( A \) is said to be Hermitian if \( A = A^* \) and unitary if \( AA^* = I \), where \( I \) is the identity matrix. Thus, if \( F = R \), then the words "Hermitian" and "unitary" merely mean "symmetric" and "orthogonal" respectively. We denote by \( A \succeq B \) (\( A \succ B \) resp.) that \( A \) and \( B \) are Hermitian of size \( n \) and \( A - B \) is nonnegative definite (positive definite resp.) and denote by \( Q \) and \( Q^+ \) the set of rational numbers and the set of positive rational numbers respectively.

Let \( A \succeq 0 \) and \( r = \frac{p}{q} \), where \( p \) and \( q \) are positive integers. Then we define \( A^r = (A^{1/q})^p \), where \( A^{1/q} \) is the unique nonnegative definite \( q \)th root of \( A \). It is easy to verify that \( A^r \) is well-defined. It is known that any Hermitian matrix can be diagonalized by a unitary matrix (for \( F = R \) or \( C \), this is well-known; for \( F = H \), see [1] or [2]). Now if \( A = U \text{diag}\{a_1, \ldots, a_n\}U^* \), where \( U \) is unitary, then \( A^r = U \text{diag}\{a_1^r, \ldots, a_n^r\}U^* \) and hence
\[ A^r A^s = A^{r+s}, \]  
for all \( r, s \in Q^+ \). If \( A > 0 \), then define \( A^0 = I \) and \( A^{-r} = (A^r)^{-1} \) \( (r \in Q^+) \) and in this case equality (1) also holds for all \( r, s \in Q \).

It is known (for example, see [3]) that if \( A \succeq B \succeq 0 \), then \( A^{1/2} \succeq B^{1/2} \). The purpose of this note is to give a simple proof of the following theorems which the author is not able to trace whether are known or not.

**Theorem 1.** If \( A, B \succeq 0 \) and \( A^r \succeq B^r \) for some \( r \in Q^+ \), then \( A^s \succeq B^s \) for all \( s \in Q^+ \) such that \( r \succeq s \).

**Theorem 2.** If \( A, B \succeq 0 \) and \( A^r > B^r \) for some \( r \in Q^+ \), then \( A^s > B^s \) for all \( s \in Q^+ \) such that \( r \succeq s \).

2. PROOF OF THE THEOREMS

We first prove some lemmas.

**Lemma 1.** If \( X, Y \succeq 0 \), \( P \succeq T > 0 \), and \( XTX \succeq YPY \), then \( X \succeq Y \).

**Proof.** Since \( X, Y \succeq 0 \), \( X \) and \( Y \) can be diagonalized simultaneously [4] by a cogredient transformation. Hence there exists a nonsingular matrix \( S \) such that \( X = S^*XS \) and \( Y = S^*YS \), where \( X \) and \( Y \) are diagonal matrices. Now \( X \succeq Y \) is equivalent to \( X \succeq Y \) and \( XTX \succeq YPY \) is equivalent to \( X^T X \succeq Y^T Y \), where \( T = STS^* \) and \( P = SPS^* \). Therefore, without loss of generality, we may assume that \( X = \text{diag}\{x_1, \ldots, x_n\} \) and \( Y = \text{diag}\{y_1, \ldots, y_n\} \), where \( x_1, \ldots, x_n, y_1, \ldots, y_n \) are nonnegative real numbers. From \( XTX \succeq YPY \) we have

\[ x_i^2 t_{ii} \succeq y_i^2 p_{ii} \quad (i = 1, \ldots, n), \]  
where \( t_{ii} \) and \( p_{ii} \) are the diagonal elements of \( T \) and \( P \) respectively. Since \( P \succeq T > 0 \), we have \( p_{ii} \geq t_{ii} > 0 \). From this and inequality (2) our lemma follows. \( \blacksquare \)

By taking \( X = A^{1/2} \) and \( Y = B^{1/2} \) and \( P = T = I \), we have the following corollary.

**Corollary.** If \( A \succeq B \succeq 0 \), then \( A^{1/2} \succeq B^{1/2} \).

**Lemma 2.** If \( X \succeq Y \succeq 0 \) and \( X > 0 \), then \( Y \succeq YX^{-1}Y \).
Proof. Again we may assume that $X$ and $Y$ are of the form as in the proof of Lemma 1. Then $YX^{-1}Y = \text{diag}\{x_1^{-1}y_1^2, \ldots, x_n^{-1}y_n^2\}$ and the lemma follows immediately from these explicit expressions. 

The following lemma is known (see the proof of Lemma 2 of [4]).

**Lemma 3.** If $A, B \geq 0$ and $A = \text{diag}\{A_1, 0\}$, where $A_1$ is of size $m$ and $> 0$ and if, for any $u \in F^n$, $uAu^* = 0$ implies $uBu^* = 0$, then $B = \text{diag}\{B_1, 0\}$, where $B_1$ is of size $m$.

**Lemma 4.** If $A, B \geq 0$, $A^r \geq B^r$ and $A^s \geq B^s$, where $r, s \in Q^+$, then $A^{(r+s)/2} \geq B^{(r+s)/2}$.

**Proof.** By using the canonical forms of $A$ and $A^r$ in the previous section, it is obvious that, for any $u \in F^n$, $uAu^* = 0$ if and only if $uA^ru^* = 0$ (and similarly for $B$). Now suppose that $uAu^* = 0$. Then $uA^ru^* = 0$. Since $A^r \geq B^r \geq 0$, we have $uB^ru^* = 0$ and hence $uBu^* = 0$. Therefore, by Lemma 3, we may assume $A > 0$, and we also assume $r > s$. From equality (1) and Lemma 2, we have

$$A^{(r+s)/2}A^{-s}A^{(r+s)/2} = A^r,$$

$$\geq B^r,$$

$$= B^{(r-s)/2}B^sB^{(r-s)/2},$$

$$\geq B^{(r-s)/2}B^sA^{-s}B^sB^{(r-s)/2},$$

$$= B^{(r+s)/2}A^{-s}B^{(r+s)/2}.$$

From this and Lemma 1 our lemma follows. 

**Proof of Theorem 1.** Let $s \in Q^+$ such that $r \geq s$. Then, by the Corollary and Lemma 4, we see that there exists a sequence $s_1, s_2, \ldots$ in $Q^+$ such that $\lim_{n \to \infty} s_n = s$ and $A^{s_n} \geq B^{s_n}$ for all $n$. Hence

$$A^s = \lim_{n \to \infty} A^{s_n} \geq \lim_{n \to \infty} B^{s_n} = B^s.$$

Theorem 2 follows from Theorem 1 and the fact that if $A^r > B^r$, then $A^r > (B + \epsilon I)^r$ for sufficiently small positive $\epsilon$. 

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REFERENCES


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