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A Note on Ramsey Numbers

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Upper bounds are found for the Ramsey function. We prove $R(3, x) < cx^2/\ln x$ and, for each $k \geqslant 3$, $R(k, x) < c_k x^{k-1}/(\ln x)^{k-2}$ asymptotically in x.

The Ramsey function $R(k, x)$ is defined as the minimal integer n so that any graph on n vertices contains either a clique of size k or an independent set of size x. We show

$$R(3, x) \leqslant cx^2/\ln x \tag{1}$$

and further that for each k

$$R(k, x) \leqslant c_k x^{k-1}/(\ln x)^{k-2}. \tag{2}$$

The function $R(3, x)$ has been the object of much study. The asymptotic bounds

$$cx^2/(\ln x)^2 < R(3, x) < cx^2 \ln \ln x/\ln x$$

have been given by Erdös [2] (lower bound) and Graver and Yackel [3] (upper bound). A quite different proof of (1) is given in our paper [1].

Notation. All graphs are finite.

$n = n(G) =$ number of vertices of G;

$e = e(G) =$ number of edges of G;

$t = t(G) =$ average degree in $G = 2e/n$;

$\delta = \delta(G) =$ edge density of $G = 2e/n(n-1)$;

$\omega(G) =$ size of maximal clique in G;

$\alpha(G) =$ size of maximal independent set in G;

$\deg(P) =$ degree of (vertex) P.

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Set $r(P)$ equal to the summation of the degrees of the points Q adjacent to P. We call P a groupie if $r(P) \geqslant t \deg(P)$, where $t = t(G)$.

LEMMA 1. *Every graph G has a groupie.*

Proof. Write PIQ if P is adjacent to Q in G

$$\sum_P r(P) = \sum_P \sum_{\substack{Q \\ PIQ}} \deg(Q) = \sum_Q \sum_{\substack{P \\ PIQ}} \deg(Q) = \sum_Q \deg(Q)^2. \tag{4}$$

Set $n = n(G)$, $t = t(G)$. If $r(P) < t \deg(P)$ for all P then

$$t^2 n = \sum_P t \deg(P) > \sum_P r(P) = \sum_P \deg(P)^2. \tag{5}$$

This is contradicted by the Cauchy–Schwartz inequality

$$\sum_P \deg(P)^2 \geqslant \left(\sum_P \deg(P)\right)^2 \Big/ n = t^2 n. \tag{6}$$

THEOREM 2. *Let G be a graph with $n = n(G)$, $t = t(G)$. Assume G is trianglefree. Then*

$$\alpha(G) \geqslant 0.01(n/t) \ln t \tag{7}$$

Note. No attempt is maoe in this paper to find best possible constants. For any G the classical theorem of Turán gives

$$\alpha(G) \geqslant n/(t+1). \tag{8}$$

This implies (7) when $t < e^{99}$. The flow chart of Fig. 1 indicates the "construction" of an independent set in G. Basically, groupies are pulled out of G (unless they have very high degree in which case they are discarded) until t becomes small. At this point Turán's Theorem takes over. The formal proof is inductive. Set

$$\begin{aligned} g(n, t) &= 0.01\ (n/t) \ln t, \\ g(G) &= g(n, t), \qquad \text{where} \quad n = n(G),\ t = t(G). \end{aligned} \tag{9}$$

We prove

$$\alpha(G) \geqslant g(G) \tag{10}$$

by induction on $n(G)$. For $t < e^{99}$ we apply (8). Henceforth, assume $t \geqslant e^{99}$. Let P be a groupie of G. Set $n = n(G)$, $e = e(G)$, $t = t(G)$, $d = \deg(P)$.

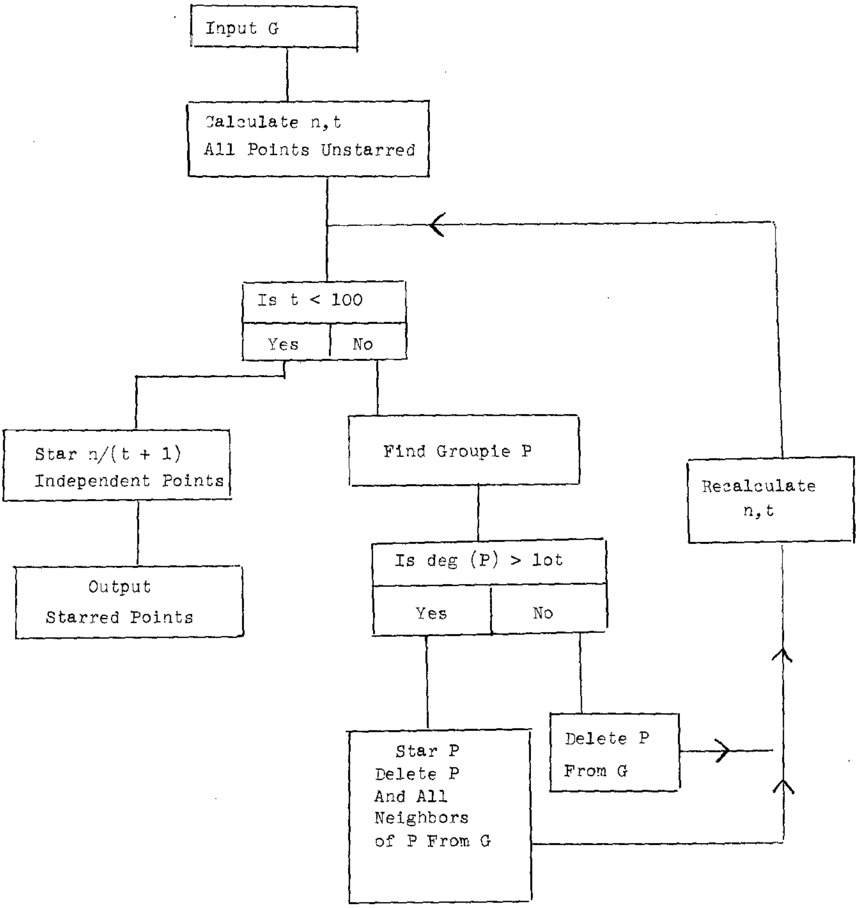

FIG. 1. Flow chart for Theorem 2.

Case 1. $d \geqslant 10t$. Set $G' = G - \{P\}$. Then

$$\begin{aligned} n' &= n(G') = n - 1, \\ t' &= t(G') \leqslant 2(e - 10t)/(n-1) = t(n-20)/(n-1). \end{aligned} \tag{11}$$

A simple (omitted) calculation gives $g(n', t') \geqslant g(n, t)$. Then

$$\begin{aligned} \alpha(G) \geqslant \alpha(G') &\geqslant g(G') \qquad \text{(by induction)} \\ &\geqslant g(G). \end{aligned} \tag{12}$$

Case 2. $d < 10t$. Delete P and all neighbors of P to form G'.

$$n' = n(G') = n - 1 - d. \tag{13}$$

Since G is trianglefree (the essential point) precisely $r(P)$ edges have been omitted.

$$\begin{aligned} e' &= e(G') = e - r(P) \leqslant e - td, \\ t' &= t(G') = 2e'/n' \leqslant t(n-2d)/(n-1-d). \end{aligned} \tag{14}$$

Now a calculation (see Remark 1 below) yields

$$g(n', t') > g(n, t) - 1. \tag{15}$$

As P is adjacent to no points of G'

$$\begin{aligned} \alpha(G) &\geqslant \alpha(G') + 1 \geqslant g(G') + 1 \qquad \text{(by induction)} \\ &\geqslant g(G), \end{aligned} \tag{16}$$

completing the proof.

Remark 1. Inequality (15), whose details we omit, is not coincidental. The deletion of the neighbors of a groupie P decreases the edge density. One almost has $\delta(G') \leqslant \delta(G)$ (Not quite because of the deletion of P itself.) Begin with a graph G with large $t(G)$. Suppose that each run through the loop of Fig. 1 (call that one time unit) produces a groupie of average degree. Suppose further that the edge density remains constant. When only half the points remained the average degree would be halved and points would be deleted from G at half the rate. The number of vertices remaining, as a function of "time," would decrease by exponential decay (versus the faster straight line decay). The process would continue until approximately $(n/t)\ln t$ independent points were found at which time $t(G)$ would become small. The constant 0.01 allows groupies of moderate degree to be selected.

The monotone behavior of $g(n, t)$ allows a more convenient form for Theorem 2.

THEOREM 2 (restatement). *Let G be a trianglefree graph with $n(G) \leqslant n$ and $1 \leqslant t(G) \leqslant t$. Then*

$$\alpha(G) \geqslant 0.01(n/t)\ln t$$

Remark 2. When $t < n^{1/3+o(1)}$ Theorem 2 is "best possible." A random graph G with n vertices and $nt/2$ edges has $\alpha(G) \lesssim (n/t)\ln t$ and $t^3/6$ triangles. Deleting all points lying on triangles gives a graph G' with $n' = n(G') \sim n$, $t' = t(G') \sim t$ and $\alpha(G') \leqslant \alpha(G) \lesssim c(n'/t')\ln t'$.

Remark 3. Erdös has asked if a result similar to Theorem 2 may be proven with the condition "G is trianglefree" replaced by "$\omega(G) < 4$." In particular, let $f_4(n, t)$ be the smallest value of $\alpha(G)$ over all G with $n(G) \leqslant n$, $t(G) \leqslant t$ and $\omega(G) < 4$. We cannot decide if

$$\lim_{t\to\infty} \lim_{n\to\infty} f_4(n, t)/(n/t) = +\infty \qquad (?).$$

THEOREM 3. $R(3, x) < 100\, x^2/\ln x$.

Proof. Let G be a trianglefree graph with n vertices and $\alpha(G) < x$. The neighbors of any point P form an independent set so $\deg(P) < x$. Hence $t(G) < x$. Theorem 2 gives

$$x > \alpha(G) > 0.01(n/x) \ln x \tag{17}$$

and therefore

$$n < 100x^2/\ln x. \tag{18}$$

Let $h = h(G)$ denote the number of triangles in G. We now extend Theorem 2 to the case $h(G)$ "small."

LEMMA 4. *Let G be a graph with $n = n(G)$, $e = e(G)$, $h = h(G)$, $t = t(G)$. Let $0 < p < 1$ with $pn \geqslant 3$. There exists an induced subgraph G' with parameters n', e', h', t' satisfying*

$$n' > np/2, \quad e' < 3ep^2, \quad h' < 3hp^3, \quad t' < 6tp. \tag{19}$$

Proof. We employ the probabilistic method. Let $\mathbf{G}'$ be the distribution on the subsets of G satisfying, for each $v \in \mathbf{G}$,

$$\operatorname{Prob}[v \in \mathbf{G}'] = p \tag{20}$$

and with these probabilities mutually independent. Then $n(\mathbf{G}')$ has Binomial distribution $B(n, p)$ with mean np and variance $np(1-p)$. Applying the classical inequality of Chebyschev

$$\begin{aligned} &\operatorname{Prob}[n(\mathbf{G}') < np/2] \\ &\quad < \operatorname{Prob}[|n(\mathbf{G}') - np| > (np(1-p))^{1/2}(np/(1-p))^{1/2}] \\ &\quad < (1-p)/np < 1/3. \end{aligned} \tag{21}$$

(In applications we can usually use the Law of Large Numbers to bound the probability in (21) by a small term.)

The random variable $e(\mathbf{G}')$ has expectation ep^2 so

$$\operatorname{Prob}[e(\mathbf{G}') \geqslant 3ep^2] < 1/3 \tag{22}$$

and similarly

$$\text{Prob}[h(\mathbf{G}') \geqslant 3hp^3] < 1/3. \tag{23}$$

Combining (21)–(23)

$$\text{Prob}[n(\mathbf{G}') > np/2 \text{ and } e(\mathbf{G}') < 3ep^2 \text{ and } h(\mathbf{G}') < 3hp^3] > 0. \tag{24}$$

Therefore there exists a specific G' with parameters n', e', h', t' satisfying $n' > np/2$, $e' < 3ep^2$, $h' < 3hp^3$ and, finally, $t' = 2e'/n' < 6tp$.

LEMMA 5. *Let $\varepsilon > 0$. Let G be a graph with $n = n(G)$, $t = t(G)$, $h = h(G)$ and $h < nt^{2-\varepsilon}$. If t is sufficiently large (dependent on ε)*

$$\alpha(G) > c'(n/t) \ln t, \tag{25}$$

where c' is a positive constant dependent on ε.

Proof. We show Lemma 5 for $c' = 0.01\, \varepsilon/48$ and $t > 12^{2/\varepsilon}$. By Lemma 4, with $p = t^{\varepsilon/4-1}$, there exists a subgraph G' with parameters n', e', t', h' satisfying

$$n' > np/2, \qquad e' < 3ep^2, \qquad h' < 3hp^3, \qquad t' < 6tp. \tag{26}$$

Delete one point from each triangle of G' to give a graph G'' with parameters n'', e'', t'', h''. As $h' < n'/2$

$$n'' > n' - h' > np/4, \qquad e'' < 3ep^2, \qquad h'' = 0 \tag{27}$$

and thus $t'' < 12tp$. We apply Theorem 2 to yield

$$\alpha(G) \geqslant \alpha(G'') \geqslant 0.01(n''/t'') \ln t'' > c'(n/t) \ln t. \tag{28}$$

Remark 4. The requirement $h < nt^{2-\varepsilon}$ in Lemma 5 is best possible in that the union of $n/(t+1)$ disjoint cliques of size $t+1$ (the Turán graph) has approximately $nt^2/6$ triangles and $\alpha \sim n/t$.

THEOREM 6. *For every $k \geqslant 2$*

$$R(k, x) \leqslant (5000)^k x^{k-1}/(\ln x)^{k-2} \tag{29}$$

for x sufficiently large (dependent on k).

Proof. Theorem 6 holds for $k = 2$ trivially and for $k = 3$ by Theorem 3. We prove (29) by induction on k. Fix ε satisfying

$$0.96(k-2)^{-1} < \varepsilon < (k-2)^{-1}. \tag{30}$$

(To prove (2) for some c_k one needs here only to assume ε is "sufficiently small.") Let G be a graph with

$$n = n(G) > (5000)^k x^{k-1}/(\ln x)^{k-2}. \tag{31}$$

Set $m = (5000)^{k-1} x^{k-2}/(\ln x)^{k-3}$. Assume $\omega(G) < k$. Every point P of G has $\deg(P) < R(k-1, x) \leqslant m$ so that $t(G) \leqslant m$.

Case 1. $h(G) < nm^{2-\epsilon}$. Applying Lemma 5, with $c' = 0.01\varepsilon/48$ (and using the lower bound of (30)),

$$\alpha(G) > c'(n/m) \ln m > x. \tag{32}$$

Case 2. $h(G) > nm^{2-\epsilon}$. Some point P lies on at least $m^{2-\epsilon}/3$ triangles. Let G' be the set of points adjacent to P. G' has at most m vertices and at least $m^{2-\epsilon}/3$ edges. Therefore G' contains a point Q of degree at least $2m^{1-\epsilon}/3$ in G'. Let G'' be the set of vertices of G' adjacent to Q. Then

$$n(G'') > 2m^{1-\epsilon}/3 > R(k-2, x) \tag{33}$$

since, by (30), ε is sufficiently small. If G'' contains a clique on $(k-2)$ points the addition of P and Q would give a clique of size k in G. As this was assumed not to hold G'' contains no such clique and hence G'', and therefore G, contains an independent set of x points.

In either case, $\alpha(G) \geqslant x$, completing the proof. A slight alteration of the proof of Theorem 6 yields the following result, whose proof we delete.

THEOREM 7. *Fix $\varepsilon > 0$. For every $k \geqslant 2$ there exists c_k so that for x sufficiently large either*

$$R(k, x) < c_k R(k-1, x) x/\ln x$$

or

$$R(k-1, x) < R(k-2, x) x^{\epsilon}.$$

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