Simultaneous and non-simultaneous blow-up for a cross-coupled parabolic system

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Received 28 April 2005
Available online 17 April 2006
Submitted by C.V. Pao

Abstract

This paper deals with a parabolic system, cross-coupled via a nonlinear source and a nonlinear boundary flux. We get a necessary and sufficient condition for the existence of non-simultaneous blow-up. In particular, four different simultaneous blow-up rates are obtained in different regions of parameters, described by an introduced characteristic algebraic system. It is observed that different initial data may result in different simultaneous blow-up rates even in the same region of parameters.

Keywords: Simultaneous blow-up; Non-simultaneous blow-up; Characteristic algebraic system; Blow-up rate; Critical exponent

1. Introduction

In this paper, we consider the cross-coupled parabolic system

\[
\begin{align*}
    &u_t = \Delta u + u^m + v^p, \quad v_t = \Delta v, \quad (x, t) \in \Omega \times (0, T), \\
    &\frac{\partial u}{\partial \eta} = 0, \quad \frac{\partial v}{\partial \eta} = u^q + v^n, \quad (x, t) \in \partial \Omega \times (0, T), \\
    &u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), \quad x \in \Omega,
\end{align*}
\]

\[(1.1)\]

Supported by National Natural Science Foundation of China.

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doi:10.1016/j.jmaa.2006.03.013
where \( m, n \geq 0, p, q > 0; \) \( \Omega \) is a bounded domain in \( \mathbb{R}^N \) with smooth boundary \( \partial \Omega \); \( u_0(x) \) and \( v_0(x) \) are positive smooth functions satisfying the compatible conditions.

Nonlinear parabolic systems like (1.1) come from chemical reactions, heat transfer, etc., where \( u \) and \( v \) represent, e.g., concentrations of two kinds of chemical reactants, or temperatures of two different materials during heat propagations. The existence and uniqueness of local classical solutions to (1.1) is well known [15].

Recently, Souplet and Tayachi [26], Rossi and Souplet [23] have studied the simultaneous and non-simultaneous blow-up for the model with coupled inner sources

\[
\begin{align*}
  u_t &= \Delta u + u^m + v^p, \\
  v_t &= \Delta v + u^q + v^n \\
\end{align*}
\]

in \( \Omega \times (0, T) \) (1.2) with \( m, n, p, q > 1, \Omega \subseteq \mathbb{R}^N \). In particular, the coexistence of simultaneous and non-simultaneous blow-up was established, and two different simultaneous blow-up rates were obtained.

In [31], the authors obtained more results for such a system, where the coupled inner sources in (1.2) were replaced by coupled boundary flux of the same form. The results include the necessary and sufficient conditions for the existence of non-simultaneous blow-up for radial solutions, as well as four different simultaneous blow-up rates in different regions of parameters. Moreover, two new cases for the coexistence of simultaneous and non-simultaneous blow-up were firstly considered. As for the new model (1.1) considered in this paper, the coupling consists of both the source and the boundary flux, which is more complicated to treat than that in [31] without sources.

For the special case of (1.1) without \( u^m, v^n \), the blow-up criterion, blow-up rate and set were known [6,27].

Currently, Brändle, Quirós and Rossi have studied non-simultaneous and simultaneous blow-up for two kinds of nonlinear diffusion systems with “product” type of coupled nonlinear inner source [2] and nonlinear boundary flux [1], respectively, instead of (1.1) with “sum” type.

Phenomena of non-simultaneous blow-up for coupled nonlinear parabolic systems were observed and studied by many authors also [19–21,24]. There have been much more studies related to the subjects on critical exponents, blow-up rates, blow-up sets, and blow-up profiles [4,5,7,9–11,13,14].

To state the main results of this paper, we introduce the following characteristic algebraic system [29,30]

\[
\begin{pmatrix}
  \theta_1 m - 1 & (1 - \theta_1) p \\
  2(1 - \theta_2) q & 2(\theta_2 n - 1)
\end{pmatrix}
\begin{pmatrix}
  \alpha \\
  \beta
\end{pmatrix}
= \begin{pmatrix}
  1 \\
  1
\end{pmatrix}
\]

(1.3)

with \( \theta_1, \theta_2 \in \{0, 1\} \), namely,

\[
(\alpha, \beta) = \begin{cases}
  (\alpha_1, \beta_1) = \left( \frac{p+2}{2(pq-1)}, \frac{2q+1}{2(pq-1)} \right) & \text{for } \theta_1 = 0, \ \theta_2 = 0; \\
  (\alpha_2, \beta_2) = \left( \frac{1}{m-1}, \frac{2q+1-m}{2(m-1)} \right) & \text{for } \theta_1 = 1, \ \theta_2 = 0; \\
  (\alpha_3, \beta_3) = \left( \frac{p+2-2n}{2(n-1)}, \frac{1}{2(n-1)} \right) & \text{for } \theta_1 = 0, \ \theta_2 = 1; \\
  (\alpha_4, \beta_4) = \left( \frac{1}{m-1}, \frac{1}{2(n-1)} \right) & \text{for } \theta_1 = 1, \ \theta_2 = 1.
\end{cases}
\]

(1.4)

We will describe four different simultaneous blow-up rates via \( \alpha_i, \beta_i, i = 1, 2, 3, 4 \).

This paper is organized as follows. The critical exponent is given in Section 2. Section 3 deals with the conditions for simultaneous and non-simultaneous blow-up. The four different blow-up rates will be considered in Section 4. In the last section, we give some remarks to illustrate the main results of this paper.
2. Critical exponent

It is easy to get from the known results [6,13,28] with the comparison principle that all positive solutions of (1.1) blow up if \( \max\{m, n, pq\} > 1 \).

On the other hand, suppose \( \max\{m, n, pq\} \leq 1 \). Construct
\[
\tilde{u} = \tilde{v}^p = A \exp\{k_1 t + k_2 h(x)\},
\]
where the positive function \( h(x) \in C^2(\Omega) \cap C^1(\tilde{\Omega}) \) solves
\[
\Delta h(x) = \lambda = \frac{\partial\Omega}{|\Omega|}, \quad \frac{\partial h}{\partial \eta} = 1, \quad x \in \partial\Omega,
\]
with \( |\nabla h| \leq \delta \) for some constant \( \delta > 0 \). A simple computation shows
\[
\tilde{u}(x, 0) \geq u_0(x), \quad \tilde{v}(x, 0) \geq v_0(x), \quad x \in \Omega \text{ for } A > \max\{1, \|u_0\|_{\infty}, \|v_0\|_{\infty}^p\};
\]
\[
\frac{\partial \tilde{u}}{\partial \eta} \geq 0, \quad \frac{\partial \tilde{v}}{\partial \eta} \geq \tilde{u}^q + \tilde{v}^n, \quad (x, t) \in \partial\Omega \times (0, T) \text{ for } n, pq \leq 1, \ k_2 \geq 2p, \ k_1 \geq K;
\]
\[
\tilde{u}_t \geq \Delta \tilde{u} + \tilde{u}^m + \tilde{v}^p, \quad \tilde{v}_t \geq \Delta \tilde{v}, \quad (x, t) \in \Omega \times (0, T) \text{ for } m \leq 1, \ k_1 \geq K,
\]
where \( K = \max\{k_2 \lambda + k_2^2 \delta^2 + 2, k_2 \lambda + k_2^2 \delta^2 / p\} \), and hence \( (\tilde{u}, \tilde{v}) \) is a global super solution of (1.1).

We have obtained the following theorem:

**Theorem 2.1.** All positive solutions of (1.1) are global if and only if \( \max\{m, n, pq\} \leq 1 \).

3. Simultaneous and non-simultaneous blow-up

In the sequel, we will consider radial solutions with \( u = u(r, t), \ v = v(r, t), \ r = |x|, \max\{m, n, pq\} > 1, \ \Delta u_0 + u_0^m + v_0^p, \ \Delta v_0, (u_0)_r, (v_0)_r \geq 0 \) for \( x \in B_R \subset \mathbb{R}^N \), which implies that \( u_t, u_r, v_r \geq 0 \) by the comparison principle. Denote
\[
U(t) = u(R, t), \quad V(t) = v(R, t).
\]
Throughout this paper, we will use \( C \) and \( c \) to denote positive constants independent of \( t \), which may be different from line to line. Let \( T \) be the blow-up time for (1.1).

Firstly, we give a necessary and sufficient condition for the existence of \( v \) blowing up and \( u \) remaining bounded.

**Theorem 3.1.**

(i) If \( p + 2 < 2n \), then for given \( u_0 \), there exists large \( v_0 \) such that \( v \) blows up while \( u \) remains bounded.

(ii) If \( v \) blows up and \( u \) remains bounded, then \( p + 2 < 2n \).

The proof of the theorem consists of three lemmas.

**Lemma 3.1.** If \( n > 1 \), then
\[
V(t) \leq C_T (T - t)^{-\frac{1}{2(n-1)}}, \quad t \in (0, T), \tag{3.1}
\]
where \( C_T = \tilde{C}(1 + 4C_1 T^{\frac{1}{2}})^{\frac{1}{2(n-1)}} \), positive constants \( \tilde{C}, C_1 \) depend only on \( n \) and \( B_R \).
Proof. By Green’s identity and the jump relation, we have
\[ \frac{1}{2} V(t) \geq C_2 \int_0^t \int_{\mathbb{R}_+^n} V^n(\tau)(T - \tau)^{-1/2} d\tau - 2C_1 T^{1/2} V(t), \quad 0 < z < t < T, \]
where \( C_1, C_2 \) depend only on \( B_R \). Set \( I(t) = \int_0^t V^n(\tau)(T - \tau)^{-1/2} d\tau \). Then
\[ I'(t) \geq C_2 \left( \frac{1}{2} + 2C_1 T^{1/2} \right)^{-n} I^n(t)(T - t)^{-1/2}. \]
Integrate the above inequality from \( t \) to \( T \),
\[ I(t) \leq \left[ 2(n - 1)C_2 \left( \frac{1}{2} + 2C_1 T^{1/2} \right)^{-n} \right]^{-1/2} (T - t)^{-n/2(n-1)}. \] (3.2)
On the other hand, for \( 0 < z = 2t - T < t < T \),
\[ I(t) \geq \int_0^{T/2} V^n(z)(T - \tau)^{-1/2} d\tau = (2 - \sqrt{2}) V^n(z)(T - z)^{1/2}. \] (3.3)
Combining (3.2) and (3.3), we obtain (3.1) with
\[ \tilde{C} = (2C_2)^{-n/2(n-1)} (2 - \sqrt{2})^{-1/2} (\sqrt{2}(n - 1))^{-n/2(n-1)}. \] (3.4)
Consider auxiliary problem
\[
\begin{aligned}
& z_t = \Delta z + C_T^p (T - t)^{-\frac{p}{2(n-1)}} + M_0, \quad (x, t) \in B_R \times (0, T), \\
& \frac{\partial z}{\partial \eta} = 0, \quad (x, t) \in \partial B_R \times (0, T), \\
& z(x, 0) = u_0(x), \quad x \in B_R,
\end{aligned}
\] (3.5)
where \( n > 1, M_0 \geq 0, C_T \) is defined in (3.1).

Lemma 3.2. If \( p + 2 < 2n \), then for given \( \varepsilon > 0 \) and \( u_0 \), we can take \( T \) small such that
\[ \sup_{0 < t < T} \| z(\cdot, t) \|_\infty \leq \| u_0 \|_\infty + \varepsilon. \] (3.6)
Prove. Let \( G(x, y, t, \tau) \) be Green’s function for the heat equation in \( B_R \) satisfying \( \frac{\partial G}{\partial \eta} = 0 \) on \( \partial B_R \) [8,16,18]. By Green’s identity,
\[ z(x, t) \leq \| u_0 \|_\infty + \int_0^t \left( C_T^p (T - \tau)^{-\frac{p}{2(n-1)}} + M_0 \right) d\tau \]
\[ \leq \| u_0 \|_\infty + \frac{2(n-1)}{2n-2-p} C_T^p T^{\frac{2n-2-p}{2(n-1)}} + M_0 T. \]
For given \( \varepsilon > 0 \), take \( T \) small such that
\[
\frac{2(n-1)}{2n-2-p} C_T^p T^{\frac{2n-2-p}{2(n-1)-p}} + M_0 T < \varepsilon. \tag{3.7}
\]

This proves (3.6). \(\square\)

**Lemma 3.3.** If \(p + 2 < 2n\), then for given \(\varepsilon > 0\) and \(u_0\), we can obtain (3.6) by taking \(T\) sufficiently small for \(\tilde{z}\) solving

\[
\begin{align*}
\tilde{z}_t &= \Delta \tilde{z} + C_T^p (T-t)^{-\frac{p}{2(n-1)}} + \tilde{z}^m, \quad (x, t) \in B_R \times (0, T), \\
\partial \tilde{z} / \partial \eta &= 0, \quad (x, t) \in \partial B_R \times (0, T), \\
\tilde{z}(x, 0) &= u_0(x), \quad x \in B_R,
\end{align*}
\tag{3.8}
\]

with \(n > 1\) and \(C_T\) defined by (3.1).

**Proof.** For given \(\varepsilon > 0\) and \(u_0\), let \(z\) be a solution of (3.5) with \(M_0 > (\|u_0\|_\infty + \varepsilon)^m\). If we choose \(T\) so small that (3.7) holds, then \(z\) satisfies (3.6), and hence

\[
z_t \geq \Delta z + C_T^p (T-t)^{-\frac{p}{2(n-1)}} + z^m, \quad (x, t) \in B_R \times (0, T).
\]

By the comparison principle, \(\tilde{z} \leq z\) in \(B_R \times (0, T)\), and hence (3.6) holds for \(\tilde{z}\) also. \(\square\)

**Proof of Theorem 3.1.** (i) Since \(n > 1\), the solution \((u, v)\) of (1.1) must blow up at a finite time by Theorem 2.1. For any given \(u_0\) and \(\varepsilon > 0\), choose \(v_0\) large such that the blow-up time \(T\) satisfies (3.7). By Lemma 3.1, \(u\) satisfies

\[
\begin{align*}
ut &\leq \Delta u + C_T^p (T-t)^{-\frac{p}{2(n-1)}} + um, \quad (x, t) \in B_R \times (0, T).
\end{align*}
\]

Let \(\tilde{z}\) be a solution of (3.8). Then \(u \leq \tilde{z}\) in \(B_R \times (0, T)\) by the comparison principle. It is easy to see that (3.6) holds for \(\tilde{z}\) by Lemma 3.3. Hence \(v\) does blow up at \(t = T\).

(ii) Since \(u\) is bounded, we know \(n > 1\). Otherwise, \(v\) would be bounded also, a contradiction. Thus, \(c \leq V(t)(T-t)^{\frac{1}{2(n-1)}} \leq C, t \in (0, T)\). Similarly to the discussion in [19,20], for \(x_0 \in \partial B_R\), it is proved in [12] that the blow-up limit is nontrivial, i.e.,

\[
\liminf_{t \to T} \inf_{|x| \leq K} v(x_0 + x\sqrt{T-t}, t)(T-t)^{-\frac{1}{2(n-1)}} \neq 0,
\]

which means that there exists a constant \(c\) such that

\[
v(x_0 + x\sqrt{T-t}, t) \geq c(T-t)^{-\frac{1}{2(n-1)}}, \quad |x| \leq K.
\]

By Green’s identity and the jump relation, for \(0 < \tau < t < T\),

\[
U(t) \geq c \int_{\tilde{z}} (T-\tau)^{-\frac{p}{2(n-1)}} \int_{B_R \cap \{|y-x_0| \leq K\sqrt{T-t}\}} \Gamma(x_0 - y, t-\tau) dy d\tau \\
\geq c \int_{\tilde{z}} (T-\tau)^{-\frac{p}{2(n-1)}} d\tau.
\]

The boundedness of \(u\) requires \(p + 2 < 2n\). \(\square\)

Next, we give a necessary and sufficient condition for the existence of \(u\) blowing up and \(v\) remaining bounded.
Theorem 3.2.

(i) If $u$ blows up and $v$ remains bounded, then $2q + 1 < m$.
(ii) If $2q + 1 < m$ with $N = 1$, then for given $v_0(R)$, there exists large $u_0$ such that $u$ blows up while $v$ remains bounded.

The following upper estimate on a parabolic inequality obtained by Souplet and Tayachi with $N = 1$ [25] is very important for the discussion in the sequel.

Lemma 3.4. [25, Theorem 4 and Lemma 3.4] Let $Q_T = (-R, R) \times (0, T)$ and $m > 1$. Assume $w = w(r) \in C^{2,1}(Q_T)$ satisfying

$$w_t - w_{xx} \geq w^m$$

in $Q_T$, with $w, w_t \geq 0, w_r \leq 0$ in $Q_T$. Then

$$w(0, t) \leq C^*(T - t)^{-\frac{1}{m-1}}, \quad t \in (0, T).$$

(3.9)

Moreover, the constant $C^*$ depends only on $m$ if $w(x, 0) \geq \varepsilon_0$ with some $\varepsilon_0 > 0$.

Similarly to Lemmas 3.2 and 3.3, we have a lemma [31] for problem

$$\begin{align*}
   &z_t = \Delta z, \
   &\frac{\partial z}{\partial \eta} = (C^*)^q (T - t)^{-\frac{q}{m-1}} + z^n, \
   &z(x, 0) = z_0(x),
\end{align*}$$

(3.10)

where $C^*$ is defined in (3.9).

Lemma 3.5. [31] If $2q + 1 < m$, then for given $\varepsilon > 0$ and $z_0(R)$, we can let $T$ be small such that the solution of (3.10) satisfies

$$\sup_{0 < t < T} \|z(\cdot, t)\|_{\infty} \leq \|z_0\|_{\infty} + \varepsilon.$$ 

(3.11)

Proof of Theorem 3.2. (i) Since $u$ blows up and $v$ is bounded, we have $m > 1$. By Green’s identity,

$$U(t) \leq U(z) + CT + \int z U^m(\tau) d\tau.$$ 

For $z \in (0, T)$ satisfying $U(z) \geq CT$, choose $t$ such that $U(t) \geq 3U(z)$. Then $U(z) \leq CU^m(z)(T - z)$, and hence $U(t) \geq c(T - t)^{-\frac{1}{m-1}}, t \in (0, T)$. By Green’s identity and the jump relation,

$$V(t) \geq c \int_0^t \int_{\partial B_R} \Gamma(x, y, t, \tau) U^q(\tau) dS_y d\tau \geq c \int_0^t (T - \tau)^{-\frac{2q + m - 1}{2(m-1)}} d\tau.$$ 

The fact of $v$ being bounded implies $2q + 1 < m$. 
(II) For \( m > 1 \), \((u, v)\) must blow up at some finite time \( T \). Let \( z \) solve (3.10) with \( N = 1 \). By Lemma 3.5, for given \( \varepsilon > 0 \) and \( v_0(R) \), (3.11) holds for \( z \) provided \( T \) small enough (i.e., \( u_0 \) large enough with compatible \( v_0 \)) and suitable \( z_0 \geq v_0 \).

Let \( w(x, t) = u(R - x, t), (x, t) \in [0, R] \times [0, T) \). Then \( w \) satisfies

\[
\begin{align*}
    w_t &\geq w_{xx} + w^m, \quad w \geq 0, \quad w_t \geq 0, \quad w_x \leq 0
\end{align*}
\]

with \( w_x(0, t) = -u_x(R, t) = 0 \). By Lemma 3.4, \( u(R, t) = w(0, t) \leq C^*(T - t)^{-q/(m - 1)} \), and hence \( v_x(R, t) \leq (C^*)^q(T - t)^{-q/(m - 1)} + v^n(R, t) \). We obtain \( v \leq z \) by the comparison principle, and hence \( u \) has to blow up at \( t = T \). \( \square \)

**Corollary 3.1.** For \( N = 1 \), simultaneous blow-up occurs for all positive initial data if and only if \( m \leq 2q + 1 \) and \( 2n \leq p + 2 \).

For \( N \geq 1 \), we show a sufficient condition for the simultaneous blow-up with any positive initial data:

**Theorem 3.3.** If \( m < 2q + 1 \) and \( 2n < p + 2 \), then simultaneous blow-up occurs for all positive initial data.

The theorem results from the following lemma, which will play an important role in the next section also.

**Lemma 3.6.**

(i) If \( m < 2q + 1 \), then there exists a constant \( C > 0 \) such that

\[
U(t) = C^* v(t), \quad t \in (0, T),
\]

where \( \gamma = \min\{q + (1 - m)/2, (2q + 1)/(p + 2), q/n\} \). In particular, \( v \) blows up.

(ii) If \( 2n < p + 2 \), then there exists a constant \( C > 0 \) such that

\[
V(t) = C^* u(t), \quad t \in (0, T),
\]

where \( \mu = \min\{p + 2 - 2n, (p + 2)/(2q + 1), p/m\} \). In particular, \( u \) blows up.

**Proof.** Clearly, \( U(t), V(t) \) are continuous, nondecreasing in \( t \in (0, T) \), and \( U(t) + V(t) \) blows up at time \( T \) for max\{\( m, n, pq \)\} > 1.

Similarly to the discussion in [26,31], assume, e.g., (3.12) is not true. Then there exists a sequence \( t_j \to T \) as \( j \to +\infty \) such that \( (U(t_j))^{-\gamma} V(t_j) \to 0 \) as \( j \to +\infty \). Since \( \gamma > 0 \), it follows that \( U(t_j) \) diverges as \( j \to +\infty \). Let \( \lambda_j = (U(t_j))^{-(q - \gamma)} \). We can check that \( q > \gamma \), and hence \( \lambda_j \to 0 \) as \( j \to +\infty \).

Let \( \hat{x}_j \in \partial B_R \) such that \( u(\hat{x}_j, t_j) = U(t_j) \). Scale \( (u, v) \) to \((\phi^{\lambda_j}, \psi^{\lambda_j})\) as follows

\[
\begin{align*}
    \phi^{\lambda_j}(y, s) &= \lambda_j^{\frac{1}{q - \gamma}} u(\lambda_j R_j y + \hat{x}_j, \lambda_j^2 s + t_j), \\
    \psi^{\lambda_j}(y, s) &= \lambda_j^{\frac{1}{q - \gamma}} v(\lambda_j R_j y + \hat{x}_j, \lambda_j^2 s + t_j)
\end{align*}
\]

for \((y, s) \in \Omega_{\lambda_j} \times (-t_j/\lambda_j^2, (T - t_j)/\lambda_j^2)\), where \( \Omega_{\lambda_j} = \{y \in \mathbb{R}^N: \lambda_j R_j y + \hat{x}_j \in B_R\} \), and \( R_j \) is an orthonormal transformation in \( \mathbb{R}^N \) that maps \((-1, 0, 0, \ldots, 0)\) into the outer normal vector
to \( B_R \) at \( \hat{x}_j \). Clearly, \((-1, 0, \ldots, 0)\) is the outer normal vector to \( \Omega_{\lambda_j} \) at \((0, 0, \ldots, 0)\), and \( \Omega_{\lambda_j} \) approaches (locally) the half-space \( \mathbb{R}^n_+ = \{ y_1 > 0 \} \) as \( \lambda_j \to +\infty \).

If we restrict \( s \) to \((-t_j/\lambda_j^2, 0]\), then

\[
0 \leq \varphi^{\lambda_j} \leq 1, \quad \varphi^{\lambda_j}(0, 0) = 1; \quad 0 \leq \psi^{\lambda_j} \leq (U(t_j))^{-\gamma} V(t_j) \to 0 \quad \text{as} \quad j \to +\infty, \quad (3.14)
\]

and \( (\varphi^{\lambda_j}, \psi^{\lambda_j}) \) solves the following system:

\[
\begin{cases}
\varphi_s = \Delta \varphi + \lambda_j \frac{2 + \frac{1}{q - \gamma} - \frac{m}{\gamma}}{2 - \frac{m}{\gamma}} \varphi^m + \lambda_j \frac{2 + \frac{1}{q - \gamma} - \frac{m}{\gamma}}{2 - \frac{m}{\gamma}} \psi^p, & \varphi_s = \Delta \psi \quad \text{in} \quad \Omega_{\lambda_j} \times (-t/\lambda_j^2, 0], \\
\frac{\partial \varphi}{\partial y} |_{\partial \Omega_j} = 0, & \frac{\partial \psi}{\partial y} |_{\partial \Omega_j} = \varphi^q + \lambda_j \frac{1}{\gamma} - \frac{m}{\gamma} \psi^n \quad \text{for} \quad s \in (-t/\lambda_j^2, 0].
\end{cases}
\]

By the definition of \( \gamma \), all the powers of \( \lambda_j \) in the above system are nonnegative and will tend to 0 or 1 as \( j \to +\infty \). By interior–boundary Schauder’s estimates, we can find a subsequence converging uniformly on compact subsets of \( \mathbb{R}^n_+ \times (-\infty, 0] \) to \((\varphi, \psi)\), which satisfies

\[
\begin{cases}
\varphi_s = \Delta \varphi + \varepsilon_1 \varphi^m + \varepsilon_2 \psi^p, & \varphi_s = \Delta \psi, \quad (y, s) \in \mathbb{R}^n_+ \times (-\infty, 0],
\\
-\frac{\partial \varphi}{\partial y} |_{y_1 = 0} = 0, & -\frac{\partial \psi}{\partial y} |_{y_1 = 0} = \varphi^q + \varepsilon_3 \psi^n, \quad s \in (-\infty, 0],
\end{cases}
\]

with \( \varepsilon_i = 0 \) or 1 \((i = 1, 2, 3)\). Observe \( \psi \equiv 0, \varphi(0, 0) = 1 \) by (3.14), a contradiction. \( \Box \)

Now, we give the conditions for non-simultaneous blow-up under any initial data.

**Theorem 3.4.**

(i) If \( m \leq 1 \) and \( p + 2 < 2n \), then \( v \) blows up and \( u \) remains bounded for every positive initial data.

(ii) If \( n \leq 1, 2q + 1 < m, \) and \( N = 1 \), then \( u \) blows up and \( v \) remains bounded for every positive initial data.

**Proof.** We prove (i) only. For \( n > 1 \), \((u, v)\) will blow up at time \( T \). We claim that \( v \) must blow up. Otherwise, no component of \((u, v)\) would blow up since \( m \leq 1 \). It is easy to check that \( V(t) \leq C(T - t)^{-\frac{1}{2n - 1}}, \) \( t \in (0, T) \). Let \( G \) be Green’s function satisfying \( \frac{\partial G}{\partial \eta} = 0 \) on \( \partial B_R \). By Green’s identity,

\[
U(t) \leq U(z) + \int \int_{B_R} G(x, y, t, \tau) \left( C(T - \tau)^{-\frac{p}{2n - 1}} + U^m(\tau) \right) dy d\tau \\
\leq U(z) + CT^{\frac{2n - 2 - p}{2n - 2}} + C(T - z)U^m(t). \quad (3.15)
\]

Now, we claim \( u \) remains bounded up to time \( T \). If not, there should exist \( z_j \to T \) such that \( U(z_j) > 1, C(T - z_j) < \frac{1}{4}, \) and \( U(z_j) \to +\infty \) as \( j \to +\infty \). Taking \( t_j \) with \( U(z_j) + CT^{\frac{2n - 2 - p}{2n - 2}} < \frac{1}{2} U(t_j) \), we have \( U(t_j) < \frac{1}{2} U(t_j) \) by (3.15), a contradiction. \( \Box \)

Inspired by [1], we have a theorem on the coexistence of both simultaneous and non-simultaneous blow-up.
Theorem 3.5. If \( p + 2 < 2n, 2q + 1 < m \) and \( N = 1 \), then there may occur both simultaneous and non-simultaneous blow-up.

We first show that the set of initial data for non-simultaneous blow-up is open \([1]\).

Lemma 3.7. The set of \((u_0, v_0)\) such that \(v\) blows up and \(u\) remains bounded (or \(u\) blows up and \(v\) remains bounded with \(N = 1\)) is open in the \(L^\infty\)-topology.

Proof. Let \((u, v)\) be a solution of (1.1) with initial data \((u_0, v_0)\) such that \(v\) blows up at \(t = T\) while \(u\) remains bounded, say \(u < M\). We only need to find a \(L^\infty\)-neighborhood of \((u_0, v_0)\) such that any solution \((\tilde{u}, \tilde{v})\) of (1.1) coming from this neighborhood maintains the property that \(v\) blows up while \(u\) remains bounded.

From Theorem 3.1, we have \(p + 2 < 2n\). Take \(M_0 > (M + 1)^m\). Let \(v\) solve

\[
\begin{aligned}
\psi_t &= \Delta \psi, \\
\frac{\partial \psi}{\partial \eta} &= v^n, \\
v(x, 0) &= v_0(x), \quad x \in B_R,
\end{aligned}
\]  

(3.16)

where \(T_v\) is the maximal existence time of (3.16). We can make \(T_v\) arbitrarily small by taking \(v_0\) large enough. Now take \(v_0\) large that

\[
M_0 > \left( M + 1 + \frac{2(n - 1)}{2n - 2 - p} C_T^p T_v^{\frac{2n - 2 - p}{2n - 2}} + M_0 T_v \right)^m
\]  

(3.17)

with \(C_T\) defined as (3.1). Let \((\tilde{u}, \tilde{v})\) be a solution of

\[
\begin{aligned}
\tilde{u}_t &= \Delta \tilde{u} + \tilde{u}^m + \tilde{v}^p, \\
\frac{\partial \tilde{u}}{\partial \eta} &= \frac{\partial \tilde{v}}{\partial \eta} = \Delta \tilde{v}, \\
\tilde{u}(x, 0) &= \tilde{u}_0(x), \quad \tilde{v}(x, 0) = \tilde{v}_0(x), \quad x \in B_R,
\end{aligned}
\]  

(3.18)

where \(\tilde{v}_0 \geq v_0, \tilde{u}_0(x) > 0\), and \(T_0\) is the blow-up time of (3.18). No matter what \(\tilde{u}_0\) is, we always have \(\tilde{v} \geq v\) and \(T_0 \leq T_v\) by the comparison principle. It follows from (3.17) that

\[
M_0 > \left( M + 1 + \frac{2(n - 1)}{2n - 2 - p} C_T^p T_0^{\frac{2n - 2 - p}{2n - 2}} + M_0 T_0 \right)^m
\]  

(3.19)

where \(C_T\) is defined as (3.1). For any \(\varepsilon_0 > 0\), denote

\[
N_v(\varepsilon_0) = \{ v_0: \|v_0(x) - v(x, T - \varepsilon_0)\|_\infty < 1 \}.
\]

Observing \(v\) blows up at \(t = T\), we can take \(\varepsilon_0\) small enough such that \(T_v\) satisfies (3.17) for all \(v_0 \in N_v(\varepsilon_0)\). For such \(\varepsilon_0\), define

\[
N_u(\varepsilon_0) = \{ u_0: \|\tilde{u}_0(x) - u(x, T - \varepsilon_0)\|_\infty < 1 \}.
\]

Then \(T_0\) satisfies (3.19) whenever \((\tilde{u}_0, \tilde{v}_0) \in N_u(\varepsilon_0) \times N_v(\varepsilon_0)\). Combining (3.18) with Lemma 3.1, we obtain

\[
\begin{aligned}
\tilde{u}_t &\leq \Delta \tilde{u} + C_{T_0}^p (T_0 - t)^{-\frac{p}{2(n - 1)}} + \tilde{u}^m, \quad (x, t) \in B_R \times (0, T_0), \\
\frac{\partial \tilde{u}}{\partial \eta} &= 0, \quad (x, t) \in \partial B_R \times (0, T_0), \\
\tilde{u}(x, 0) &= \tilde{u}_0(x), \quad x \in B_R.
\end{aligned}
\]
Next, consider
\[
\begin{aligned}
\ddot{u} &= \Delta \ddot{u} + C^p_{T_0}(T_0 - t)^{-\frac{p}{2n-1}} + M_0, \quad (x, t) \in B_R \times (0, T_0), \\
\frac{\partial \ddot{u}}{\partial \eta} &= 0, \quad (x, t) \in \partial B_R \times (0, T_0), \\
\ddot{u}(x, 0) &= \ddot{u}_0(x), \\
\end{aligned}
\]
with $M_0$ satisfying (3.19). We have
\[
\ddot{u}(x, t) \leq M + 1 + \frac{2(n - 1)}{2n - 2 - p} C^p_{T_0} T_0^{\frac{2n-2-p}{2n-1}} + M_0 T_0, \quad t \in (0, T_0).
\]
Thus, $\dddot{u} \leq M_0$, and so $\ddot{u} \leq \dddot{u} \leq M_0^{1/m}$ for $t \in (0, T_0)$.

Assume $(\ddot{u}, \dot{v})$ is a solution of (1.1) with $(\ddot{u}(x, t), \dot{v}(x, t)) = (\ddot{u}(x, t - T + \varepsilon_0), \dot{v}(x, t - T + \varepsilon_0))$, $t \in [T - \varepsilon_0, T - \varepsilon_0 + T_0)$, where $(\ddot{u}(x, 0), \dot{v}(x, 0)) \in N_u(\varepsilon_0) \times N_v(\varepsilon_0)$. According to the continuity of bounded solutions with respect to initial data, we can obtain that there must exist a neighborhood of $(u_0, v_0)$, any initial data lying in which yields that $v$ blows up and $u$ remains bounded. □

**Proof of Theorem 3.5.** The set of $(u_0, v_0)$ such that $u$ blows up and $v$ remains bounded is nonempty (Theorem 3.2), and so is the set of initial data for $v$ blowing up and $u$ being bounded (Theorem 3.1). Moreover, Lemma 3.7 concludes that such sets are open. Clearly, the two open sets are disjoint. That is to say, there exists $(u_0, v_0)$ such that $u$ and $v$ blow up simultaneously. □

4. Blow-up rates and set

Let us determine the asymptotic behavior of solutions near the blow-up time. We should study the interaction among the nonlinear terms $u^m, v^p, u^q, v^q$. At first, consider the case where $v^p, u^q$ dominate the system.

**Theorem 4.1.** Assume $m \leq \frac{2p_0 + p}{p+2}, n \leq \frac{pq + 2q}{2q+1}$, and either $\max\{\alpha_1, \beta_1\} > \frac{N}{2}$, or $\max\{\alpha_1, \beta_1\} = \frac{N}{2}$ with $p, q \geq 1$. Then
\[
c \leq U(t)(T - t)^{\alpha} \leq C, \quad c \leq V(t)(T - t)^{\beta} \leq C
\]
with $(\alpha, \beta) = (\alpha_1, \beta_1)$.

The conditions $m \leq \frac{2p_0 + p}{p+2}, n \leq \frac{pq + 2q}{2q+1}$ with $\max\{m, n, pq\} > 1$ guarantee $pq > 1$, and hence $\alpha_1, \beta_1 > 0$. The proof of the theorem consists of two lemmas.

**Lemma 4.1.** If $m \leq \frac{2p_0 + p}{p+2}, n \leq \frac{pq + 2q}{2q+1}$, and either $\max\{\alpha_1, \beta_1\} > \frac{N}{2}$, or $\max\{\alpha_1, \beta_1\} = \frac{N}{2}$ with $p, q \geq 1$, then $U(t) \leq C(T - t)^{-\alpha_1}$ and $V(t) \leq C(T - t)^{-\beta_1}$.

**Proof.** At first, we scale the solutions. By Theorem 3.3, $U(t), V(t) \to +\infty$ monotonically as $t \to T$. For $U(t) > \|u_0\|_\infty$, there exists $\hat{x} \in \partial B_R$ satisfying $u(\hat{x}, t) = U(t)$. Denote $\lambda = (U(t))^{-1/(2\alpha_1)}$. Clearly, $\lambda \to 0$ as $t \to T$. Let
\[
\varphi^\lambda(y, s) = \lambda^{2\alpha_1}u(\lambda R y + \hat{x}, \lambda^2 s + t), \quad \psi^\lambda(y, s) = \lambda^{2\beta_1}v(\lambda R y + \hat{x}, \lambda^2 s + t)
\]
for any \((y, s) \in \hat{\Omega}_\lambda \times (-t/\lambda^2, (T - t)/\lambda^2)\), where \(\Omega_\lambda = \{y \in \mathbb{R}^N: \lambda R y + \hat{x} \in B_R\}\); \(R\) is an orthonormal transformation defined in the proof of Lemma 3.6. Then \((\varphi^\lambda, \psi^\lambda)\) solves

\[
\begin{cases}
\varphi_s = \Delta \varphi + \lambda^{2\alpha_1 + 2 - 2\alpha_1 m} \varphi^m + \psi^p, & \varphi_s = \Delta \psi \text{ in } \Omega_\lambda \times \left(-\frac{t}{\lambda^2}, \frac{T - t}{\lambda^2}\right), \\
\partial \varphi / \partial \eta_y = 0, & \partial \psi / \partial \eta_y = \varphi^q + \lambda^{2\beta_1 + 2 - 2\beta_1 n} \psi^n \text{ on } \partial \Omega_\lambda \times \left(-\frac{t}{\lambda^2}, \frac{T - t}{\lambda^2}\right),
\end{cases}
\]

and satisfies \(\varphi(0, 0) = 1, 0 \leq \varphi^\lambda \leq 1, 0 \leq \psi^\lambda \leq (U(t))^{-\beta_1 / \alpha_1} V(t)\) for \((y, s) \in \hat{\Omega}_\lambda \times (-t/\lambda^2, 0)\). By \(m \leq (2pq + p)/(p + 2), n \leq (pq + 2q)/(2q + 1)\) and (1.4), we can check that \(2\alpha_1 + 2 - 2\alpha_1 m > 0\) and \(2\beta_1 + 1 - 2\beta_1 n > 0\).

Similarly, we can apply the same scaling procedure for \(v\).

Since \(m \leq \frac{2pq + p}{p + 2}, n \leq \frac{pq + 2q}{2q + 1}\), there exists \(\delta > 0\) by Lemma 3.6 with \(\gamma = \frac{1}{\mu} = \frac{\beta_1}{\alpha_1}\) that

\[
\delta \leq \left(U(t)\right)^{-\frac{1}{2\mu_\lambda}} \left(V(t)\right)^{\frac{1}{2\mu_\lambda}} \leq \delta^{-1}, \quad t \in [0, T).
\]  

Next, we give the estimate on doubling of \(U(t)\). For any \(t \in (0, T)\), define \(t^+ = \min\{t' \in (t, T): U(t') = 2U(t)\}\), and set \(s_\lambda = \lambda^{-2}(t^+ - t)\). It is easy to see that \(\max_{y \in \hat{\Omega}_\lambda} \varphi^\lambda(y, s_\lambda) \leq U(t^+)/U(t) = 2\) and \(\max_{y \in \hat{\Omega}_\lambda} \varphi^\lambda(y, s_\lambda) < 2\) for \(s \in [-t/\lambda^2, s_\lambda]\). Then \(0 \leq \varphi^\lambda \leq 2, 0 \leq \psi^\lambda \leq 2^{\beta_1 / \alpha_1} \delta^{-2\beta_1}\) for \((y, s) \in \hat{\Omega}_\lambda \times [-t/\lambda^2, s_\lambda]\). By using Schauder’s estimates in (4.2),

\[
\|\varphi^\lambda\|_{C^{2+\sigma,1+\frac{2\lambda}{2}}(\hat{\Omega}_\lambda \times [0,s_\lambda])} \leq C, \quad \|\psi^\lambda\|_{C^{1+\sigma,1+\frac{2\lambda}{2}}(\hat{\Omega}_\lambda \times [0,s_\lambda])} \leq C
\]

for some \(\sigma \in (0, 1)\) and \(C > 0\) independent of \(\lambda\).

We claim \(s_\lambda \leq H\) for some \(H > 0\). Otherwise, there would be a sequence \(t_j \to T\) such that \(s_{\lambda_j} = \lambda^{-2}(t_j^+ - t_j) \to +\infty\). Define \(\hat{x}_j, (\varphi^\lambda_j, \psi^\lambda_j)\) as before such that

\[
0 \leq \varphi^\lambda_j \leq 2, \quad 0 \leq \psi^\lambda_j \leq \lambda^{-2} V(t_j + \lambda^2 s_{\lambda_j}), \quad (y, s) \in \hat{\Omega}_{\lambda_j} \times [-t_j/\lambda_j^2, s_{\lambda_j}].
\]  

By using (4.3) and (4.5), \(0 \leq \varphi^\lambda_j \leq 2, 0 \leq \psi^\lambda_j \leq 2^{\beta_1 / \alpha_1} \delta^{-2\beta_1}\). Uniform Schauder’s estimates for \((\varphi^\lambda_j, \psi^\lambda_j)\) as (4.4) yield a subsequence converging to \((\varphi, \psi)\), which solves

\[
\begin{cases}
\varphi_s = \Delta \varphi + \varepsilon_1 \varphi^m + \psi^p, & \varphi_s = \Delta \psi, \quad (y, s) \in \mathbb{R}^N_+ \times \mathbb{R}, \varepsilon_1 \in [0, 1], \\
\partial \varphi / \partial y_1 \bigg|_{y_1=0} = 0, & \partial \psi / \partial y_1 \bigg|_{y_1=0} = \varphi^q + \varepsilon_2 \psi^n, \quad s \in \mathbb{R}, \varepsilon_2 \in [0, 1]
\end{cases}
\]

with

\[
\varphi(0, 0) = 1, \quad 0 \leq \varphi \leq 2, \quad 0 \leq \psi \leq 2^{\beta_1 / \alpha_1} \delta^{-2\beta_1} \text{ in } \mathbb{R}^N_+ \times \mathbb{R}.
\]

However, all nontrivial nonnegative solutions of (4.6) with \(\varepsilon_1 = \varepsilon_2 = 0\) blow up either \(\max\{\alpha_1, \beta_1\} > \frac{N}{2}\), or \(\max\{\alpha_1, \beta_1\} = \frac{N}{2}\) with \(p \geq 1\) [7]. This contradicts (4.7).

Let \(t_0 = t\) and \(t_1 = t^+ = \lambda^2 s_\lambda + t_0\). Then \(t_1 - t_0 = \lambda^2 s_\lambda\) and \(U(t_1) = 2U(t_0)\). Define recursively \(t_j = t_{j-1}^+\) to get a sequence \(t_j \to T\) satisfying \(t_j - t_{j-1} = (\lambda_{j-1})^2 s_{\lambda_{j-1}}, U(t_j) = 2U(t_{j-1})\), and thus \(t_j - t_{j-1} \leq H_2 (j) (1 - 2^{-1/\alpha_1})^{-\alpha_1} (T - t)^{-\alpha_1}, t \in (0, T)\). Summing these inequalities yields \(U(t) \leq H^{\alpha_1} (1 - 2^{-1/\alpha_1})^{-\alpha_1} (T - t)^{-\alpha_1}, t \in (0, T)\). We obtain \(V(t) \leq C(T - t)^{-\beta_1}\) for \(t \in (0, T)\) by (4.3).

Lemma 4.2. If \(m \leq \frac{2pq + p}{p + 2}\) and \(n \leq \frac{pq + 2q}{2q + 1}\), then \(c(T - t)^{-\alpha_1} \leq U(t), c(T - t)^{-\beta_1} \leq V(t)\).
Proof. Let $G$ be Green’s function in $B_R$ satisfying $\frac{\partial G}{\partial \eta} = 0$ on $\partial B_R$. Then

$$\int_{\partial B_R} G(x, y, t, \tau) dS_y \leq \hat{C}(t - \tau)^{-\frac{1}{2}}$$

(4.8)

for a $\hat{C} > 0$ depending only on $B_R$. By Green’s identity and (4.3), we have

$$V(t) \leq V(z) + \hat{C} \int_0^t (V^n(\tau) + \delta^{-2\alpha_1 q} V^{\frac{q}{p+1}}(\tau))(t - \tau)^{-\frac{1}{2}} d\tau.$$  

(4.9)

Since $V(t) \to +\infty$ as $t \to T$, we can choose $z < t$ such that $V(z) = V(t)/2 > 1$. It follows from (4.9) with $n \leq q\alpha_1/\beta_1$ that $V(t) \geq c(T - t)^{-\beta_1}$. The similar estimate for $U(t)$ is true also. \qed

Now we deal with the situation, where $v^n, v^p$ dominate the system.

**Theorem 4.2.** If $\frac{pq + 2q}{2q + 1} < n < \frac{1}{2}p + 1$ for $pq > 1$ (or if $1 < n < \frac{1}{2}p + 1$ for $pq \leq 1$), and $m \leq 1$, then (4.1) holds with $(\alpha, \beta) = (\alpha_3, \beta_3)$.

**Proof.** By Lemma 3.1, the upper bound for $v$ follows. Similarly to Lemma 4.2, we have the lower bound for $u$. Using (3.12) with $\gamma = q/n$, we get the lower bound for $v$.

Next estimate the upper bound for $u$. By Green’s identity,

$$U(t) \leq U(z) + C \int_z^t (U^m(\tau) + C(T - \tau)^{-p\beta_3}) d\tau$$

$$\leq U(z) + CU^m(t)(T - z) + C(T - t)^{1 - p\beta_3}.$$  

Take $z$ such that $C(T - z) < 1/4$ and $4U(z) \leq U(t)$. Due to $m \leq 1$ and $1 - p\beta_3 = -\alpha_3$, we have $U(t) \leq C(T - t)^{-\alpha_3}$. \qed

**Theorem 4.3.** If $\frac{pq + 2q}{2q + 1} < n < \frac{1}{2}p + 1$ for $pq > 1$ (or if $1 < n < \frac{1}{2}p + 1$ for $pq \leq 1$), and $1 < m < \frac{p}{p + 2 - 2n}$, then for any fixed $u_0$, there exists large $v_0$ such that (4.1) holds with $(\alpha, \beta) = (\alpha_3, \beta_3)$.

**Proof.** Similarly to Theorem 4.2, $U(t) \geq c(T - t)^{-\alpha_3}$, $V(t) \leq C(T - t)^{-\beta_3}$, $t \in (0, T)$.

Consider the upper bound for $u$. Introduce an auxiliary problem

$$\begin{cases}
\frac{\partial w}{\partial \eta} = 0, & (x, t) \in \partial B_R \times (0, T), \\
w_t = \Delta w + C_T^p (T - t)^{-p\beta_3} + C_0 (T - t)^{-m\alpha_3}, & (x, t) \in B_R \times (0, T), \\
w(x, 0) = u_0(x), & x \in B_R,
\end{cases}$$

where $C_0 > (\frac{2n - 2}{p + 2 - 2n} \tilde{C}^p)^m$, $\tilde{C}$ is defined in (3.4), and $T$ is the blow-up time of (1.1). For fixed $u_0$, we can make $T$ small by taking $v_0$ large. By Green’s identity,

$$W(t) \leq \left[\|u_0\|_\infty T^{\alpha_3} + \frac{2n - 2}{p + 2 - 2n} \tilde{C}^p (1 + 4C_1 T^{\frac{1}{2}})^\frac{p}{p-1} + \frac{2n - 2}{p + 2 - 2n} C_0 T^{\frac{p - m(p + 2 - 2n)}{2(2n - 1)}}\right](T - t)^{-\alpha_3},$$
and hence
\[ w_t = \Delta w + C_T^p(T - t)^{-p\beta_3} + C_0(T - t)^{-m\alpha_3} \geq \Delta w + C_T^p(T - t)^{-p\beta_3} + u^m \]
provided \( T \) small (\( v_0 \) large) such that
\[ C_0 > \left[ \| u_0 \|_\infty T^{\alpha_3} + \frac{2n - 2}{p + 2 - 2n} \tilde{C}_p (1 + 4C_1 T^\frac{p}{p-n}) \frac{C_0 T^{m(p-2-2n)}}{2(n-1)^2} \right]^m. \]
On the other hand, \( u_t \leq \Delta u + C_T^p(T - t)^{-p\beta_3} + u^m \) in \( B_R \times (0, T) \). We have \( U(t) \leq W(t) \leq C(T - t)^{-\alpha_3} \) by the comparison principle.

Finally, we show \( V(t) \geq c(T - t)^{-\beta_3} \) under \( m < \frac{p}{p+2-2n} \). If not, there would be \( \varepsilon_j \to 0 \) and \( t_j \to T \) such that \( V(t_j) < \varepsilon_j (T - t_j)^{-\beta_3} \). By Green’s identity,
\[ U(t) \leq U(z) + C(U^m(t) + V^p(t))(t - z). \]
It is known that [3] for \( t \) closed to \( T \), there exist \( 0 < z < t < T \) and \( M > 0 \) such that \( U(z) = U(t)/2 > 1 \) and \( t - z \leq M(T - t) \). Then
\[ U(t) \leq C(U^m(t) + V^p(t))(T - t). \]
Using the blow-up rate of \( u \) with \( t = t_j \), we have
\[ c(T - t_j)^{-\alpha_3} \leq C(T - t_j)^{-m\alpha_3 + 1} + \varepsilon_j^p(T - t_j)^{-p\beta_3 + 1}. \]
That is to say, there should be either \(-m\alpha_3 + 1 \leq -\alpha_3\), or \(-p\beta_3 + 1 < -\alpha_3\). However, the definition (1.4) yields \(-p\beta_3 + 1 = -\alpha_3\), and the condition \( m < \frac{p}{p+2-2n} \) with (1.4) implies \(-m\alpha_3 + 1 > -\alpha_3\), a contradiction. \( \square \)

In the following three theorems on blow-up rates, we should assume \( N = 1 \). At first consider the situation, where \( u^m, u^q \) dominate the system.

**Theorem 4.4.** If \( \frac{2pq + p}{p+2} \leq m < 2q + 1 \) for \( pq > 1 \) (or \( 1 < m < 2q + 1 \) for \( pq \leq 1 \)) and \( n \leq 1 \) with \( N = 1 \), then (4.1) holds with \( (\alpha, \beta) = (\alpha_2, \beta_2) \).

**Proof.** By Lemma 3.4, \( U(t) \leq C^*(T - t)^{-\frac{1}{p-n}} = C^*(T - t)^{-\alpha_2}, t \in (0, T). \) The lower bounds for \( u \) and \( v \) can be obtained similarly to the argument of Lemma 4.2.

Next estimate the upper bound for \( v \). Inspired by [22], introduce an auxiliary problem
\[
\begin{cases}
  w_t = w_{xx}, & (x, t) \in (0, R) \times \{ t > 0 \}, \\
  w_x(R, t) = (w^k + w^n)(R, t), & w_x(0, t) = 0, \quad t > 0, \\
  w(x, 0) = w_0(x) > 0, & x \in (0, R),
\end{cases}
\]
where \( k = 1 + \frac{1}{2\beta_2} \geq 1 \geq n \) and \( w_0 \geq 0 \). Choose \( w_0 \) such that \( w \) blows up at the blow-up time \( T \) of (1.1), where \( T \) is determined by \( (u_0, v_0) \). Clearly, \( c_0 \leq W(t)(T - t)^{\beta_2} \leq C_0 \) for \( t \in (0, T) \), where \( W(t) = w(R, t) \).

For such \( w_0 \), let \( l < 1 \) such that \( w_0 - lv_0 > 0 \) and \( c_0^k - (C^*)^q l > 0 \), where \( C^* \) is defined in (3.9) and depends only on \( m \). Since \( (w - lv)_x = (w - lv)_{xx} \), we claim that \( w > lv \) for \( (x, t) \in [0, R] \times [0, T] \). Otherwise, suppose that \( t_0 > 0 \) is the first time such that \( (w - lv)(R, t_0) = 0 \). Due to \( v_x(R, t_0) \leq (C^*)^q(T - t_0)^{-q\alpha_2} + v^q(R, t_0) \) with \( n \leq 1 \),
\[
(w - lv)_x(R, t_0) \geq \left( c_0^k - (C^*)^q l \right)(T - t_0)^{-q\alpha_2} + (1 - l^{1-n})w^n(R, t_0) > 0.
\]
This contradicts \( (w - lv)_x(R, t_0) \leq 0 \). Hence \( V(t) \leq W(t)/l \leq C(T - t)^{-\beta_2}. \) \( \square \)
Theorem 4.5. If \( \frac{2pq+p}{p+2} \leq m < 2q + 1 \) for \( pq > 1 \) (or \( 1 < m < 2q + 1 \) for \( pq \leq 1 \)) and \( 1 < n < \frac{2q}{2q+1-m} \) with \( N = 1 \), then for any fixed \( v_0(R) \), there exists large \( u_0 \) such that (4.1) holds with \( (\alpha, \beta) = (\alpha_2, \beta_2) \).

Proof. By Lemma 3.4, \( U(t) \leq C^*(T - t)^{-\frac{1}{m-1}}, t \in (0, T) \).

As for the estimate for the upper bound of \( v \), consider the auxiliary problem

\[
\begin{align*}
\frac{dw}{dt} &= w_{xx}, \\
\frac{dw}{dt}(R, t) &= (C^*)^q (T - t)^{-q\alpha_2} + C_0(T - t)^{-n\beta_2}, \\
w_x(0, t) &= 0, \\
w_x(w, 0) &= \tilde{v}_0(x),
\end{align*}
\]

where \( \tilde{v}_0 \geq v_0, C_0 > [4q\hat{C}(C^*)^q / (2q + 1 - m)]^n, \hat{C} \) is defined in (4.8). For fixed \( v_0(R) \), we can make \( T \) arbitrarily small by taking \( u_0 \) sufficiently large. By using Green’s identity,

\[
W(t) \leq \|\tilde{v}_0\|_\infty + \frac{4q}{2q + 1 - m} \hat{C}[(C^*)^q + C_0T^{\frac{2q-(2q+1-m)n}{2(m-1)}}](T - t)^{\frac{2q+1-m}{2(m-1)}}
\]

\[
\leq \left[ \|\tilde{v}_0\|_\infty T^{\beta_2} + \frac{4q}{2q + 1 - m} \hat{C} (C^*)^q + \frac{4q}{2q + 1 - m} \hat{C} C_0T^{\frac{2q-(2q+1-m)n}{2(m-1)}} \right] (T - t)^{-\beta_2}
\]

for \( n < \frac{2q}{2q+1-m} \) and \( 1 < m < 2q + 1 \). Then

\[
w_x(R, t) = (C^*)^q (T - t)^{-q\alpha_2} + C_0(T - t)^{-n\beta_2} \geq (C^*)^q (T - t)^{-q\alpha_2} + w^n
\]

provided \( T \) small (\( u_0 \) large) such that

\[
C_0 > \left[ \|\tilde{v}_0\|_\infty T^{\beta_2} + \frac{4q}{2q + 1 - m} \hat{C} (C^*)^q + \frac{4q}{2q + 1 - m} \hat{C} C_0T^{\frac{2q-(2q+1-m)n}{2(m-1)}} \right]^n.
\]

On the other hand, \( v_x(R, t) \leq (C^*)^q (T - t)^{-q\alpha_2} + v^n(R, t) \) in \( (0, T) \). So, by the comparison principle, we have \( V(t) \leq W(t) \leq C(T - t)^{-\beta_2}, t \in (0, T) \).

Similarly to the proof of Theorem 4.4, \( V(t) \geq c(T - t)^{-\beta_2} \) in \( (0, T) \). The lower bound for \( u \) can be obtained as that of the lower bound for \( v \) in Theorem 4.3. \( \square \)

At last, consider the situation, where \( u^n \) and \( v^n \) play the main role. According to Theorem 3.5, simultaneous blow-up may occur for particular initial data in the region of \( p + 2 < 2n, 2q + 1 < m \).

Theorem 4.6. If \( p + 2 < 2n \) and \( 2q + 1 < m \) with \( N = 1 \), then (4.1) holds with \( (\alpha, \beta) = (\alpha_4, \beta_4) \) for simultaneous blow-up.

Proof. Due to \( m > 1 \) and Lemma 3.4, \( U(t) \leq C^*(T - t)^{-\frac{1}{m-1}}, t \in (0, T) \). By Lemma 3.1, we have the upper bound for \( v \) directly.

By using Green’s identity and the upper bound for \( v \),

\[
U(t) \leq U(z) + CU^m(t) - z + C(t - z)^{1 - \frac{p}{2(n-1)}}.
\]

Taking \( z \) such that \( U(z) = U(t)/3 > CT^{-1 - \frac{p}{2(n-1)}} \) for \( p + 2 < 2n \), we have \( U(z) \geq c(T - z)^{-\alpha_4} \). The lower estimate for \( v \) is true similarly. \( \square \)
Theorem 4.8. If $u$ blows up while $v$ remains bounded ($v$ blows up while $u$ remains bounded), then $c \leq U(t)(T-t)^{\alpha_1} \leq C (c \leq V(t)(T-t)^{\beta_1} \leq C)$, $t \in (0, T)$.

Inspired by [13], we have the following theorem on blow-up set. Denote the blow-up sets of $u$ and $v$ by $B(u)$ and $B(v)$, respectively.

Theorem 4.8.

(i) Under the conditions of Theorems 4.2, 4.3, or Theorem 4.1 with $\alpha_1 + 1 \neq \alpha_1 m$, it is true that $B(u) = B(v) = \partial B_R$.

(ii) Under the conditions of Theorems 4.1, 4.4–4.7, we always have $B(v) = \partial B_R$.

**Proof.** We prove (i) only. With no loss suppose $R = 1$. Assume, e.g., the conditions of Theorem 4.1 hold, and hence $U(t) \leq C(T-t)^{-\alpha_1}, V(t) \leq C(T-t)^{-\beta_1}$. Construct

$$w(r, t) = AM^{\alpha_1}/[(1 - r^2)^2 + M(T-t)]^{\alpha_1},$$

$$z(r, t) = BM^{\beta_1}/[(1 - r^2)^2 + M(T-t)]^{\beta_1}$$

in $[0, 1] \times [t_0, T)$ with $B \geq 2^{\beta_1} C$, and $A \geq \max\{2^{\alpha_1} C, 3B^p/\alpha_1\}$. Since $\alpha_1 + 1 > \alpha_1 m$, we can choose $t_0$ close to $T$ such that $M(T-t_0) = 1$ and

$$M \geq \max\{12N + 12 + 48\alpha_1, 4N + 4 + 16\beta_1, 2(3A^{m-1}/\alpha_1)^{1/(\alpha_1 + 1 - \alpha_1 m)}\}. \tag{4.10}$$

By a simple computation, we have $w(r, t_0) \geq u(r, t_0)$, $z(r, t_0) \geq v(r, t_0)$, $r \in [0, 1]$ and

$$w(1, t) = A(T-t)^{-\alpha_1} \geq C(T-t)^{-\alpha_1} \geq u(1, t), \quad t \in [t_0, T),$$

$$z(1, t) = B(T-t)^{-\beta_1} \geq C(T-t)^{-\beta_1} \geq v(1, t), \quad t \in [t_0, T).$$

Due to (4.10), we know for $(r, t) \in (0, 1) \times (t_0, T)$ that

$$w_t - w_{rr} - \frac{N - 1}{r} w_r - w^m - z^p$$

$$\geq A\alpha_1 M^{\alpha_1} [(1 - r^2)^2 + M(T-t)]^{-\alpha_1 - 1}$$

$$\times \left[M - (4N + 4 + 16\alpha_1) - \frac{B^p M}{A\alpha_1} - 2^{\alpha_1 + 1 - \alpha_1 m} A^{m-1} \alpha_1^{-1} M^{m\alpha_1 - \alpha_1} \geq 0,$$

$$z_t - z_{rr} - \frac{N - 1}{r} z_r \geq B\beta_1 M^{\beta_1} [(1 - r^2)^2 + M(T-t)]^{-\beta_1 - 1} (M - 4N - 4 - 16\beta_1) \geq 0.$$ 

By the comparison principle, we get $u \leq w, v \leq z$ for $(r, t) \in [0, 1] \times [t_0, T)$. □

**5. Remarks**

Similarly to [31], we illustrate the main results of this paper by Fig. 1 with notations: $G$—global existence; $S$—simultaneous blow-up only; $N$—non-simultaneous blow-up only; $C$—coexistence of simultaneous and non-simultaneous blow-up; $E$—non-simultaneous blow-up under suitable initial data.
It is clear that all the regions in Fig. 1 have been discussed in this paper. The necessary and sufficient condition for blow-up is \( \max\{m, n, pq\} > 1 \) (Theorem 2.1). Furthermore, the necessary and sufficient condition for simultaneous blow-up with any positive initial data is \( m \leq 2q + 1 \) and \( 2n \leq p + 2 \) for \( N = 1 \) (Corollary 3.1).

For (1.1), we obtain all of the results parallel to those in [31], although there are some technical difficulties here caused by the cross-coupled nonlinear terms. This extends the related work in [23,26] also. For example, the regions \( C_2 \) and \( C_3 \) for coexistence of simultaneous and non-simultaneous blow-up (Theorems 3.1, 3.2, 4.3, and 4.5) were not considered there [23,26]. Four different simultaneous blow-up rates (defined in (1.4)) are obtained (Theorems 4.1–4.6). Moreover, there can be two different simultaneous blow-up rates even in the same \( S_6 \), i.e., \( (\alpha, \beta) = (\alpha_2, \beta_2) \) with large \( u_0 \), and \( (\alpha, \beta) = (\alpha_3, \beta_3) \) with large \( v_0 \). It follows from Theorems 4.1–4.5 that \( (\alpha_1, \beta_1) = (\alpha_2, \beta_2) \) on \( \bar{S}_1 \cap (\bar{S}_2 \cup \bar{S}_4) \), \( (\alpha_1, \beta_1) = (\alpha_3, \beta_3) \) on \( \bar{S}_1 \cap (\bar{S}_3 \cup \bar{S}_5) \).

The introduced characteristic algebraic system (1.3) plays important roles in this paper. Firstly, the four different simultaneous blow-up rates are described as \( O((T - t)^{-\alpha_i}) \), \( O((T - t)^{-\beta_i}) \) (\( i = 1, 2, 3, 4 \)). Secondly, the critical exponent of (1.1) can be stated by \( \max_{i=1,2,3,4}\{1/\alpha_i, 1/\beta_i\} = 0 \), where in addition to (1.4), we define \( 1/\alpha_2 = m - 1, 1/\beta_2 = +\infty \) for \( 2q + 1 = m \), and \( 1/\alpha_3 = +\infty, 1/\beta_3 = 2n - 2 \) for \( p + 2 = 2n \). In fact, \( \max\{m, n, pq\} \leq 1 \) (the necessary and sufficient condition for global solutions) in Theorem 2.1 is equivalent to \( 1/\alpha_i, 1/\beta_i \leq 0 \) (\( i = 1, 2, 3, 4 \)). Lastly, we find from Theorems 3.1–3.5 that the non-simultaneous blow-up established in this paper requires \( \alpha_i \cdot \beta_i < 0, i = 2 \) or 3 (noticing always \( \alpha_1 \cdot \beta_1 > 0 \) by (1.4)). Indeed, concerning non-simultaneous blow-up, together with (1.4), Theorem 3.1 and (i) of Theorem 3.4 require \( \alpha_3 \cdot \beta_3 < 0 \); Theorem 3.2 and (ii) of Theorem 3.4 require \( \alpha_2 \cdot \beta_2 < 0 \), and both \( \alpha_3 \cdot \beta_3 < 0 \) and \( \alpha_2 \cdot \beta_2 < 0 \) for Theorem 3.5. While in Theorem 3.3 it is assumed that both \( \alpha_3 \cdot \beta_3 > 0 \) and \( \alpha_2 \cdot \beta_2 > 0 \) for only simultaneous blow-up.

In this paper we apply a combination of various methods to system (1.1), including the scaling method and Green’s identity method with construction of appropriate auxiliary functions. This makes us consider radial solutions in ball-domains such that, e.g., the solution attains its uniform maximum on the whole boundary as required for the related arguments. It would be interesting to extend the results to more general domains.
In addition, it is observed that for some results of this paper, such as Theorems 3.2, 3.4, 3.5, 4.4–4.6, we treat $N = 1$ only, because these conclusions rely heavily on the upper estimate in Lemma 3.4 established by Souplet and Tayachi [25, Theorem 4 and Lemma 3.4], where $N = 1$ was assumed.

References