A LEAST-SQUARES FINITE ELEMENT METHOD
FOR THE NAVIER-STOKES EQUATIONS

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Abstract—A finite element method based on a least-squares variational principle is developed for
the velocity-vorticity-pressure formulation of the Navier-Stokes equations. The method can be imple-
mented so that only symmetric, positive definite matrices are encountered. Also, the method seems
to be nearly optimally accurate.

1. INTRODUCTION

A finite element method based on a least-squares variational principle is developed for the ap-
proximate solution of the stationary, incompressible Navier-Stokes equations. These equations
are cast into a first-order system of partial differential equations involving the velocity, vorticity,
and pressure as dependent variables. In three-dimensions one has seven unknown scalar fields.
However, the application of a least-squares principle along with, for example, a Newton lineariza-
tion, results in symmetric, positive definite, linear systems, at least in a neighborhood of the
solution. Thus, if properly implemented, one can expect to only encounter symmetric, positive
definite, linear systems in the solution procedure. A further advantage of this method is that
a single piecewise polynomial finite element space may be used for all test and trial functions.
A final advantage resulting from the use of a least squares principle is that, unlike some other
methods involving the vorticity, no artificial numerical boundary conditions for the vorticity need
be devised.

The method is based on the velocity-vorticity-pressure form of the Navier-Stokes equations:

\[ \text{div } u = 0 \quad \text{in } \Omega, \]  
\[ \omega = \text{curl } u \quad \text{in } \Omega, \]  
\[ \nu \text{curl } \omega + \omega \times u + \text{grad } p = f \quad \text{in } \Omega, \]  

where \( \Omega \subset \mathbb{R}^3 \) denotes the flow domain with \( \Gamma \) its boundary, \( u, \omega, \) and \( p \) denote the velocity, vorticity, and total pressure, respectively, \( \nu \) the inverse of the Reynolds number, and \( f \) a given
body force. (The method discussed here is similar to the ones of [1-10]; however, there are
also crucial differences in the formulation, discretization, linearization, and solution procedures.)
From both the points of view of analyses and computations, it is advantageous (see the references)
to explicitly add to (1)-(3) the seemingly redundant relation

\[ \text{div } \omega = 0 \quad \text{in } \Omega. \]  

The system (1)-(4) should be supplemented with boundary conditions. Here, we only consider

\[ u = 0 \quad \text{on } \Gamma. \]  

Note that (5) implies that

\[ \omega \cdot n = 0 \quad \text{on } \Gamma, \]  

where \( n \) denotes the unit outer normal to \( \Omega \).

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2. THE LEAST-SQUARES PRINCIPLE

We use the standard notation and definitions for the Sobolev spaces \( H^s(\Omega), s \in \mathbb{R} \); the standard associated inner products and norms are denoted by \((\cdot, \cdot)_s\) and \(\| \cdot \|_s\), respectively; \( H^s(\Omega) \) denotes the vector-valued Sobolev spaces whose inner products and norms will also be denoted by \((\cdot, \cdot)_s\) and \(\| \cdot \|_s\), respectively. Of course, \( L^2(\Omega) = H^0(\Omega) \). We introduce the vector spaces:

\[
V_0 = \{ v \in H^1(\Omega) \mid v|_{\Gamma} = 0 \}, \\
Z_0 = \{ \zeta \in H^1(\Omega) \mid (\zeta \cdot n)|_{\Gamma} = 0 \}, \\
Q_0 = \left\{ q \in H^1(\Omega) \mid \int_{\Omega} q \, d\Omega = 0 \right\}.
\]

We use (1)-(4) in a natural way to define the least-squares functional:

\[
\mathcal{J}(u, \omega, \rho) = \int_{\Omega} \left[ (\text{curl } u - \omega)^2 + |\text{div } u|^2 + |\text{div } \omega|^2 + |\nu \text{curl } \omega + \omega \times u + \text{grad } \rho - f|^2 \right] \, d\Omega.
\]  

(7)
The least-squares principle is then given by

\[
\text{seek } u \in V_0, \omega \in Z_0, \text{ and } \rho \in Q_0 \text{ such that }
\]

\[
\mathcal{J}(u, \omega, \rho) \leq \mathcal{J}(v, \zeta, q) \quad \forall \ v \in V_0, \zeta \in Z_0, \text{ and } q \in Q_0.
\]  

(8)

Standard techniques from the calculus of variations may be used to deduce that any solution \((u, \omega, \rho)\) of (8) necessarily satisfies:

\[
\int_{\Omega} \left[ (\text{curl } u - \omega) \cdot \text{curl } v + \text{div } u \text{div } v \right. \\
+ (\nu \text{curl } \omega + \text{grad } \rho + \omega \times u - f) \cdot (\omega \times v) \left. \right] \, d\Omega = 0 \quad \forall \ v \in V_0,
\]

(9)

\[
\int_{\Omega} \left[ \text{div } \omega \text{div } \zeta - (\text{curl } u - \omega) \cdot \zeta \right. \\
+ (\nu \text{curl } \omega + \text{grad } \rho + \omega \times u - f) \cdot (\nu \text{curl } \zeta + \zeta \times u) \left. \right] \, d\Omega = 0 \quad \forall \ \zeta \in Z_0,
\]

(10)

\[
\int_{\Omega} (\nu \text{curl } \omega + \text{grad } \rho + \omega \times u - f) \cdot \text{grad } q \, d\Omega = 0 \quad \forall \ q \in Q_0.
\]

(11)

3. FINITE ELEMENT METHODS

For the sake of simplicity, we only consider the case wherein \( \Omega \) is a polyhedral domain. Starting with the weak formulation (9)-(11), a conforming finite element method can be defined in a completely standard manner. We choose a finite dimensional subspace \( S^h \subset H^1(\Omega) \) parametrized by \( h \). For example, for a given positive integer \( r \), \( S^h \) could consist of continuous (over \( \Omega \)) piecewise polynomials of degree less than or equal to \( r \) with respect to a subdivision of \( \Omega \) into finite elements. In this case, the parameter \( h \) may be related to the size of the grid. We assume that the space \( S^h \) satisfies the usual regularity properties. We then define the spaces:

\[
V_0^h = \{ v \in S^h \mid v_i \in S^h, i = 1, 2, 3 \}, \\
Z_0^h = \{ \zeta \in S^h \mid \zeta \cdot n = 0 \text{ on } \Gamma \} \subset Z_0, \\
Q_0^h = \left\{ q \in S^h \mid \int_{\Omega} q \, d\Omega = 0 \right\} \subset Q_0.
\]

Note that all of the discrete variables, i.e., \( h \) and the components of \( u^h \) and \( \omega^h \), are approximated by the same degree continuous piecewise polynomials defined with respect to a single grid.

The discrete problem we consider is then to seek \( u^h \in V_0^h, \omega^h \in Z_0^h, \text{ and } p^h \in Q_0^h \) such that \((9)-(11)\) are satisfied for all \( v \in V_0^h, \omega \in Z_0^h, \text{ and } q \in Q_0^h \), respectively. This problem may also be derived directly as the necessary conditions for the finite dimensional least-squares principle:

\[
\text{seek } u^h \in V_0^h, \omega^h \in Z_0^h, \text{ and } p^h \in Q_0^h \text{ such that }
\]

\[
\mathcal{J}(u^h, \omega^h, p^h) \leq \mathcal{J}(v^h, \zeta^h, q^h) \quad \forall \ v^h \in V_0^h, \zeta^h \in Z_0^h, \text{ and } q^h \in Q_0^h.
\]  

(12)
4. NEWTON'S METHOD

The discrete equations resulting from (12) are equivalent to a nonlinear system of algebraic equations that must be solved in an iterative manner. There are many methods that one might use for such a purpose; here we only consider Newton's method, which in the present context, is defined as follows. Given initial guesses $u^{(0)}$, $\omega^{(0)}$, and $p^{(0)}$ for $u^h$, $\omega^h$, and $p^h$, respectively, the sequence of Newton iterates $\{u^{(k)}, \omega^{(k)}, p^{(k)}\}_{k>0}$ is generated recursively by solving, for $k = 1, 2, \ldots$, the system:

$$
J^h \left[ \left( \text{curl } u^{(k)} - \omega^{(k)} \right) \cdot \text{curl } v + \text{div } u^{(k)} \text{div } v 
+ \left( \nu \text{curl } \omega^{(k)} + \text{grad } p^{(k)} + \omega^{(k)} \times u^{(k-1)} + \omega^{(k-1)} \times u^{(k)} \right) \cdot (\omega^{(k-1)} \times v) 
+ \left( \nu \text{curl } \omega^{(k-1)} + \text{grad } p^{(k-1)} + \omega^{(k-1)} \times u^{(k-1)} \right) \cdot (\omega^{(k-1)} \times v) \right] \, \text{d}\Omega
= \int_{\Omega} \left( \nu \text{curl } \omega^{(k-1)} + \text{grad } p^{(k-1)} + 2\omega^{(k-1)} \times u^{(k-1)} \right) \cdot (\omega^{(k-1)} \times v) \, \text{d}\Omega 
\forall \, v \in V_0^h,
$$

$$
J^h \left[ \left( \text{div } \omega^{(k)} \text{div } \zeta - \left( \text{curl } u^{(k)} - \omega^{(k)} \right) \cdot \zeta 
+ \left( \nu \text{curl } \omega^{(k)} + \text{grad } p^{(k)} + \omega^{(k)} \times u^{(k-1)} + \omega^{(k-1)} \times u^{(k)} \right) \cdot (\nu \text{curl } \zeta) 
+ \left( \nu \text{curl } \omega^{(k)} + \text{grad } p^{(k)} + \omega^{(k)} \times u^{(k-1)} + \omega^{(k-1)} \times u^{(k)} \right) \cdot (\zeta \times u^{(k-1)}) 
+ \left( \nu \text{curl } \omega^{(k-1)} + \text{grad } p^{(k-1)} + \omega^{(k-1)} \times u^{(k-1)} \right) \cdot (\zeta \times u^{(k-1)}) \right] \, \text{d}\Omega
= \int_{\Omega} \left[ \nu (f + \omega^{(k-1)} \times u^{(k-1)}) \cdot \text{curl } \zeta 
+ \left( \nu \text{curl } \omega^{(k-1)} + \text{grad } p^{(k-1)} + 2\omega^{(k-1)} \times u^{(k-1)} \right) \cdot (\zeta \times u^{(k-1)}) \right] \, \text{d}\Omega 
\forall \, \zeta \in Z_0^h,
$$

$$
J^h \int_{\Omega} \left( \nu \text{curl } \omega + \text{grad } p + \omega \times u + \omega \times u \right) \cdot \text{grad } q \, \text{d}\Omega
= \int_{\Omega} (f + \omega \times u) \cdot \text{grad } q \, \text{d}\Omega 
\forall \, q \in Q_0^h.
$$

This system of linear algebraic equations that determines the $k^{th}$ Newton iterate from the $(k - 1)^{st}$ looks rather formidable. However, as discussed below, it also has some very good features.

5. RESULTS

We outline some results that we have obtained in connection with the least-squares algorithm discussed above. Details will be reported on elsewhere.

It is easy to see that the Newton equations constitute a symmetric linear algebraic system. Moreover, in a neighborhood of a solution of (12), the coefficient matrix of that system is positive definite. Thus, in a neighborhood of a solution of (12), the Newton system is symmetric and positive definite. This feature is independent of the value of $\nu$, i.e., of the value of the Reynolds number. These observations, along with the guaranteed local and quadratic convergence of Newton's method, are potentially very advantageous. In particular, if the Newton linearization is properly coupled to an appropriate continuation method, one should encounter only symmetric, positive definite, linear systems. One may then use standard iterative methods, e.g., conjugate gradients, to solve these linear systems; this, of course, is a great advantage, especially for three-dimensional problems.

The system (1)–(6) is elliptic, but does not satisfy the complementing conditions of [11,12] if we require that all of $u$, $\omega$, and $p$ belong to $H^1(\Omega)$. As a result, one cannot, using standard techniques, derive error estimates for finite element approximations that employ merely continuous basis functions. In addition, other methods based on vorticity formulations of the Navier-Stokes equations are known to yield very poor vorticity approximations. However, we have carried out
an extensive numerical study of the accuracy of the least-squares finite element method applied to (1)–(6) and have found that, at the least, approximations are nearly optimally accurate in the sense that the rates of convergence are nearly those of the best approximations to the solutions out of the finite element space employed. Note that all variables are approximated to the same accuracy, and that the same degree polynomials with respect to the same grid may be used for all variables.

REFERENCES