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Integral Extensions of Rings Satisfying a Polynomial Identity

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INTRODUCTION

Our main purpose is to study noetherian rings satisfying a polynomial identity. Recall that a prime P.I. ring R has a central simple quotient ring RK where K is the quotient field of the center of R. The central integral closure of R is obtained by adjoining to R all elements of K which satisfy a monic polynomial with coefficients in R. If R is noetherian prime P.I. we show that being centrally integrally closed and (Krull) dimension 1 is equivalent to being a finite module over its center which is a Dedekind domain. In higher dimensions the center need not be noetherian but will still be a Krull domain. R need not be finite over its center but will be integral over it. Our present proof of the above results uses our Theorem 2, which asserts that if R is noetherian prime P.I. and $r \in R$, then the coefficients of the characteristic polynomial of r are integral over R. We show that a prime P.I. ring with A.C.C. on centrally generated ideals, and finitely generated (as an algebra) over a central subring, has a finite central integral extension which is a finite module over its center. This allows us to deduce for instance, that a prime affine (i.e., f.g. over a field) P.I. ring has A.C.C. on centrally generated ideals iff it is right and left noetherian. As a final application we deduce that a noetherian affine P.I. ring is catenary.

We add that "going up," "lying over," and "incomparability" are proved in Theorem 1 for integral extensions of P.I. rings. I would like to thank Lance Small for his helpful suggestions concerning the exposition of this paper.

1. SOME PRELIMINARY FACTS ABOUT P.I. RINGS

We recall a few results about a prime P.I. ring R. It has a central simple quotient ring Q by Posner's theorem, and $Q = R \cdot K$, where K is the center

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of Q, indeed K is the quotient field of the center C of R. The last result is a consequence of central polynomials. A two sided ideal I of R (indeed any subring which is an order in Q) meets the center of R, since $IK = Q$ [14]. R is contained in a finite free C-module [7]. If R is noetherian, then R is a finite C module iff C is noetherian (the implication $\Rightarrow$ being due to [4]).

2. Properties of Integral Extensions

Recall that if R is a subring of S, S is said to be an extension of R if $S = R \cdot S^R$ where $S^R = \{s \in S : rs = sr, \text{ all } r \in R\}$. One good reason for considering overrings of this type is that prime ideals of S contract to prime ideals of R. S is a central extension if $S = R \cdot \text{Center}(S)$. $S \supseteq R$ is integral over R if for every $s \in S$, we have $s^n + r_{n-1}s^{n-1} + \cdots + r_0 = 0$ or $s^n + s^{n-1}r_{n-1} + \cdots + r_0 = 0$ for some $r_i \in R$. For example the central integral closure of a prime P.I. ring (as defined in the introduction) is a central extension. The following lemma shows that it is indeed an integral extension, and therefore the largest integral central extension of R in its quotient ring.

Lemma 1. *Let the R-module $M = \sum_{i=1}^{n} m_i R$ be also an $R[t]$ module, where $t \in \text{Center } R[t]$, $m_i \in M^R$, and $\text{ann}_{R[t]}M = 0$, then t is integral over R.*

Proof. We have $tm_i = \sum_{j=1}^{n} m_j r_{ij}$, where $r_{ij} \in R$ for all i, j. Thus for each $i \leq n$, $\sum_{j=1}^{n} f_{ij}(t)\, m_j = 0$, with f_{ii} a monic polynomial in t of degree strictly greater than $\deg f_{ij}$ if $i \neq j$. This implies that

$$0 = f_{nn}(t)\left(\sum_{j=1}^{n} f_{ij}(t)\, m_j - \sum_{j=1}^{n} f_{nj}(t)\, m_j\right) f_{in}(t)$$

$$= \sum_{j=1}^{n-1} (f_{nn}(t) f_{ij}(t) - f_{nj}(t) f_{in}(t))\, m_j \,.$$

Letting $f'_{ij}(t) = f_{nn}(t) f_{ij}(t) - f_{nj}(t) f_{ij}(t)$ we observe that $f'_{ii}(t)$ is a monic polynomial in t and $\deg f'_{ii}(t) > \deg f_{ij}(t)$ for $j \neq i$. Repeating this procedure we obtain $m_1 g_1(t) = 0$ for some monic polynomial $g_1(t)$. After obtaining such a $g_i(t)$ for each m_i, we see that $g_1(t)\, g_2(t),\ldots, g_n(t)$ annihilates M and so is 0. Q.E.D.

Corollary. *If $R \subseteq T$, $t \in \text{Center}(T)$ and t is integral over R, then every element of $R[t]$ is integral over R.*

Next we extend the basic result of Krull on lifting primes, to apply to integral extensions of a P.I. ring (see [10]).

THEOREM 1. *If T is an integral extension of a ring R, satisfying a polynomial identity, then the following hold*:

1. *If $P \subseteq P_0$ are prime ideals of R and Q is prime in T, with $Q \cap T = P$, then there is a prime ideal $Q_0 \supseteq Q$ of T with $Q_0 \cap R = P_0$ (Going Up).*

2. *If $Q \cap R = Q' \cap R$ for Q and Q' primes of T, then $Q \not\subsetneq Q'$ and $Q' \not\subsetneq Q$. (Incomparability).*

3. *For any prime P in R there is a prime Q in T with $Q \cap R = P$ (Lying Over).*

Proof. Let $Q' \subseteq T$ be maximal (by Zorn) among ideals containing Q_0, such that $Q' \cap R \subseteq P_0$. Q' is obviously prime, and therefore so is $Q' \cap R$ (since T is an extension of R). Suppose $Q' \cap R \subsetneq P_0$. Then in the prime P.I. ring $\bar{R} = R/(Q' \cap R)$, we take a central element $\bar{z} \in \bar{P}_0$, $\bar{z} \neq \bar{0}$ (obtained from the evaluation of a central polynomial for $\bar{R}$ on $\bar{P}_0$). Then we claim $\overline{zT} \cap \bar{R} \subseteq \bar{P}_0$, where $\bar{T} = T/Q'$. For if $\overline{zt} \in \bar{R}$, then since T is integral over R, $t^n + r_{n-1}t^{n-1} + \cdots + r_0 = 0$, for some $r_i \in R$. Note that since T is an extension of R, Center $R \subseteq$ Center T so $(\overline{zt})^n + \bar{r}_{n-1}\bar{z}(\overline{zt})^{n-1} + \cdots + \bar{z}^n\bar{r}_n = \bar{0}$, i.e., $(zt)^n \in P_0$. Thus $\overline{zT} \cap \bar{R}$ is a nil ideal modulo $\bar{P}_0$, and is therefore contained in $\bar{P}_0$ (since R/P_0 is prime P.I. and so does not admit nil ideals). Therefore $(zT + Q) \cap R \subseteq P_0$, contradicting the maximality of Q, and thus proving 1.

For 2 suppose $Q_1 \supsetneq Q_2$ are prime ideals of T and $Q_1 \cap R = Q_2 \cap R$. Let $\bar{R} = R/Q_2 \cap R \subseteq \bar{T} = T/Q_2$. Take $q \in Q_1$, such that q is regular modulo Q_2. We have $\bar{q}^n + \bar{r}_{n-1}\bar{q}^{n-1} + \cdots + \bar{r}_0 = 0$, $r_i \in R$. We may assume $\bar{r}_0 \neq 0$ since $\bar{q}$ is regular; but then $r_0 \in Q_1 \cap R$ but $r_0 \notin Q_2 \cap R$.

3 is an immediate consequence of 1, since we may take Q' maximal (by Zorn) in T such that $Q' \cap R \subseteq P$, and then apply going up to the primes of R, $Q' \cap R \subseteq P$. Q.E.D.

COROLLARY 1. *If either R or T has classical Krull dimension then so does the other and* $\dim R = \dim T$.

Proof. Recall that in order to define dimension (possibly infinite) one defines $\mathscr{P}_{0,R}$ to be the maximal primes of R and

$$\mathscr{P}_{\alpha,R} = \{P \in \operatorname{Spec} R : P \notin \mathscr{P}_{\beta,R} \text{ for } \beta < \alpha \text{ and for all } Q' \in \operatorname{Spec} R,$$
$$Q' \supsetneq P \Rightarrow Q \in \mathscr{P}_{\beta,R} \text{ for some } \beta < \alpha\}.$$

The dimension if it exists is the first α such that $\operatorname{Spec} R = \bigcup_{\beta \leqslant \alpha} \mathscr{P}_{\beta,R}$. We claim $\mathscr{P}_{\alpha,T} = \{Q \in \operatorname{Spec} T : Q \cap R \in \mathscr{P}_{\alpha,R}\}$. We assume it is true for any $\beta < \alpha$. Then if $Q \in \mathscr{P}_{\alpha,T}$ certainly $Q \cap R \notin \mathscr{P}_{\beta,R}$ some $\beta < \alpha$. If $P' \supsetneq Q \cap R$, then we can take $Q' \supset Q$, Q' lying over P', we have $Q \in \mathscr{P}_{\beta,T}$ for some $\beta < \alpha$;

and thus by induction $P' = Q' \cap R \in \mathscr{P}_{\beta,R}$, so $Q \cap R \in \mathscr{P}_{\alpha,R}$. On the other hand if $Q \in \operatorname{Spec} T$ and $Q \cap R \in \mathscr{P}_{\alpha,R}$, then certainly $Q \notin \mathscr{P}_{\beta,T}$ for $\beta < \alpha$, and so suppose $Q' \supsetneq Q$. We know by 2 $Q' \cap R \supsetneq Q \cap R$ so that $Q' \cap R \in \mathscr{P}_{\beta,R}$ for $\beta < \alpha$ and so $Q' \in \mathscr{P}_{\beta,T}$, by induction. This means $Q \in \mathscr{P}_{\alpha,R}$. The final conclusion of the corollary is now clear. Q.E.D.

The reader should be warned that "Going Up" fails miserably if we assume merely that T is an integral overring of R. One also has the following Proposition of which we shall make occasional use. Unlike the extension case the inclusion can be strict.

PROPOSITION 1. *If T is an integral overring of R then* Jac $T \cap R \subseteq$ Jac R, *where* Jac *denotes the Jacobson radical.*

Proof. If $j \in \operatorname{Jac} T \cap R$, then $t(1-j) = 1$ for some $t \in T$. But $t^n + r_{n-1}t^{n-1} + \cdots + r_0 = 0$, some $r_i \in R$, and so

$$t = -(r_{n-1} + r_{n-2}(1-j) + \cdots + r_0(1-j)^{n-1}) \in R.$$

Q.E.D.

The next Proposition is a generalization of a result of Sirsov [13]. His theorem was for the case of a commutative ring R.

PROPOSITION 2. *If $T = R\{x_1, \ldots, x_n\}$ is a finitely generated extension of R* (i.e., $x_i \in T^R$) *and T satisfies a multilinear identity of degree m, and if all monomials in $x_1, \ldots, x_n$ of degree $\leqslant (m/2)^2$ are integral over R, then T is a finite R module.*

Proof. We leave the extension of the argument when R is commutative in [12, p. 152] as an exercise for the reader.

We next consider polynomial rings. We wish to know that adjoining a central indeterminate leaves a ring centrally integrally closed. We would also like to be able to say this for a twisted polynomial ring. In that case it will be so if the original ring were commutative but need not be so otherwise. The first step in either case is to know that if R is prime P.I. with quotient ring Q, and we have fg^{-1} is in the center of the quotient ring of $R[x, \sigma]$ (the twisted polynomial ring over R, with σ an automorphism of R) then if fg^{-1} is integral over $R[x, \sigma]$, it should at least fall into $Q[x, \sigma]$. It does.

LEMMA 2. *If S is a principal right ideal domain, it is centrally integrally closed.*

Remark. By [9] $Q[x, \sigma]$ is a principal right ideal domain.

Proof. If $ab^{-1} \in$ Center Q', where Q' is the quotient ring of S, then $ab^{-1} = b^{-1}a$. We know $aS + bS = cS$. We have $(c^{-1}b)^{-1}c^{-1}a = b^{-1}a$, so we may assume $aS + bS = S$, i.e., $b^{-1} = b^{-1}as_1 + s_2$, $s_1, s_2 \in S$. Now if $(ab^{-1})^n = (ab^{-1})^{n-1} r_{n-1} + \cdots + r_0$, then we also have $(b^{-1})^n = (b^{-1}a)^n (s_1)^n + (b^{-1}a)^{n-1} t_{n-1} + \cdots + t_0$, with $t_i \in S$. Substituting the former in the latter and multiplying through by b^{n-1} yields $b^{-1} \in S$. Q.E.D.

We also have the following result.

PROPOSITION 3. *If $R \subseteq T$, x is a commutative indeterminate, $f(x) \in$ Center $T[x]$, and $f(x)$ is integral over $R[x]$, then the coefficients of $f(x)$ are integral over R.*

Proof. As in the commutative case we pass to a ring $T' \supseteq T$ such that if f is monic $f(x) = (x - \beta_1) \cdots (x - \beta_n)$ where $\beta_i \in$ Center T'. T' is obtained by passing first to $T_1 = T[x]/(f(x))$ (we have $T \cap (f(x)) = 0$ since $\deg (f(x) \cdot t(x)) = \deg f(x) + \deg t(x) > 0$.) In T_1, $f(x) = g(x)(x - \alpha_1)$ and we repeat.

Now take any $f(x) \in$ Center $T(x)$ such that $g(f(x)) = 0$ where $g(y) \in R[x][y]$ is monic, $g(y) = y^m + f_{m-1}y^{m-1} + \cdots + f_0$. We form $g(y - x^r)$ where $r \geqslant \deg f_i$, $\deg g$, $\deg f$. Then $g(y - x^r) = \sum_{i=0}^{n} g_i(x)\, y^i$ where g_0 is monic. Now $f + x^r$ is monic so by the first part of the proof, it factors into $\prod_1^r (x - \alpha_i)$, $\alpha_i \in$ Center T'. The α_i are integral over R since $g_0(\alpha_i) = 0$. By Lemma 1 the coefficients of f are integral over R. Q.E.D.

Remark. If R is commutative and we form $R[x, \alpha]$, then $f(x) \in$ Center $Q[x, \alpha]$ implies that the coefficients of $f(x)$ are central, as are the powers of x appearing in f. The above argument then applies. Thus combining it with the preceding Lemma we would have that R integrally closed implies $R[x, \alpha]$ is centrally integrally closed.

EXAMPLE. Let

$$R = \begin{pmatrix} k[x] & (x^2) \\ (x^2) & k[x] \end{pmatrix}$$

where k is a field. R is centrally integrally closed, but if σ is the automorphism of R given by conjugation with $X = \left(\begin{smallmatrix} 0 & x \\ x & 0 \end{smallmatrix}\right)$, then $R[y, \sigma]$ is not centrally integrally closed. For Xy lies in the center of the quotient of $R[y, \sigma]$, and $(Xy)^2 \in R[y, \sigma]$.

3. NOETHERIAN P.I. RINGS

The first result shows that the center of a centrally integrally closed prime P.I. ring R with A.C.C. on two sided ideals is a Krull ring. We then show that in this case R is integral over its center. Indeed we show that the coefficients of the characteristic polynomial, of any element of R, are integral over R.

We recall that a Krull domain is one in which localization at minimal primes yields a discrete rank 1 valuation ring, and it is a locally finite intersection of these localizations.

PROPOSITION 4. *If R is prime, with* A.C.C. *on centrally generated ideals, and centrally integrally closed, then the center Z of R is a Krull domain.*

Before proving this we require a lemma about the set of zero divisors of Z/cZ called $\mathscr{Z}(Z/cZ)$.

LEMMA 3. *If R has* A.C.C. *on centrally generated ideals then $\mathscr{Z}(Z/cZ) = \bigcup_{i=1}^{n} \mathfrak{p}_i$, where each $\mathrm{P}_i = \mathrm{ann}_Z x_i + cZ/cZ$, $x_i \in Z$, and the P_i are prime.*

Proof. We have $Z/cZ \subseteq R/cR$ since $cR \cap Z = cZ$. Thus Z/cZ enjoys the maximum condition on annihilator ideals.

Now proceed as in the usual commutative case to put any $x \in \mathscr{Z}(Z/cZ)$ into a maximal annihilator which is naturally prime. So $\mathscr{Z}(Z/cZ)$ is a union of maximal annihilators say $\mathrm{ann}_Z(x_i + cZ/cZ)$, indeed it is the union of the finite number, corresponding to those x_i required to generate $\sum_i x_i R$.

Proof of Proposition. If $\mathfrak{p}$ is any height one prime of Z, and if $c \in \mathfrak{p}$, we have $\mathfrak{p} \subseteq \mathfrak{p}_j$ for some $\mathfrak{p}_j$ as in the statement of the lemma. Let $M = \mathfrak{p}_j Z_{\mathfrak{p}_j}$. Then $M^{-1}M = Z_{\mathfrak{p}_j}$, for otherwise $M^{-1}M \subseteq M$ so that $M^{-1}MR_{\mathfrak{p}_j} \subseteq MR_{\mathfrak{p}_j}$ implying $M^{-1} \subseteq R_{\mathfrak{p}_j}$ by Lemma 1. But $x_j c^{-1} \in M^{-1}$ and $x_j c^{-1} \notin Z_{\mathfrak{p}_j}$. Thus M is invertible and so $Z_{\mathfrak{p}_j}$ is a D.V.R. That $Z = \bigcap\{Z_{\mathscr{A}} : \mathscr{A}$ is a maximal prime $\subseteq \mathscr{Z}(Z/cZ), c \in Z\}$ is always true, and local finiteness follows from the fact that $c \in Z$ is contained in only finitely many minimal primes, namely the $\mathfrak{p}_i$ of the lemma. Q.E.D.

Remark. Minimal prime ideals of the center always lift to prime ideals of R. Also if R has A.C.C. any prime $\mathfrak{p}$ maximal in $\mathscr{Z}(Z/cZ)$ lifts to R. For if $\mathfrak{p} = \mathrm{ann}_Z(x + cZ/cZ)$, then take $M \subseteq R$ maximal such that $M \supseteq \mathrm{ann}_R(x + cR/cR)$ and $M \cap Z = \mathfrak{p}$. We can do this since

$$\mathrm{ann}_R(x + cR/cR) \cap Z = \mathrm{ann}_Z(x + cZ/cZ) = \mathfrak{p}.$$

M is the required prime lying over $\mathfrak{p}$.

If $f = \sum_\nu \alpha_\nu x_1^{\nu_1} \cdots x_n^{\nu_n}$ is a polynomial in commuting indeterminates, one can construct (following Formanek in [6]) a polynomial $p_f = \sum_\nu \alpha_\nu X^{\nu_1} Y_1 X^{\nu_2} Y_2 \cdots X^{\nu_n} Y_n$ where the X, Y_i are noncommuting indeterminates. If $\delta = \sum_{i<j\leqslant n} (x_i - x_j)^2$ and g is symmetric in $x_1, \ldots, x_n$, then Amitsur observed the following consequence of Formanek's method.

LEMMA (Amitsur, [1]). *If X is an $n \times n$ matrix over a field with all its eigenvalues $\lambda_1, \ldots, \lambda_n$ distinct, then for any matrices $Y_1, \ldots, Y_n$*

$$p_{g\delta}(X, Y_1, \ldots, Y_n) = g(\lambda_1, \ldots, \lambda_n)\, p_\delta(X, Y_1, \ldots, Y_n),$$

and $p_\delta(X, E_{12}, E_{23}, \ldots, E_{nn}) \neq 0$ where E_{ij} denotes the matrix with a 1 in the (i,j)th component.

Proof. It suffices to check the lemma when X is diagonal since $A^{-1}XA$ is diagonal for some A. The functions are linear in the Y_i's so we need only check it with $Y_i = E_{j_ik_i}$. If $g\delta(x_1, \ldots, x_n) = \sum_\nu \beta_\nu x_1^{\nu_1} \ldots x_n^{\nu_n}$

$$\begin{aligned} P_{g\delta}(A^{-1}XA, E_{j_1k_1}, \ldots, E_{j_nk_n}) &= \sum \beta_\nu \lambda_{j_1}^{\nu_1} \ldots \lambda_{j_n}^{\nu_n} E_{j_1k_1} \cdots E_{j_nk_n} \\ &= g\delta(\lambda_{j_1}, \ldots, \lambda_{j_n})\, E_{j_1k_1} \cdots E_{j_nk_n} \\ &= g(\lambda_1, \ldots, \lambda_n)\, \delta(\lambda_{j_1}, \ldots, \lambda_{j_n})\, E_{j_1k_1} \ldots E_{j_nk_n} \\ &= g(\lambda_1, \ldots, \lambda_n)\, p_\delta(A^{-1}XA, E_{j_1k_1}, \ldots, E_{j_nk_n}). \end{aligned}$$

Q.E.D.

THEOREM 2. *If R is a prime* P.I. *ring with* A.C.C. *on centrally generated ideals, then the coefficients of the characteristic polynomial of any element of R, are integral over R.*

Proof. If $r \in R$, let $S(r)$ denote the coefficient of the characteristic polynomial of r, obtained by letting the elementary symmetric function S act on the eigenvalues of r. If $s \in R$ has all its eigenvalues distinct (obtainable by taking such an element in the central simple quotient ring of R and multiplying by a central element so that if falls into R), then $r + ys$ has distinct eigenvalues, if y is a central indeterminate. To see this let v be another central indeterminate. The discriminant $\delta(vr + ys) = g(v)$ is a polynomial function of v (obtained by writing δ as a polynomial function of the elementary symmetric functions). We know $g(0) \neq 0$, so since $C =$ Center (R) is infinite (otherwise our result is well known to be true), we have $g(c^{-1}) \neq 0$ for some $c \in C$. Thus $c^{-1}r + ys$ has distinct eigenvalues, and so does $c(c^{-1}r + ys) = r + ycs$ (and so does $r + ys$). Let I be the ideal of $R[y]$ generated by $p_{S^m\delta}(r + ys, x_1, x_2, \ldots, x_n)$, with $x_i \in R[y]$, and m any natural number, then $S(r+ys)I \subseteq I$. For we have, $S(r+ys)\, p_{S^m\delta}(r+ys, x_1, \ldots, x_n) = p_{S^{m+1}\delta}(r + ys, x_1, \ldots, x_n)$. If $0 \neq h(y)$ is a central element of I, then, let $h(y) = f(y)y^j$ where $f(y) \in C[y]$ and $f(0) \neq 0$. Now $(S(r+ys))^m f(y) \in C[y]$ for any m, so $(S(r))^m f(0) \in C$. But now if $\sum_{i=0}^{\infty} (S(r))^i f(0)R = \sum_{i=0}^{k} (S(r))^i f(0)R$, then $(S(r))^{k+1} f(0)) = \sum_{i=0}^{k} (S(r))^i f(0)\, r_i$, with the $r_i \in R$, i.e., $S(r)^{k+1} - (S(r))^k r_k - \cdots - r_0 = 0$. Q.E.D.

COROLLARY 1. *If R is prime* P.I., *centrally integrally closed, and satisfies* A.C.C. *on centrally generated ideals, then R is integral over its center which is a Krull domain.*

Proof. Immediate from Proposition 4 and Theorem 2.

We still have some information when R satisfies no noetherian condition:

COROLLARY 2. *If R is prime* P.I., *then the coefficients of the characteristic polynomial of any element of R are in the complete integral closure of the center of R.*

Proof. The proof of the corollary is contained in the proof of the theorem.

Robson and Small proved in [13] a statement similar to $2 \Rightarrow 3$ of the following theorem; namely they showed that if 2-sided ideals are projective then 3 holds.

THEOREM 3. *If R is prime* P.I. *with* A.C.C. *on two sided ideals, the following are equivalent:*

1. *R is centrally integrally closed and has Krull dimension* 1.
2. *Centrally generated ideals are projective.*
3. *R is a finite module over its center which is a Dedekind domain.*

Proof. $1 \Rightarrow 3$. The center of R is a Krull domain by Proposition 4. But Corollary 1 says R is integral over its center. The going up theorem tells us that the center must be 1-dimensional. But 1-dimensional Krull domains are Dedekind domains.

$3 \Rightarrow 2$. This is immediate since R is contained in a finite free module over its center Z [7], and so R_Z is projective, since Z is hereditary. If $I = \sum_{j=1}^{n} z_j R$, with $z_j \in Z$, then $I' = \sum_{j=1}^{n} z_j Z$ satisfies $I' \oplus Y = Z^{(n)}$. Tensoring with R_Z yields the result since $I' \otimes R_Z \simeq I'R_Z = I$.

$2 \Rightarrow 1$. The usual dual basis argument of the commutative case shows centrally generated ideals are invertible. If k is in the center of the quotient ring of R and k satisfies a monic polynomial of degree n with coefficients in R, then the ideal $I = \cap\{k^{-j}R \cap R : j < n\}$ satisfies $kI \subseteq I$. Thus $k(I \cap Z)R \subseteq k(I \cap Z)R$. Letting $I_1 = (I \cap Z)R$, we have $kR = kI_1I_1^{-1} \subseteq I_1I_1^{-1} = R$, so R is centrally integrally closed. Suppose R (and hence Z) is not of dimension 1. Take $\mathfrak{m} \supsetneq \mathfrak{p} \supsetneq 0$ primes in Z and $M \supsetneq P \supsetneq 0$ primes lying over them in R (we have R integral over Z by the first part and Corollary 1). If $z \in \mathfrak{m} \backslash \mathfrak{p}$, then we wish to show $z^i(R/P) = z^{i+1}(R/P)$ for some i. This would be clearly a contradiction (since then $1 = zr + p \in M$). The increasing sequence of ideals of $R(z^iR + R)^{-1}p$, with $p \in \mathfrak{p}$, must stop. When it does, $(z^iR + \mathfrak{p}R)^{-1} = (z^{i+1}R + \mathfrak{p}R)^{-1}$, so we have $z^iR + \mathfrak{p}R = z^{i+1}R + \mathfrak{p}R$ as required. Q.E.D.

We can have 1-dimensional noetherian prime P.I. rings which are not finite modules over their centers. For example if K_1, $K_2 \subseteq L$ are fields such that $[L : K_1] < \infty$, $[L : K_2] < \infty$, X is a commutative indeterminate, then

$$\begin{pmatrix} K_1[X] + XL[X] & XL[X] \\ XL[X] & K_2[X] + XL[X] \end{pmatrix}$$

is such a ring if $[L: K_1 \cap K_2] = \infty$.

We also note that in [5] K. Fields gives examples of rings satisfying the above theorem, yet of arbitrary finite global dimension.

We note the following ring R has Krull dimension 2, is right and left noetherian, prime, P.I., and is centrally integrally closed. It is not a finite module over its center. Let A be the 2-dimensional noetherian integrally closed domain constructed in [11], with automorphism σ, such that the ring $A^\sigma = \{a: a^\sigma = a\}$ is not noetherian, and $\sigma^2 =$ identity.

$$R = \left\{\begin{pmatrix} a & b \\ b^\sigma & a^\sigma \end{pmatrix} : a, b \in A\right\}.$$

Noether's partial solution of Hilbert's Fourteenth Problem says that the fixed ring of an affine domain (i.e., a commutative domain finitely generated over a field), under the action of a finite group of automorphisms, must be affine. We might thus expect to obtain better results when our noetherian prime P.I. ring is affine (i.e., finitely generated over a central subfield). We recall further that in [15] we showed that a 1-dimensional prime affine P.I. ring was a finite module over its center. Here is an example of a 2-dimensional one which is not finite over its center, although as the next proposition shows, centrally integrally closed noetherian affine P.I. rings are finite modules over their centers.

Let $B_1 = Q[6^{1/2}][(2)^{1/2} + X, Y, (2)^{1/2}Y]$ and $B_2 = Q[6^{1/2}][(3)^{1/2} + X, Y, (3)^{1/2}Y]$. The ring

$$R = \begin{pmatrix} B_1 & YB_1 + (2)^{1/2}YB_1 \\ YB_1 + (2)^{1/2}YB_1 & B_2 \end{pmatrix}$$

is the desired ring. For first note that

$$\begin{aligned} YB_1 + (2)^{1/2}\, YB_1 &= \{f \in Q[(2)^{1/2}, (3)^{1/2}][X^iY^j : j \geq 1] : f(X, 0) = 0\} \\ &= YB_2 + (3)^{1/2}\, YB_2 . \end{aligned}$$

"$\subseteq$" is clear and "$\supseteq$" is because if $k \in Q[(2)^{1/2}, (3)^{1/2}]$

$$kX^iY^j = ((2)^{1/2} + X)^i\, Y^{j-1}kY - \{\text{terms in } X^sY^j \text{ with } s < i\}$$

so induction yields the equality. We claim that $B_1 \cap B_2 = Q[6^{1/2}] + YB_1 + (2)^{1/2}YB_1$. "$\supseteq$" is clear, so take $\sum_{i=0}^{n} a_i((2)^{1/2}+X)^i+f = \sum_{i=0}^{m} b_i((3)^{1/2}+X)^i+g$ where $f, g \in YB_1 + (2)^{1/2} YB_1$, and $a_i, b_i \in Q[6^{1/2}]$, $a_n \neq 0$. Setting $Y = 0$, we see $m = n$ and $a_n = b_m$. If $n = 0$, everything is fine. Suppose $n \geq 1$, then equating the coefficients of X^{n-1}, $a_n n(2)^{1/2} + a_{n-1} = b_n n(3)^{1/2} + b_{n-1}$, i.e., $(2)^{1/2} - (3)^{1/2} = (b_{n-1} - a_{n-1})/a_n n \in Q[6^{1/2}]$ a contradiction. It is clear that $B_1 \cap B_2$ is not noetherian and it is the center of R. R is noetherian since it is a finite right $B_1 \oplus B_2$ module. It is clearly affine and 2-dimensional since B_1 and B_2 are.

We also note that there are prime ideals $P_1 \supsetneq P_2$ in the above ring R such that $P_1 \cap$ Center $(R) = P_2 \cap$ Center (R). For take $Q_2 = YB_1 + (2)^{1/2} YB_1 \subsetneq Q_1 \subsetneq B_1$ with Q_1 a prime ideal. Then

$$\begin{pmatrix} Q_1 & Q_2 \\ Q_2 & B_2 \end{pmatrix} \supsetneq \begin{pmatrix} Q_2 & Q_2 \\ Q_2 & B_2 \end{pmatrix}$$

and both prime ideals intersect the center in the maximal ideal $Q_2 \subseteq B_1 \cap B_2$.

We note further, that not only is the center $B_1 \cap B_2$ not noetherian, but it fails to satisfy the principal ideal theorem. For the maximal ideal Q_2 of $B_1 \cap B_2$, satisfies $(Q_2)^2 \subseteq Y(B_1 \cap B_2)$, yet

$$Q_2 \supsetneq (XQ[(2)^{1/2}, (6)]^{1/2}[X, Y]) \cap B_1 \cap B_2 \supsetneq 0,$$

so Q_2 is not a minimal prime.

We have however that the central integral closure of a prime affine P.I. ring with A.C.C. on centrally generated ideals, is a finite module over its center. Thus normalization (i.e., integral closure) is a powerful tool for the study of these rings.

PROPOSITION 5. *If $R = A\{\xi_1, \ldots, \xi_n\}$ is a prime* P.I. *ring, finitely generated over a central subring A, and if R satisfies* A.C.C. *on centrally generated ideals then, there is a finite central extension R' of R such that R' is a finite module over its center.*

Proof. R satisfies a multilinear identity of degree m. We form R' by adjoining to R the coefficients of the characteristic polynomials of all monomials of degree $\leqslant (m/2)^2$ in $\xi_1, \ldots, \xi_n$. By Theorem 2 R' is a finite module over R. By Proposition 2 [Sirsov] R' is a finite module over the ring obtained by adjoining to A the above coefficients. Q.E.D.

COROLLARY 1. *If R is prime affine, with* A.C.C. *on centrally generated ideals then it is right and left noetherian, if A is noetherian.*

Proof. Pass to R' as above. R' is right and left noetherian, so by Eisenbud's generalization [4] of Eakins theorem, R is right and left noetherian. Q.E.D.

COROLLARY 2. *If R is centrally integrally closed, noetherian, and finitely generated over a central subring, then it is a finite module over its center.*

COROLLARY 3. *If R satisfies* A.C.C. *on 2-sided ideals and is finitely generated over a central subring, then $\bigcap_{n=1}^{\infty} (J)^n$ is nilpotent, where $J = \mathrm{Jac}(R)$.*

Remark. The example of Herstein $\begin{pmatrix} D & Q \\ 0 & Q \end{pmatrix}$ where D denotes the localization of the integers at some prime, satisfies the hypothesis of the Corollary, and $\bigcap_{n=1}^{\infty} (J)^n \neq 0$.

Proof. If P is any prime ideal of R then we claim $\bigcap_n J^n \subseteq P$. For if $R' \supseteq R/P$ as in the proposition $J \subseteq \mathrm{Jac}(R')$, so $J^{\omega} \subseteq (\mathrm{Jac}\ R')^{\omega} \subseteq (\mathrm{Jac}(R_m'))^{\omega} = (mR_m')^{\omega} = 0$ where m is any maximal ideal of center R'. Thus J^{ω} is contained in the prime radical and so is nilpotent. Q.E.D.

We recall that Jategaonkar and Formanek generalized Eisenbud's theorem to modules. Namely if R is a finite extension of a subring S (i.e., $R = \sum_1^n a_i S$, $a_i \in S^R$) then noetherian R-modules are still noetherian when regarded as S-modules.

COROLLARY 4. *If $R = A\{\xi_1, ..., \xi_n\}$ is prime* P.I. *and satisfies* A.C.C. *on two-sided ideals then it is right and left noetherian.*

Proof. Pass to R' as in the theorem, then R' satisfies A.C.C. on two-sided ideals. $R' \otimes_{Z'} R'^{op}$ is a finite extension of $Z'(Z' = \text{Center } R')$ $R'_{R' \otimes_Z R'^{op}}$ is noetherian, and hence so is $R'_{Z'}$. Thus R' is right and left noetherian and therefore so is R.

With what we now have available the following important fact is easy.

PROPOSITION 6. *The central integral closure of a prime noetherian affine* P.I. *ring R is a finite R-module.*

Proof. Pass to R' as in the previous theorem. The center Z' of R' is now affine. The integral closure Z'' is well known to be a finite module over Z', and the central integral closure of R' is just $R'Z''$. Q.E.D.

Our final major result is the following theorem, which tells us in particular that noetherian affine P.I. rings are catenary i.e., all saturated chains of primes, between two given primes, have the same length. It also tells that all maximal primes have the same height, namely equal to the dimension of the ring.

THEOREM[1] 4. *If R is a prime affine noetherian* P.I. *ring, and P is any prime ideal, then* $\dim R = \dim R/P + \operatorname{ht} P$.

Proof. It is enough to prove it when $\operatorname{ht} P = 1$. Let R' be the central integral closure of R, and P' be a height 1 prime of R' lying over P. Since $\dim R/P = \dim R'/P'$, we need only check the theorem in this case. Let $Z' = \text{Center } R'$, then since Z' is affine, it suffices to check that $P' \cap Z'$ has height 1; for we know $\dim R'/P' = \dim Z'/P' \cap Z'$ and we could now apply the usual commutative result. Suppose $P' \cap Z' = m$ is not minimal. Localize to form R'_m (i.e., $R'Z_m'$). Let R'' denote a maximal integral extension of $Z_m' = C$ containing R'_m (exists by Zorn). Note that the center of R'' is still C which is noetherian, so R'' is a finite module over C. We want all the maximal ideals of R'' to be not minimal. For if so, $M_i \supsetneq Q_i \supsetneq 0$, M_i's maximal and the Q_i's prime, then $\prod_{Q_i \supseteq mR''} (Q_i \cap R') \subseteq \text{Jac } R'' \cap R' \subseteq \text{Jac } R' \subseteq P'$ so without loss in generality $Q_1 \cap R' \subseteq P'$. The inclusion is proper, so we need to show $Q_1 \cap R'$ is prime. If not then P' is minimal over $Q_1 \cap R'$ (by assumption) so $(P'X)^n \subseteq Q_1 \cap R'$ where $X \not\subseteq Q_1 \cap R'$. But take $z \in P' \cap C \backslash Q_1 \cap C$ (can do by INC), so $z^n X^n \subseteq Q_1 \cap R'$. Thus $z^n R'' X^n \subseteq Q_1$ contradicting the fact that Q_1 is prime.

Thus we may assume some maximal ideal M of R'' is minimal. Take $c \in M \cap C$ and let $\operatorname{rad} cR = M \cap Y$ (there are other minimal over primes since we can lift primes minimal over cC). $(MY)^n \subseteq cR''$ for some n. Let $R''' = \{q \in Q(R)\colon qm^t \subseteq R'', \text{ some } t\}$. The above asserts $R''' \supsetneq R''(Y^n c^{-1} \subseteq R''')$. But center $R''' = C$. For if k is in the center of R''', then $km^i \subseteq C$, implying $k \in C$, since $\operatorname{ht} m \geq 2$ and C is integrally closed. Then R''' is a finite module over its noetherian center C, providing a contradiction. Q.E.D.

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[1] This result has now been proved without the noetherian assumption and will appear in a subsequent paper, by the author, in this journal.

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