Classes of chromatically equivalent graphs and polygon trees

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Abstract

We give the definition of complete classes of chromatically equivalent graphs and some results on this topic. We give an invariant for generalized polygon tree under chromatic equivalence, which is useful in searching for chromatically equivalent graphs. As a consequence we show that \( \{C_{k_1}, \ldots, C_{k_j}, k\{K_2\}\} \) is a complete class of chromatically equivalent graphs, which solves a problem raised in Whitehead Jr (1988).

0. Introduction

The graphs considered here are finite, undirected, simple and loopless. If two graphs \( G \) and \( H \) have the same chromatic polynomial, i.e., \( P(G, \lambda) = P(H, \lambda) \), we say that \( G \) and \( H \) are chromatically equivalent. If \( P(G, \lambda) = P(H, \lambda) \) implies that \( H \) is isomorphic to \( G \) we say that \( G \) is chromatically unique. There has been much work on searching for chromatically unique graphs, see [5].

Read [6] proved that a graph \( G \) of order \( n \) is a tree iff \( P(G, \lambda) = \lambda(\lambda - 1)^{n-1} \). A tree of order \( n \) can be viewed as a graph obtained from \( n - 1 \) edges by overlapping on vertices. Similarly, a graph obtained from triangles by overlapping on edges is called a 2-tree. Whitehead [7] showed that a graph \( G \) of order \( n \) is a 2-tree iff \( P(G, \lambda) = \lambda(\lambda - 1)(\lambda - 2)^{n-2} \). Whitehead's result has been extended in two ways in [1] and [2]: instead of triangles overlapping in edges, we consider complete graphs, \( K_q \)'s, overlapping in \( K_{q-1} \) and cycles, \( C_k \)'s, overlapping in edges. A question was raised in [8] whether we can extend the results to a graph obtained from a class of cycles, not necessarily with same length, overlapping in edges.

Motivated by these results, we consider classes of graphs with the same chromatic polynomial. In Section 1, we define maximal class of chromatically equivalent graphs.
and give some necessary conditions. In Section 2, we define complete classes of chromatically equivalent graphs and extend a result in [9].

Chao and Zhao [4] proved that if two connected graphs are chromatically equivalent, then either both of them or none of them contain subgraphs homeomorphic to $K_4$. In Section 3, we study the graphs which contain no subgraphs homeomorphic to $K_4$, and show that they are just generalized polygon trees (for definition, see Section 3). In Section 4, we introduce the definition of intercourse number of a generalized polygon tree and show that it is an invariant of generalized polygon tree under chromatic equivalence. The conclusion is very useful in study of chromatic unique graphs and complete classes of chromatically equivalent graphs. As a consequence we show that $\{\{C_i, ..., C_k\}, k\{K_2\}\}$ is a complete class of chromatically equivalent graphs, which solves Problem 2 raised in [8].

1. Maximal classes of chromatically equivalent graphs

A generalized $\theta$-graph $\theta_{i,j,k}$ is a graph which consists of three paths with lengths $i$, $j$ and $k$ joining two vertices. A cycle is said to be a mini-cycle if it is a cycle without chords. A connected graph is said to be forest-like if any two cycles of the graph have at most one edge in common. In [3], Chao and Whitehead proved the following result.

**Theorem 1.** All forest-like connected graphs with order $n$, size $m$ and the same number of mini-cycles of each length are chromatically equivalent.

Now we first generalize the above theorem.

**Theorem 2.** Let $G_0, G_1, ..., G_p$ be $p + 1$ graphs and $K_{i_1}, ..., K_{i_p}$ be $p$ complete graphs. Suppose that $G_0$ and $G_1$ overlap in $K_{i_1}$ to get a graph called $G'_1$, $G'_1$ and $G_2$ overlap in $K_{i_2}$ to get a graph called $G'_2$, ..., finally, $G'_{p-1}$ and $G_p$ overlap in $K_{i_p}$ to get a graph denoted by $G$. Then the chromatic polynomial of $G$ depends only on the class of graphs $\mathcal{G} = \{G_0, ..., G_p\}$ and $\mathcal{X} = \{K_{i_1}, ..., K_{i_p}\}$, but does not depend on the position of $K_{i_j}$ in $G_{j-1}$ and $G_j$, $j = 1, ..., p$, or on the order of overlapping.

**Proof.** $P(G, \lambda) = \prod_{i=0}^p P(G_i, \lambda)/\prod_{i=1}^p P(K_{i_j}, \lambda)$. This theorem means that both $\mathcal{G}$ and $\mathcal{X}$ are unordered sets. The only requirement is that we can execute the overlapping process to the end, and when $G_{j-1}$ and $G_j$ overlap in $K_{i_j}$, they can overlap in any $K_{i_j}$ subgraph of $G_{j-1}$ and $G_j$. The resultant graphs have the same chromatic polynomial.

Let $\mathcal{G} = \{G_0, ..., G_p\}$ and $\mathcal{X} = \{K_{i_1}, ..., K_{i_p}\}$. Then all graphs obtained from $G_0, ..., G_p$ by overlapping in $K_{i_1}, ..., K_{i_p}$ in different positions and different orders form a class of graphs. We denote it by $\{\mathcal{G}, \mathcal{X}\}$. For any graph $G$, we can always find two
classes of graphs \( \mathcal{G} \) and \( \mathcal{K} \) such that \( G \in \{ \mathcal{G}, \mathcal{K} \} \). Particularly we can take \( \mathcal{G} = \{ G \} \) and \( \mathcal{K} = \emptyset \).

Let \( \mathcal{G} = \{ C_{i_0}, \ldots, C_{i_t}, K_2, \ldots, K_2 \} \) and \( \mathcal{K} = \{ K_2, \ldots, K_2, K_1, \ldots, K_1 \} \), where the number of \( K_2 \) in \( \mathcal{G} \) is \( p \), the number of \( K_2 \) in \( \mathcal{K} \) is \( k \) and the number of \( K_1 \) in \( \mathcal{K} \) is \( p \).

Then \( \{ \mathcal{G}, \mathcal{K} \} \) is the class of graphs described in Theorem 1 and we can easily get

\[ \text{Theorem 1.} \]

Given any \( \mathcal{G} \) and \( \mathcal{K} \), can \( \{ \mathcal{G}, \mathcal{K} \} \) form a class of graphs? In general, the answer is 'No'. For instance, \( \{ \{K_2, K_3\}, \{K_4\} \} \) is not a class of graphs. Denoting the clique number of \( G \) by \( \omega(G) \), we can easily get the following lemma.

**Lemma 3.** Let \( \mathcal{G} = \{ G_0, \ldots, G_p \} \) and \( \mathcal{K} = \{ K_i, \ldots, K_i \} \), where \( \omega(G_0) \geq \cdots \geq \omega(G_p) \) and \( i_1 \geq \cdots \geq i_p \). Then \( \{ \mathcal{G}, \mathcal{K} \} \) is a class of graphs if and only if

\[ \omega(G_j) \geq i_j, \quad j = 1, \ldots, p \]

Now we define the clique-like number \( \omega' \) of a graph \( G \).

\[
\omega'(G) = \begin{cases} 
\omega(G) & \text{if } G \neq K_i, \\
1 - 1 & \text{if } G = K_i.
\end{cases}
\]  

We know that \( \{ \{K_4\}, \emptyset \} = \{ \{K_4, K_3\}, \{K_3\} \} = \{ \{K_4, K_3, K_2\}, \{K_3, K_2\} \} = \{ \{K_4\} \cup k\{K_1\}, k\{K_1\} \} \). That is to say that the expressions of a class of graphs are not unique. In the example of \( \{ \{K_4, K_3, K_2\}, \{K_3, K_2\} \} \), \( K_3 \) and \( K_2 \) are redundant factors. In order to avoid such troubles, we introduce the following definition.

**Definition 4.** If \( \{ \mathcal{G}, \mathcal{K} \} = \{ \mathcal{G}', \mathcal{K}' \} \) implies \( \mathcal{G} \subseteq \mathcal{G}' \) and \( \mathcal{K} \subseteq \mathcal{K}' \), then we say \( \{ \mathcal{G}, \mathcal{K} \} \) is a proper expression of a class of graphs.

**Lemma 5.** Let \( \mathcal{G} = \{ G_0, \ldots, G_p \} \) and \( \mathcal{K} = \{ K_i, \ldots, K_i \} \), where \( \omega'(G_0) \geq \cdots \geq \omega'(G_p) \) and \( i_1 \geq \cdots \geq i_p \). Then \( \{ \mathcal{G}, \mathcal{K} \} \) is a proper expression of a class of graphs if and only if

\[ \omega'(G_j) \geq i_j, \quad j = 1, \ldots, p. \]

If \( G \) can be obtained from \( G_1 \) and \( G_2 \) by overlapping in \( K_i \) where \( \omega'(G_1) \geq \omega'(G_2) \geq i \), then \( G \) is said to be separable. For convenience sake in our discussion, we always avoid separable graphs in \( \mathcal{G} \) when we write a class of graphs \( \{ \mathcal{G}, \mathcal{K} \} \).

Let \( G \) be obtained from \( C_3 \) and \( C_3 \) by overlapping in \( K_i \), i.e., \( G \in \{ \{C_3, C_3\}, \{K_1\} \} = \{ \mathcal{G}_1, \mathcal{K}_1 \} \). We can also see that \( G \in \{ \{C_3, C_3, K_2\}, \{K_2, \{K_1\} \} = \{ \mathcal{G}_2, \mathcal{K}_2 \} \). It is easy to see that \( \{ \mathcal{G}_1, \mathcal{K}_1 \} \subset \{ \mathcal{G}_2, \mathcal{K}_2 \} \) and \( \{ \mathcal{G}_1, \mathcal{K}_1 \} \neq \{ \mathcal{G}_2, \mathcal{K}_2 \} \).

**Definition 6.** If \( \{ \mathcal{G}, \mathcal{K} \} \subset \{ \mathcal{G}', \mathcal{K}' \} \) implies \( \{ \mathcal{G}, \mathcal{K} \} = \{ \mathcal{G}', \mathcal{K}' \} \), then we say that \( \{ \mathcal{G}, \mathcal{K} \} \) is a maximal class of the chromatically equivalent graphs.
We can see that \( \{G_2, X_2\} \) is a maximal class, but \( \{G_1, X_1\} \) is not. In fact \( |\{G_2, X_2\}| = 3 \) and \( |\{G_1, X_1\}| = 1 \).

We have to emphasize here that in the overlapping process, in order to get maximal class of chromatically equivalent graphs, we allow a graph to overlap with \( K_r \), in \( K_r \), which is just itself.

**Theorem 7.** Let \( \mathcal{G} = \{G_0, \ldots, G_p\} \) and \( \mathcal{X} = \{K_{i_1}, \ldots, K_{i_k}\} \), where \( \omega'(G_0) \geq \cdots \geq \omega'(G_p) \) and \( i_1 \geq \cdots \geq i_p \). Then \( \{\mathcal{G}, \mathcal{X}\} \) is a maximal class if and only if

\[ \omega'(G_j) = i_j, \quad j = 1, \ldots, p. \]

**Proof (Necessity).** Because \( \{\mathcal{G}, \mathcal{X}\} \) is a graph class, by Lemma 5, \( \omega'(G_j) \geq i_j, j = 1, \ldots, p \). If the condition is not satisfied, let \( k = \max\{j|\omega'(G_j) > i_j\} \) and \( \mathcal{G}' = \mathcal{G} \cup \{K_{i_{k+1}}\}, \mathcal{X}' = \mathcal{X} \cup \{K_{i_{k+1}}\} \). Then \( \{\mathcal{G}', \mathcal{X}'\} \) is a class of graphs and \( \{\mathcal{G}, \mathcal{X}\} \subseteq \{\mathcal{G}', \mathcal{X}'\} \). We only need to show that there exists a graph \( H \in \{\mathcal{G}, \mathcal{X}\} \) and \( H \in \{\mathcal{G}', \mathcal{X}'\} \). Without loss of generality, we may assume \( k = p \). By Lemma 3, \( \{\mathcal{G}', \mathcal{X}' \setminus \{K_{i_p}\}\} \) is also a class of graphs. Let \( H_1 \in \{\mathcal{G}', \mathcal{X}' \setminus \{K_{i_p}\}\} \) and \( H \) be obtained from \( H_1 \) and \( K_{i_{k+1}} \) by overlapping in \( K_{i_p} \). Then \( H \in \{\mathcal{G}', \mathcal{X}'\} \) and \( H \in \{\mathcal{G}', \mathcal{X}'\} \), as required.

**Sufficiency.** Now suppose that \( \mathcal{G} \) and \( \mathcal{X} \) satisfy the condition. If \( \{\mathcal{G}, \mathcal{X}\} \) is not a maximal class, i.e., there exist \( \mathcal{G}' \) and \( \mathcal{X}' \) such that \( \{\mathcal{G}, \mathcal{X}\} \subseteq \{\mathcal{G}', \mathcal{X}'\} \) and \( \{\mathcal{G}, \mathcal{X}\} \neq \{\mathcal{G}', \mathcal{X}'\} \). Let \( \mathcal{G}_1 = \mathcal{G} \cup \{H_1, \ldots, H_k\}, \mathcal{X}_1 = \mathcal{X} \cup \{K_{r_1}, \ldots, K_{r_k}\} \), where \( \omega'(H_1) \geq \cdots \geq \omega'(H_k) \) and \( r_1 \geq \cdots \geq r_k \). We claim that \( \omega'(H_j) \geq r_j, j = 1, \ldots, k \).

When \( j = 1 \), assume that \( \omega'(G_s) = i_s, s = 1, \ldots, i \), so \( \omega'(H_1) \geq r_1 \). When \( j = 2 \), assume that \( \omega'(G_s) \geq \omega'(H_2) \geq \omega'(G_{i+1}) \). Since \( r_1 \geq r_2 \) and \( \omega'(G_s) = i_s, s = i+1, \ldots, i' \), we have \( \omega'(H_j) \geq r_j \). By the same reason, \( \omega'(H_j) \geq r_j \) also hold for \( j > 2 \). Therefore,

\[
\sum_{H \in \mathcal{G}} |V(H)| \geq \sum_{K \in \mathcal{X}} |V(K)|.
\]

If \( G' \in \{\mathcal{G}, \mathcal{X}\} \) and \( G'' \in \{\mathcal{G}', \mathcal{X}'\} \), then

\[
|V(G'')| = \sum_{G \in \mathcal{G}} |V(G)| - \sum_{K \in \mathcal{X}} |V(K)|
= \sum_{G \in \mathcal{G}} |V(G)| - \sum_{K \in \mathcal{X}} |V(K)| + \sum_{H \in \mathcal{G}_1} |V(H)| - \sum_{K \in \mathcal{X}_1} |V(K)|
> |V(G'')|.
\]

This inequality implies that \( G' \) is not chromatically equivalent to \( G'' \). This is a contradiction which shows that \( \{\mathcal{G}, \mathcal{X}\} \) is a maximal class. \( \square \)

By Theorem 7, if a class of graphs \( \{\mathcal{G}, \mathcal{X}\} \) is given, we can always get a maximal class \( \{\mathcal{G}', \mathcal{X}'\} \) such that \( \{\mathcal{G}, \mathcal{X}\} \subseteq \{\mathcal{G}', \mathcal{X}'\} \). The method is just to add complete graphs to both \( \mathcal{G} \) and \( \mathcal{X} \) until the condition of Theorem 7 is satisfied.
2. Complete classes of chromatically equivalent graphs

By Lemma 3, all graphs belonging to a class of graphs \( \mathcal{G}, \mathcal{N} \) are chromatically equivalent. Now we ask the following question: if \( H \) is chromatically equivalent to the graphs in \( \{ \mathcal{G}, \mathcal{N} \} \) (or simply, \( H \) is chromatically equivalent to \( \{ \mathcal{G}, \mathcal{N} \} \)) can we assert that \( H \in \{ \mathcal{G}, \mathcal{N} \} \)? If \( \{ \mathcal{G}, \mathcal{N} \} \) is not a maximal class, the answer is clearly 'No' (i.e., Theorem 9). But what is the case if we consider the maximal classes? In general, the answer is negative. For example, as shown in [10], \( W_6 \) is chromatically equivalent to \( \{ \{ K_4, K_4, \}, \{ K_3, K_2 \} \} \) but \( W_6 \notin \{ \{ K_4, K_4, \}, \{ K_3, K_2 \} \} \), where it is easy to see that \( \{ \{ K_4, K_4, \}, \{ K_3, K_2 \} \} \) is a maximal class. Now the further question is: when is the answer positive?

**Definition 8.** If the chromatic equivalence of \( H \) to \( \{ \mathcal{G}, \mathcal{N} \} \) implies \( H \in \{ \mathcal{G}, \mathcal{N} \} \), then \( \{ \mathcal{G}, \mathcal{N} \} \) is said to be a complete class of chromatically equivalent graphs.

We can see that this is a significant concept. Although any graph in \( \{ \mathcal{G}, \mathcal{N} \} \) is not chromatically unique if \( |\{ \mathcal{G}, \mathcal{N} \}| \geq 2 \), when studying the chromaticity, we can see that a complete class plays the same role as a chromatically unique graph. In fact, study on complete classes has been done for a few years and some interesting results have been obtained. For instance, it was proved [3] that \( \theta \)-graphs are chromatically unique, i.e., \( \{ \{ C_i, C_j \}, \{ K_2 \} \} \) is a complete class for any given \( i \) and \( j \). It was also proved that \( q \)-trees [2] and \( n \)-gon trees [1] have the above property, i.e. \( \{ (p+1)K_n \}, p\{ K_{n-1} \} \) and \( \{ (p+1)C_n \}, p\{ K_2 \} \) are all complete classes.

**Theorem 9.** A complete class is a maximal class.

**Proof.** The conclusion can be easily obtained from the proof of Necessity of Theorem 7.

Theorem 9 brings us some convenience in searching for complete classes.

In [9], it was proved that \( \{ \{ C_4 \} \cup k\{ K_3 \}, k\{ K_1 \} \} \) and \( \{ \{ K_n \} \cup k\{ K_2 \}, k\{ K_1 \} \} \) are both complete classes. Now we generalize these results.

**Theorem 10.** If \( \{ \mathcal{G}, \mathcal{N} \} \) is a complete class, then \( \mathcal{G} \cup k\{ K_2 \}, \mathcal{N} \cup k\{ K_1 \} \) is also a complete class.

**Proof.** Suppose that \( G \in \{ \mathcal{G}, \mathcal{N} \} \) where \( \{ \mathcal{G}, \mathcal{N} \} \) is a maximal class and \( G \) is chromatically equivalent to \( \{ \mathcal{G} \cup k\{ K_2 \}, \mathcal{N} \cup k\{ K_1 \} \} \). By Theorem 1 in [6], the nontrivial blocks of \( G \) are as many as those of \( H \in \{ \mathcal{G} \cup k\{ K_2 \}, \mathcal{N} \cup k\{ K_1 \} \} \). It is easy to see that \( k\{ K_2 \} \subseteq \mathcal{G} \) and \( k\{ K_1 \} \subseteq \mathcal{N} \). Let \( G_1 \in \{ \mathcal{G} \setminus k\{ K_2 \}, \mathcal{N} \setminus k\{ K_1 \} \} \). Then \( G_1 \) is chromatically equivalent to \( \{ \mathcal{G}, \mathcal{N} \} \). Since \( \{ \mathcal{G}, \mathcal{N} \} \) is a complete class, \( G_1 \in \{ \mathcal{G}, \mathcal{N} \} \). Therefore \( \{ \mathcal{G}, \mathcal{N} \} = \{ \mathcal{G} \cup k\{ K_2 \}, \mathcal{N} \cup k\{ K_1 \} \} \) and \( G \in \{ \mathcal{G} \cup k\{ K_2 \}, \mathcal{N} \cup k\{ K_1 \} \} \), i.e., \( \{ \mathcal{G} \cup k\{ K_2 \}, \mathcal{N} \cup k\{ K_1 \} \} \) is a complete class. □
Using Theorem 10, we can get many complete classes, such as \( \{W_{n+1}\} \cup k\{K_2\} \), \( k\{K_1\} \), \( \{C_i, C_i\} \cup k\{K_2\} \), \( k\{K_2\} \cup k\{K_1\} \), \( \{p\{K_2\} \cup k\{K_2\} \), \( (p-1)\{K_{n-1}\} \cup k\{K_1\} \) and \( \{p\{C_n\} \cup k\{K_2\} \), \( (p-1)\{K_2\} \cup k\{K_1\} \) etc.

It seems that we have much work to do on this topic and we make the following conjectures.

**Conjecture 11.** If \( G \) is chromatically unique, nonseparable and \( \omega'(G) = k \), then \( \{p\{G\}, (p-1)\{K_k\}\} \) is a complete class.

**Conjecture 12.** Let \( \mathcal{G} = \{G_0, \ldots, G_p\} \). If \( \mathcal{G}, \mathcal{K} \) is a complete class and \( \omega'(G_0) \geq 1 \), then \( \{\mathcal{G} \cup k\{K_{i+1}\}, \mathcal{K} \cup k\{K_i\}\} \) is also a complete class.

In the following two sections, we consider the converse of Theorem 1. The question is: is \( \{C_i, \ldots, C_i, k\{K_2\}\} \) a complete class? By Theorem 10, we know that this question is equivalent to the following one: Is \( \{C_i, \ldots, C_i, k\{K_2\}\} \) a complete class? Since the conclusion holds for \( i_0 = \ldots = i_k \) and for \( k = 1 \), the question attracts the attention of some researchers. In [8], it was raised as a problem. Now we prove that it also holds for any positive integer \( k \).

### 3. Some properties of graphs having no subgraph homeomorphic to \( K_4 \)

Let \( \mathcal{G} = \{C_i, \ldots, C_i\}, \mathcal{K} = k\{K_2\} \) and \( G \in \{\mathcal{G}, \mathcal{K}\} \). Then we call \( G \) a polygon tree. In the following, we always let \( y = \lambda - 1 \). First we give some known results.

**Theorem 13** (Chao and Zhao [4]). Let \( G \) be a connected graph with more than 3 vertices and \( P(G) = yT(G, y) \). Then

1. \( T(G, 0) = 0 \) if and only if \( G \) has at least one cut point;
2. \( |T(G, 0)| = 1 \) if and only if \( G \) is a 2-connected graph and has no subgraph homeomorphic to \( K_4 \).
3. \( |T(G, 0)| \geq 2 \) if and only if \( G \) is a 2-connected graph and has at least one subgraph homeomorphic to \( K_4 \).

**Lemma 14** (Chao and Li [1]). A nonplanar graph has a subgraph homeomorphic to \( K_4 \).

Now for a path \( P = v_0 \ldots v_n \) which is a subgraph of \( G \), we call \( v_0 \) and \( v_n \) the end vertices, others the interior vertices. If the degrees of all interior vertices are 2, we say that \( P \) is a simple path of \( G \). Suppose that \( G \) is a 2-connected planar graph. We define \( r(G) \) as the number of interior regions of \( G \). Let the interior regions of \( G \) be \( C_i, \ldots, C_i \). Then \( G \) can be obtained by the following process. (If needed, we can change the order
of cycles.) Let \( H_1 = C_{i_1} \). For \( j = 2, \ldots, r \), let \( H_j \) be obtained from \( H_{j-1} \) and \( C_{i_j} \) by overlapping in path \( P_{i_j} \) and \( G = H_r \).

Now if \( G \) can be obtained from the cycle class \( \mathcal{G} \) by overlapping in paths, we say that \( \mathcal{G} \) is a decomposition of \( G \). Clearly \( |\mathcal{G}| = r(G) \). Of course a planar graph \( G \) can be decomposed into different classes. For example, the generalized \( \theta \)-graph \( \theta_{abc} \) can be obtained from \( C_{a+b} \) and \( C_{b+c} \) by overlapping in \( P_b \) and can be obtained from \( C_{a+b} \) and \( C_{a+c} \) by overlapping in \( P_a \). Even if for the same decomposition, \( G \) can be obtained by different overlapping processes.

**Definition 15.** We call \( G \) a generalized polygon tree, if \( G \) can be decomposed into cycle class \( \mathcal{G} = \{ C_{i_1}, \ldots, C_{i_r} \} \) and there exist an overlapping process: \( H_1 = C_{i_1}, H_j \) is obtained from \( H_{j-1} \) and \( C_{i_j} \) by overlapping in path \( P_{i_j} \), where in each step of overlapping \( P_{i_j} \) is a simple path of \( H_{j-1} \), \( j = 2, \ldots, r \).

Therefore a polygon tree is a special case of generalized polygon trees: \( P_{i_j} \) is an edge for every \( j \).

It is easy to prove the following theorem.

**Theorem 16.** Let \( G \) be a generalized polygon tree and \( \mathcal{G} \) be a decomposition of \( G \). Then \( \chi(G) \leq 3 \) and \( \chi(G) = 2 \) if and only if \( |\mathcal{V}(C)| \) is even for every \( C \in \mathcal{G} \).

**Proof.** Using the inductive method, we can prove \( \chi(G) \leq 3 \). If \( \chi(G) = 2 \), then \( G \) is a bipartite graph and \( |\mathcal{V}(C)| \) is even for every \( C \in \mathcal{G} \). If \( |\mathcal{V}(C)| \) is even for every \( C \in \mathcal{G} \), we can prove \( G \) is bipartite by induction.

**Theorem 17.** A 2-connected graph \( G \) has no subgraphs homeomorphic to \( K_4 \) if and only if \( G \) is a generalized polygon tree.

**Proof (Necessity).** Suppose that \( G \) has no subgraphs homeomorphic to \( K_4 \). By Lemma 14, \( G \) is a planar graph. If \( r(G) = 1 \), the conclusion clearly holds. Assume that the conclusion holds for all graphs \( H \) with \( r(H) < k \) and let \( r(G) = k \). We take a region \( C \) of \( G \) and two vertices \( u, v \) on \( C \) with degrees at least 3, where there is a simple path \( uv_1 \ldots u_p v \) and \( C \setminus \{v_1, \ldots, v_p\} = P \). (If \( uv \in E(G) \), let \( P = C \setminus uv \) Then \( G \) can be obtained from \( H = G \setminus \{v_1, \ldots, v_p\} \) and \( C \) by overlapping in \( P \), where \( r(H) = k - 1 \). If \( P \) is a simple path of \( H \), then \( G \) is obtained form \( H \) by replacing a section of a path in \( H \) by two paths in parallel, and this cannot effect the connectivity of the graph. Hence \( H \) is 2-connected. The conclusion holds for \( H \) and hence for \( G \). If \( P \) is not simple, then there is a block in \( H \), say \( H' \), which is 2-connected and has no subgraph homeomorphic to \( K_4 \).

By the inductive assumption, \( H' \) is a generalized polygon tree, i.e. \( H' \) is a cycle or can be obtained from a generalized polygon tree \( H'' \) and a cycle \( C \) by overlapping in path \( P' \), where \( P' \) is a simple path of \( H'' \). Let \( P' = x_1x_2 \ldots x_t \). Since \( P \) is not simple in \( H \), but \( P' \) is, \( \{u, v\} \not\subseteq \{x_1, \ldots, x_t\} \). If \( u \in \{x_2, \ldots, x_{t-1}\} \) and \( v \not\in \{x_1, x_2, \ldots, x_t\} \), it is easy to see \( G \) contains a \( K_4 \) homeomorphism, a contradiction. Hence \( u, v \not\subseteq \{x_2, \ldots, x_{t-1}\} \) and \( P' \)
is a simple path of $G$. Now $G$ can be obtained from a subgraph $G'$ and the cycle $C'$ by overlapping in the path $P'$, where $P'$ is a simple path of $G'$ and $G'$ is a 2-connected graph with no subgraph homeomorphic to $K_4$ and $r(G')=k-1$. Therefore $G'$ is a generalized polygon tree and so is $G$.

(Sufficiency). Now suppose that $G$ is a generalized polygon tree. We prove that $G$ has no subgraphs homeomorphic to $K_4$. If $r(G)=1$ or 2, the conclusion clearly holds. Suppose that the conclusion holds for all graphs $H$ with $r(H)<k$ and let $r(G)=k$. Then $G$ is obtained from $H \subseteq G$ and a cycle $C$ by overlapping in path $P$, which is a simple path of $H$. Since $H$ is a generalized polygon tree and $r(H)<k$, by the inductive assumption, $H$ has no subgraphs homeomorphic to $K_4$. Assume that $G$ has a subgraph $F$ homeomorphic to $K_4$.

Let $Q=C \setminus P$, where $Q$ is a simple path of $G$. Since $H$ has no subgraphs homeomorphic to $K_4$ but $G$ does, at least one edge on $Q$ must appear in $F$. It is easy to see that $Q$ must be contained in one of the six paths of $F$. In the subgraph homeomorphic to $K_4$ replace $Q$ by $P$. This gives another homeomorph, which is now a subgraph of $H$. This is a contradiction. Hence $G$ has no subgraphs homeomorphic to $K_4$. The conclusion holds.

4. The chromatic polynomials of generalized polygon trees

Definition 18. Let $G$ be a generalized polygon tree. A pair $(u, v)$ of nonadjacent vertices of $G$ is called an intercourse pair if there are at least three internally-disjoint $u-v$ paths in $G$. The intercourse number of $G$, $\sigma(G)$, is defined as the number of intercourse pairs of vertices in $G$.

Theorem 19. Suppose that $G$ is a generalized polygon tree. Then $\sigma(G) \leq r(G) - 1$, and $\sigma(G) = 0$ if and only if $G$ is a polygon tree.

Proof. $G$ can be obtained from $r$ cycles by $r-1$ times of overlapping in paths. We only need to prove that one overlapping step produces at most one intercourse pair.

The conclusion holds for $r(G) = 1$ and 2. Suppose that it holds for all graphs $H$ with $r(H)<k$ and let $r(G)=k$. If $G$ is obtained from $H \subseteq G$ and a cycle $C$ by overlapping in a simple path $P-v_0 \ldots v_r$. Let $S(G)$ (resp. $S(H)$) be the set of intercourse pairs of $G$ (resp. $H$). Then $S(H) \subseteq S(G)$ and $S(G) \setminus S(H) \subseteq \{(v_0, v_3)\}$. So $\sigma(G) \leq \sigma(H) + 1 \leq r(H) = r(G) - 1$.

It is easy to prove that $\sigma(G) = 0$ if and only if $G$ is a polygon tree.

Theorem 20. Let $G$ be a generalized polygon tree, $|V(G)|=p$, $r(G)=r$ and

$$P(G) = \frac{y}{(y+1)^r-1} Q(G)$$

where $Q(G) = a_0 + a_1 y + \cdots$. Then $a_0 = (-1)^p$ and $a_1 = (-1)^{p-1} \sigma(G)$.

Instead of this, we prove the following general conclusion.
Theorem 21. Suppose that $G$ is a connected graph and every block of $G$ is either a generalized polygon tree or $K_2$. We consider a $K_2$ as an interior region. Let $|V(G)| = p$, the number of blocks of $G$ be $b$, the number of interior regions of $G$ be $r(G) = r$ and

$$P(G) = \frac{y}{(y+1)^{r-1}} Q(G)$$

where $Q(G) = a_0 + a_1 y + \cdots$. We have the following conclusion.

If $b \geq 2$, then $a_0 = \cdots = a_{b-2} = 0$ and $|a_{b-1}| = 1$; if $b = 1$, we have $a_0 = (-1)^p$, and $G = K_2$ implies $a_1 = 1$, otherwise $a_1 = (-1)^{b-1} \sigma(G)$.

Proof. We deduce the conclusion by induction on $r(G)$. If $r(G) = 1$, then $G$ is a cycle $C_p$ or $K_2$, and if $G$ is a cycle, then $\sigma(G) = 0$. We have

$$P(G) = y^p + (-1)^p y$$

$$= \frac{y}{y^0} (y^{p-1} + (-1)^p),$$

$$Q(G) = y^{p-1} + (-1)^p.$$ 

Hence $a_0 = (-1)^p$. If $G$ is $C_p (p \geq 3)$, then $a_1 = 0$ and if $G$ is $K_2$, then $p = 2$ and $a_1 = 1$. Now suppose that the conclusion holds for every graph $H$ satisfying the condition with $r(H) < r$ and let $G$ be a graph which also satisfies the condition with $r(G) = r$.

When $b \geq 2$, assume that the blocks of $G$ are $G_1, \ldots, G_b$ and $G$ can be obtained from $G_1, \ldots, G_b$ by overlapping in vertices. Let $r(G_i) = r_i$, $i = 1, \ldots, b$. Then $\sum r_i = r$ and we can get

$$P(G) = \frac{1}{(y+1)^{b-1}} p(G_1) \cdots P(G_b),$$

$$\frac{y}{(y+1)^{r-1}} Q(G) = \frac{1}{(y+1)^{b-1}} \frac{y}{(y+1)^{r-1}} Q(G_1) \cdots \frac{y}{(y+1)^{r-1}} Q(G_b),$$

$$Q(G) = y^{b-1} Q(G_1) \cdots Q(G_b).$$

So $a_0 = \cdots = a_{b-2} = 0$ and $|a_{b-1}| = 1$. The conclusion holds.

Suppose that $b = 1$. If there are two vertices $u, v \in V(G)$ with $s(s \geq 3)$ paths joining them and $u, v \in E(G)$, then $G$ can be obtained from $G_1$ and $G_2$ by overlapping on an edge $uv$, where $|V(G_1)| = p_1$, $|V(G_2)| = p_2$, $p_1 + p_2 = p + 2$, $r(G_1) = r_1$, $r(G_2) = r_2$, $r_1 + r_2 = r$ and $\sigma(G_1) + \sigma(G_2) = \sigma(G)$. Therefore

$$P(G) = \frac{1}{y(y+1)} P(G_1) P(G_2)$$

$$= \frac{1}{y(y+1)} \frac{y}{(y+1)^{r_1-1}} Q(G_1) \frac{y}{(y+1)^{r_2-1}} Q(G_2)$$

$$= \frac{y}{(y+1)^{r-1}} Q(G_1) Q(G_2).$$
And so,

\[ Q(G) = Q(G_1)Q(G_2). \]

Let \( Q(G_1) = a_0^0 + a_1^1 y + \cdots \) and \( Q(G_2) = a_0^0 + a_1^1 y + \cdots \) From the assumptions that \( a_0 = (-1)^p_1, a_0 = (-1)^p_2, a_1 = (-1)^{p_1} \sigma(G_1) \) and \( a_1 = (-1)^{p_2} \sigma(G_2) \), we can deduce that

\[ a_0 = a_0^0 a_1 = (-1)^{p_1 + p_2} = (-1)^p, \]
\[ a_1 = a_0^0 a_1 + a_0^0 a_1 \]
\[ = (-1)^{p_1 + p_2 - 1} \sigma(G_1) + (-1)^{p_1 + p_2 - 1} \sigma(G_2) \]
\[ = (-1)^{p - 1} \sigma(G). \]

Therefore the conclusion holds.

If there are two vertices \( u, v \in V(G) \) with \( s (s \geq 3) \) paths joining them and \( uv \notin E(G) \), then

\[ P(G) = P(G + uv) + P(G \cdot uw), \]
\[ \frac{y}{(y+1)^{r-1}} Q(G) = \frac{y}{(y+1)^r} Q(G + uv) + \frac{y}{(y+1)^r} Q(G \cdot uw). \]

Here \( G + uv \) can be obtained from three graphs \( G_1, G_2 \) and \( G_3 \) by overlapping in edge \( uv \), where \( |V(G_i)| = p_i, r(G_i) = r_i, i = 1, 2, 3, \sum p_i = p + 4, \sum r_i = r(G + uv) = r(G) + 1 = r + 1, \sum \sigma(G_i) = \sigma(G + uv) = \sigma(G) - 1 \). The conclusion holds for \( G_1, G_2 \) and \( G_3 \) by the inductive assumption because \( r_i < r, i = 1, 2, 3 \), where

\[ Q(G_i) = (-1)^{p_i} + (-1)^{p_i - 1} \sigma(G_i) y + \cdots, \]
\[ P(G + uv) = \frac{1}{y^2(y+1)^2} P(G_1)P(G_2)P(G_3), \]
\[ \frac{y}{(y+1)^r} Q(G + uv) = \frac{1}{y^2(y+1)^2} \prod \frac{y}{(y+1)^{r_i - 1}} Q(G_i), \]
\[ Q(G + uv) = Q(G_1)Q(G_2)Q(G_3). \]

Let \( Q(G + uv) = a_0^0 + a_1^1 y + \cdots \). Then

\[ a_0 = (-1)^{2p_i} = (-1)^p, \]
\[ a_1 = (-1)^{p_1 + r_1}(-1)^{p_2 - 1} \sigma(G_3) + (-1)^{p_1}(-1)^{p_2 - 1} \sigma(G_2)
\[ + (-1)^{p_1 + p_2}(-1)^{p_2 - 1} \sigma(G_1) \]
\[ = (-1)^{p - 1} \sigma(G_1) \]
\[ = (-1)^{p - 1} \sigma(G) - 1. \]

Since \( Q(G) = (1/(y+1))(Q(G + uv) + Q(G \cdot uw)) \) and there are at least three blocks in \( G \cdot uv \), the conclusion holds. \( \Box \)

Let the girth of graph \( G \) be \( y(G) \). We have the following result.
Lemma 22. Let $G$ and $H$ be graphs such that $P(G) = P(H)$, then $|V(G)| = |V(H)|$, $|E(G)| = |E(H)|$, $g(G) = g(H)$ and the numbers of cycles of $G$ and $H$ with the length equal to their girth are equal. Moreover if they are both planar, then $r(G) = r(H)$ and if $G$ is a generalized polygon tree, then $H$ is also a generalized polygon tree and $\sigma(H) = \sigma(G)$.

Proof. The first part of the Theorem is well known. By Euler's formula, if $G$ and $H$ are both planar, then $r(G) = r(H)$. Now if $G$ is a generalized polygon tree, by Theorem 17, $G$ has no subgraph homeomorphic to $K_4$. By Theorem 13, $|T(G, 0)| = 0$, which implies $|T(H, 0)| = 0$, and hence $H$ is a generalized polygon tree. By Theorem 21, we have $\sigma(H) = \sigma(G)$. □

Now we are ready to give the following

Theorem 23. Let $G = \{C_4, \ldots, C_k\}$, $X = \{K_2\}$. Then $\{G, X\}$ is a complete class of chromatically equivalent graphs.

Proof. Let $G \in \{G, X\}$ and $P(H) = P(G)$. Then by Lemma 22, $H$ is a generalized polygon tree and $\sigma(H) = \sigma(G) = 0$. It is easy to see that $H \in \{G, X\}$.

Note added in proof. After this paper had been accepted for publication, the author noticed that Theorem 23 of the paper was also obtained by Prof. C.D. Wakelin and Prof. D.R. Woodall in their paper 'Chromatic polynomials, polygon trees and outer-planar graphs' (J. Graph Theory, 16 (5) (1992) 459–466).

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