Graphs with Projective Linear Stabilizers

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We investigate the structure of the free amalgamated product $P_1 \ast_{P_1 \cap P_2} P_2$ in which $P_1$ and $P_2$ are isomorphic projective linear groups and $P_1 \cap P_2$ is a one-point stabilizer in the natural action of $P_1$ on the points of a projective space of dimension $n \geq 2$. We apply the results to graphs admitting a vertex-transitive automorphism group such that the subgroup stabilizing a vertex is a projective linear group. In particular, graphs of girth 4 with projective linear stabilizers are characterized.

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1. INTRODUCTION

Let $H$ be a group generated by two isomorphic subgroups, $P_1$ and $P_2$, such that $PSL(n, q) \leq P_1 \leq PGL(n, q)$ for some $n \geq 3$ and $q$ a power of some prime $p$, and $B = P_1 \cap P_2$ is a one-point stabilizer in the natural action of $P_1$ on the points of $PG(n-1, q)$. The group $H$ is called a completion of the amalgam $A = (P_1, P_2, B)$, and is a homomorphic image of the universal completion $Q = P_1 \ast_{B} P_2$, the free product of $P_1$ and $P_2$ with subgroup $B$ amalgamated. We obtain results about the amalgamated product $Q$ and then apply our results to 2-transitive graphs with projective linear stabilizers.

THEOREM 1.1. Let $Q = P_1 \ast_{B} P_2$, with $P_1 \cong P_2$, $PSL(n, q) \leq P_1 \leq PGL(n, q)$ for some $n \geq 3$ and $q$ a power of some prime $p$, and $B = P_1 \cap P_2$ is a one-point stabilizer in the natural action of $P_1$ on the points of $PG(n-1, q)$. Then one of the following holds:

(a) There exists an isomorphism $f : P_1 \rightarrow P_2$ trivial on $B$, and if $a \in P_1$, with $a \neq P_2$, but $a^2 \in B$, and $b = a^2$, then $Q$ is a semi-direct product $(ab^{-1})^B : P_1$, where the normal subgroup $(ab^{-1})^B$ is a free product on $n-1$ generators, $v = |P_1 : B|$.

(b) $n = 3$, $q = 4$ and $P_1 \nmid PGL(3, 4)$.

(c) $n = 4$ and $q = 2$.

Now let $\Gamma$ be a (not necessarily finite) connected, undirected graph and $G$ a subgroup of $Aut(\Gamma)$ acting transitively on $V(\Gamma)$, the vertex set of $\Gamma$. For each vertex $x$, we denote by $\Gamma(x)$ the set of vertices adjacent to $x$ and by $G^{\Gamma(x)}_x$ the permutation group induced by the stabilizer $G_x$ of $x$ in $G$. The graph $\Gamma$ is called $(G, s)$-transitive, or $x$-transitive, if $G$ acts transitively on the set of $s$-paths but not on the set of $(s + 1)$-paths in $\Gamma$; an $s$-path being a sequence $(x_0, x_1, \ldots, x_s)$ of $s + 1$ vertices $x_i$ such that $x_{i-1} \in \Gamma(x_i)$ if $1 \leq i \leq s$ and $x_{i-2} \neq x_i$ if $2 \leq i \leq s$, and is an $s$-circuit if $x_0 = x_s$ and $s > 0$. If $\Gamma$ is $(G, 2)$-transitive, then it is easy to show that for each vertex $x$, $G^{\Gamma(x)}_x$ is a 2-transitive permutation group. Suppose $PSL(n, q) \leq G^{\Gamma(x)}_x \leq PGL(n, q)$ for $n \geq 3$ and $q$ a power of some prime $p$, in its natural representation on the points of $PG(n-1, q)$. If $G_x$ acts faithfully on $\Gamma(x)$, and $y \in \Gamma(x)$, then $(G_x, G_y)$ is a homomorphic image of $Q = P_1 \ast_{B} P_2$, with $P_1 \cong G_x$, $P_2 \cong G_y$, and $B \cong G_x \cap G_y$. The group $(G_x, G_y)$ is an edge-transitive subgroup of $G$ of index 2 if $\Gamma$ is bipartite, and generates $G$ otherwise.

THEOREM 1.2. Let $\Gamma$ be a connected, undirected graph and $G$ a vertex-transitive subgroup of $Aut(\Gamma)$ such that for all vertices $x$ of $\Gamma$, $PSL(n, q) \leq G^{\Gamma(x)}_x \leq PGL(n, q)$ for some $n \geq 3$...
and $q$ a power of some prime $p$, in its natural representation on the points of $PG(n-1, q)$. Suppose $G_x$ acts faithfully on $G_0$. Let $(w, x, y)$ be a 2-path in $\Gamma$. Let $Q = (ab^{-1})^B : P_1$ be given as in Theorem 1.1(a), with $P_1 \cong G_x$, $P_2 \cong G_y$ and $B \cong G_0 \cap G_y$. Then $\Gamma$ is $(G, 2)$-transitive and either $G(w, x, y)$ is fixed-point free on $\Gamma(\gamma) - \{x\}$ and $n = 4, q = 2$, or $G(w, x, y)$ fixes some vertex $z \in \Gamma(y) - \{x\}$, with $z$ unique if $q \neq 2$, and one of the following holds:

(a) There exists an isomorphism $f$ from $G_x$ onto $G_0$ trivial on $G_0 \cap G_y$ and $a_1$ is an element in $G_0$ switching $w$ and $y$, and $b_1 = a_1^{-1} G(w, x, y) = 1$ and $G(w, x, y)$ fixes pointwise the path $C = (\ldots, x, w, x, y, x^k, y^k, x, y^k, \ldots)$, where $h = a_1 b_1^{-1}$. If $C$ is finite, then $C$ is a circuit of length $2m$ for some $m \geq 2$, and $G(w, x, y)$ is a homomorphic image of $Y_{n, q, m} = \langle Q | (ab^{-1})^m = 1 \rangle$ or $W_m = \langle Q | P_1 \cong PSL(3, 2), (ab^{-1})^m = c \rangle$, where $c$ is the unique nontrivial element in $P_1 \cap P_2 \cap P_2$ centralized by $a$.

(b) $n = 3, q = 4$ and $G_x \not\cong PGL(3, 4)$.

As an application of our results, we consider the nontrivial case of minimal girth — girth 4. In [2], Cameron and Praeger obtained the following result on 2-transitive graphs of girth 4:

**Theorem 1.3.** Let $\Gamma$ be a finite, connected, undirected $(G, 2)$-transitive graph of girth 4, such that for all vertices $x$ of $\Gamma$, $PSL(n, q) \leq G_{x}^{(1)} \leq PGL(n, q)$ for some $n \geq 3$ and some prime power $q$, in its natural representation on the points of $PG(n - 1, q)$. Then one of the following holds:

(a) $\Gamma$ is the incidence graph of points and hyperplanes in $PG(n, q)$ and either $PSL(n + 1, q) \leq G \leq Aut(PGL(n + 1, q))$ or $G \cong S_7$ and $(n, q) = (3, 2)$.

(b) $PGL(2n, q) \leq G \leq PGL(2n, q)$ for $n \geq 4$ and the vertices of $\Gamma$ can be identified with the maximal totally singular subspaces with respect to a quadratic form of Witt index $n$ on a $2n$-dimensional vector space over $GF(q)$, two vertices being adjacent whenever the intersection of the corresponding subspaces has codimension 1 in each.

(c) $\Gamma$ is a complete bipartite graph $K_{m, m}$ with a matching deleted, where $m = 1 + |PG(n - 1, q)|$, and either $q = 2$ and $G \cong AGL(n, 2) \times Z_2$ or $(n, q) = (3, 4)$ and $M_{22} < G \leq Aut(M_{22}) \times Z_2$.

(d) $G_x$ acts faithfully on $G_0(x)$, and if $x$ and $z$ are at distance 2, then $k = |\Gamma(x) \cap \Gamma(z)| = 2, 3, 4$ or 6. Moreover, if $n > 2$, then either $k = q = 3$ or 4 and $\Gamma(x) \cap \Gamma(z)$ is a line of $\Gamma(y)$ with one point removed, or $k = q + 1 = 3, 4, 6$, and $\Gamma(x) \cap \Gamma(z)$ is a line of $\Gamma(z)$.

Applying our results to graphs satisfying part (d) of Theorem 1.3, we have:

**Theorem 1.4.** Let $\Gamma$ be as in Theorem 1.3(d). Let $Y_{n, q, m}$ and $W_m$ be the groups given in Theorem 1.2(a). Let $(x, y)$ be an edge in $\Gamma$, $v = |\Gamma(x)|$ and $H = \langle G_x, G_y \rangle$. Then one of the following holds:

(a) $H$ is isomorphic to a quotient of $Y_{n, q, m} \cong Z_{2}^{v-1} : G_x$, $H \leq G \leq H_2$, and $\Gamma$ is a quotient of the $v$-cube.

(b) $H \cong U_2 = \cong U_3(3), G \cong G_2(2)$ and $|\Gamma| = 72$.

(c) $G \cong U_3(3).2_2$ or $U_3(3).2^{2}^{2}_2$, $|\Gamma| = 324$ and $\Gamma$ is uniquely determined.

As usual, for any group $G$, $O_p(G)$ denotes the largest normal $p$-subgroup of $G$. $Z^k_m$ denotes the direct product of $k$ copies of the cyclic group $Z_m$ of order $m$, and $K = N : M$ denotes
the semi-direct product of the groups $M$ and $N$, with $N \triangleleft K$. If $x_1, \ldots, x_m$ are vertices of a graph $\Gamma$, we set $G(x_1, \ldots, x_m) = G_{x_1} \cap \cdots \cap G_{x_m}$. We will often use the notation $G_{xy}$ in place of $G(x, y)$. If $S$ is a set of vertices of $\Gamma$, we denote by $G_S$ and $G_{[S]}$ the subgroups of $G$ fixing $S$ setwise and pointwise, respectively. For each pair of vertices $x, y$, $d(x, y)$ denotes the distance between $x$ and $y$, i.e., the length of a shortest path from $x$ to $y$. For each positive integer $i$, $\Gamma_i(x)$ is the set of vertices in $\Gamma$ at distance $i$ from $x$; we will use $\Gamma(x)$ in place of $\Gamma_1(x)$. Throughout, we let $v = |P_1 : B| = |\Gamma(x)|$.

The following preliminary lemmas describe some properties of the projective groups; we leave the proofs to the reader.

**Lemma 1.1.** Let $H$ be a subgroup of $PGL(n, q)$ (with $n \geq 3$) containing $PSL(n, q)$ considered as acting in the usual fashion on $PG(n - 1, q)$, $q$ a power of a prime $p$. Let $X$ be the set of points of $PG(n - 1, q)$, $x \in X$ arbitrary and $Y$ the set of lines passing through $x$. If $1 \neq M \triangleleft K \triangleleft H_x$, where $H_x$ denotes the stabilizer of $x$ in $H$, and $K'$ is any conjugate of $K$ in $H$ such that $K \triangleleft K'$, then either $K' \geq PSL(n - 1, q)$ or $|K'| = q + 1$ and $(n, q) = (3, 2)$ or $(3, 3), M \geq O_p(H_x)$. $|O_p(H_x)| = q^{n-1}$, $O_p(H_x)^G = 1$ and $O_p(H_x)$ acts transitively on $g - \{x\}$ for every $g \in Y$. The order of the largest subgroup of $H_x$ acting trivially on $Y$ divides $(q - 1)q^{n-1}$.

**Lemma 1.2.** Let $F = \{x_1, \ldots, x_n\}$ be a frame of $PG(n - 1, q)$; i.e., a set of $n$ points such that $PG(n - 1, q) = \{x_1, \ldots, x_n\}$. Let $H$ be a group of automorphisms of $PG(n - 1, q)$ such that $PSL(n, q) \leq H \leq PGL(n, q)$. Let $D$ be the subgroup of $H$ fixing $F$ pointwise. For each $x_j \in F - \{x_1\}$, let $\ell_j$ be the line in $PG(n - 1, q)$ containing $x_1$ and $x_j$.

(a) If $K$ is a subgroup of $O_p(H_{x_1})$ normalized by $D$, then $K$ acts either trivially or transitively on $\ell_j - \{x_1\}$.

(b) If $h$ is an element in $D$ such that $h$ fixes pointwise some $\ell_j$, then there exists $g \in O_p(H_{x_1})$ such that $[g, h] = 1$ and $g^{\ell_j} \neq 1$.

(c) Let $M = \{x_1, \ldots, x_{n-1}\}$. If $K$ is a nontrivial subgroup of $O_p(H_{x_1}) \cap H_{x_n}$ and $K \triangleleft H_{x_1} \cap H_{x_n} \cap H_M$, then $|K| = q^{n-2}$.

2. Proof of Theorem 1.1

Let $Q = P_1 \ast_p P_2$ satisfy the hypothesis of Theorem 1.1. Let $\Gamma$ be the graph whose vertices are the right-cosets of $P_1$ and $P_2$ in $Q$, where two vertices $P_1 g$ and $P_2 k$ are incident whenever $P_1 g \cap P_2 k \neq \emptyset$. Then $\Gamma$ is connected and $Q$ acts edge-transitively and faithfully, by right multiplication, on $\Gamma$. Furthermore, the stabilizer in $Q$ of the vertex $P_1 g$ is $P_1^g$, $|\Gamma(P_1)| = |P_1 : B|$, and the action of $P_1$ on $\Gamma(P_1)$ is equivalent to the action of $P_1$ on the set of right-cosets of $B$ in $P_1$ [5]. For ease of notation, we use lower case letters to represent the vertices of $\Gamma$. For each vertex $x$ in $\Gamma$, we denote by $\hat{Q}_x$ the largest subgroup of $Q_x$ such that $\hat{Q}_x \leq PGL(n, q)$.

For each vertex $x$ of $\Gamma$, we denote by $P(x)$ the projective space of dimension $n - 1$ with point set $\Gamma(x)$ induced by $\hat{Q}_x$. For each edge $(x, y)$, let $P(x, y)$ be the projective space of dimension $n - 2$ whose subspaces of dimension $m$ are the subspaces of $P(x)$ of dimension $m + 1$ containing $y$ ($0 \leq m \leq n - 2$). In particular, we denote by $[x : y]$ the set of lines of $P(x)$ containing $y$. For each element $g$ of $Q_x$, let $g^{[x : y]}$ be the permutation that $g$ induces on $[x : y]$.

Following [12], we define for each edge $(x, y)$ a map $m_{xy} : Q_{xy}^{[x : y]} \rightarrow Q_{xy}^{[y : x]}$ as follows. Given an arbitrary element $c \in Q_{xy}^{[x : y]}$, pick an element $g \in Q_{xy}$ with $g^{[x : y]} = c$ and set $m_{xy}(c) = g^{[y : x]}$. We claim that $m_{xy}$ is well defined. Suppose not. Then $Q_{xy}$ contains elements $d$ such that $d^{[x : y]} = 1$ but $d^{[y : x]} \neq 1$. Let $K = \{d \in G_{xy} : d^{[x : y]} = 1\}$. Then $|K|^{\Gamma(x)}$ divides
(q − 1)q^{n-1} and q + 1 divides |K^{[y:x]}| by Lemma 1.1. With this contradiction, we conclude that \( m_{xy} \) is well defined. Since \( m_{xy} \) is clearly an isomorphism and \( Q_{xy}^{[y:x]} \cong PSL(n−1, q) \), \( m_{xy} \) is induced by a map \( \varphi_{xy} : P(x, y) \to P(y, x) \) which is either a correlation or a collineation.

**Lemma 2.1.** Let \( x \) be a vertex in \( \Gamma \) and \( y \in \Gamma(x) \). Then \( \hat{Q}_x \cap Q_y < \hat{Q}_y \).

**Proof.** Let \( h \in \hat{Q}_x \cap Q_y \). Let \( m_{xy} \) be the isomorphism from \( Q_{xy}^{[y:x]} \) to \( Q_{xy}^{[y:x]} \) defined above. Then \( m_{xy}(h^{[y:x]}) = h^{[y:x]} \in (\hat{Q}_y \cap Q)_y^{[y:x]} \). Hence, there exists \( g \in \hat{Q}_y \cap Q_y \) such that \( g^{[y:x]} = h^{[y:x]} \), so \( h_g^{-1} \) acts trivially on \( y \). Therefore \( h_g^{-1} \) is a central collineation of \( P(y) \), so \( h_g^{-1} \in \hat{Q}_y \). Thus \( h \in \hat{Q}_y \). (Recall that \( PGL(n, q) \) is generated by all central collineations of \( PG(n−1, q) \) and a collineation is central if it fixes a point \( v \) and every subspace containing \( v \).) \( \square \)

By Lemma 1.1, there is no ambiguity as to the meaning of \( \hat{Q}(x_0, x_1, \ldots, x_m) \) for any \( m \)-path \( (x_0, x_1, \ldots, x_m) \).

For each edge \((x, y) \in \Gamma \), we claim the map \( \varphi_{xy} : P(x, y) \to P(y, x) \) is a collineation. Suppose \( \varphi_{xy} \) is a correlation. We may assume \( n \geq 4 \) since the correlation and collineation cases coincide for \( n = 3 \). Let \( F = \langle w_1, w_2, \ldots, w_n \rangle \) be a frame of \( P(y) \), so \( P(y) = \langle w_1, w_2, \ldots, w_n \rangle \). Let \( D \) be the subgroup of \( \hat{Q}_y \) fixing \( F \) pointwise. For \( 1 \leq i \leq n − 1 \), let \( S_i = F − \{ w_i \} \) and \( H_i = (S_i) \). \( D \) fixes each \( H_i \), which is a hyperplane of \( P(y, w_n) \), so \( D \) fixes each line \( L_i = \varphi_{yw_n}(H_i) \in [w_n : y] \). Since \( |D| \) divides \((q − 1)q^{n−1}\), \( D \) fixes some \( u_i \in L_i − \{ y \} \) for \( 1 \leq i \leq n − 1 \). Let \( F' = \langle y, u_1, u_2, \ldots, u_{n-1} \rangle \). Then \( P(w_n) = \langle y, u_1, u_2, \ldots, u_n \rangle \). By Lemma 2.1, \( D < \hat{Q}_{w_n} \), so \( D \) is the subgroup of \( \hat{Q}_{w_n} \) fixing \( F' \) pointwise. Let \( K = O_F(Q_{w_n}) \cap Q_{w_1} \). Since \( K < Q(w_1, y, w_n) \) and \( Q(w_1, y, w_n) > D \), by Lemma 1.2(a), \( K \) acts trivially on at most one of the \( L_i \)’s because \( |K| = q^{n-2} \). Let \( \ell \) be the line in \( P(y) \) containing \( w_1 \) and \( w_n \). Since \( Q(w_1, y, w_n) \) acts transitively on the set of hyperplanes of \( P(y, w_n) \) containing \( \ell \), \( Q(w_1, y, w_n) \) acts transitively on the set of lines in \([w_n : y]\) that are contained in the hyperplane \( M = \varphi_{yw_n}(\ell) \) in \( P(w_n, y) \). Thus, if \( K \) acts trivially on \( \ell \) in \( M \), then \( K \) acts trivially on every \( L_i \) in \( M \). Note that for \( i \neq 1, \ell \in H_i \), so \( L_i \subset M \). Since \( n \geq 4 \), \( M \) contains at least \( 2L_i \)’s. Therefore \( K \) acts nontrivially on every \( L_i \) in \( M \). Since \( Q(w_1, y, w_n) \) acts transitively on the set of hyperplanes of \( P(y, w_n) \) not containing \( \ell \), \( Q(w_1, y, w_n) \) acts transitively on the set of lines in \([w_n : y]\) not contained in \( M \). Since \( L_1 \not\subset M \), \( K \) fixes \( L_1 \) pointwise, \( K \) fixes pointwise \( \Gamma(w_n) = M \). Let \( u \) be a point in \( M \). For any \( v \not\in M \), the line \( A \) containing \( u \) and \( w \) has \( q \) points in \( \Gamma(w_n) \), otherwise \( A \subset M \). Therefore \( K \) fixes \( u \). Hence \( K \) fixes \( \Gamma(w_n) \) pointwise, so \( K = 1 \). But \( |K| = q^{n-2} \). Therefore \( K \) acts nontrivially and hence, by Lemma 1.2(a), transitively on \( L_1 − \{ y \} \). Thus, \( |K \cap Q_{u_1}| = q^{n-3} \neq 1 \) since \( n \geq 4 \). Since \( K \cap Q_{u_1} \not\subset \varphi_{y, w_n}(u_1) \cup Q_{\ell} = \varphi_{y, w_n}(u_1) \subset Q_M \), \( K \cap Q_{u_1} \) = \( q^{n-2} \) by Lemma 1.2(c). Hence, we conclude that \( \varphi_{xy} \) is a collineation for each edge \((x, y) \) in \( \Gamma \), and for every \( \ell \in [x : y] \), \( \varphi_{xy}(\ell) \subseteq [z : y] \). Then \( Q(x, z) = \varphi_{xy}(Q(z, y) \cap Q_L) \) acts transitively on \( \Gamma(z) − L \) and \( Q(x, y, z) \) contains an element inducing a \((q − 1)\)-cycle on \( L \).

**Lemma 2.2.** Let \((x, y, z)\) be a 2-path in \( \Gamma \). Let \( \ell \in [y : z] \), with \( x \in \ell \), and let \( L = \varphi_{yz}(\ell) \subseteq [z : y] \). Then \( Q(x, y, z) = \varphi_{xy}(Q(x, y) \cap Q_L) \) acts transitively on \( \Gamma(z) − L \) and \( Q(x, y, z) \) contains an element inducing a \((q − 1)\)-cycle on \( L \).

**Proof.** \( Q(x, y, z) \) acts transitively on \([z : y] − L \) since \( Q(x, y, z)^{[yz]} = (Q_{yz} \cap Q_L)^{[yz]} \cong (Q_{yz} \cap Q_L)^{[yz]} \). Thus, it suffices now to show \( Q(x, y, z) \cap Q_L \) acts transitively on \( v − \{ y \} \) for some \( v \in [z : y] − L \). Let \( F = \langle w_1, w_2, w_3, \ldots, w_n \rangle \) be a frame of \( P(y) \), with \( w_1 = z \) and \( w_2 = x \). Let \( D \) be the subgroup of \( \hat{Q}_y \) fixing \( F \) pointwise. For \( 2 \leq i \leq n \), let \( \ell_i \in [y : z] \) with \( w_i = \ell_i \). Let \( L_i = \varphi_{y, z}(\ell_i) \) for \( 2 \leq i \leq n \). Since \( D \) fixes \( L_i \) and \( |D| \) divides \((q − 1)q^{n-1} \), \( D \) fixes some \( u_i \in L_i − \{ y \} \). Let \( F' = \langle y, u_2, u_3, \ldots, u_n \rangle \).
Then $F'$ is a frame of $P(z)$. By Lemma 2.1, $D < \tilde{Q}$, so $D$ is the subgroup of $\tilde{Q}$, fixing $F'$ pointwise. Let $K = O_p(Q_{xy}) \cap Q_z$. Since $|K| = q^{n-2}$, $K$ can fix pointwise at most one $L_i$. Therefore $K$ acts nontrivially on at least $n - 2$ $L_i$s. For each $L_i$ on which $K$ acts nontrivially, $D$ induces a $(q-1)$-cycle on $L_i - \{y, u_i\}$. Therefore $Q(x, y, z) \cap Q_{L_i} \geq \langle K, D \rangle$ acts transitively on $L_i - \{y\}$ for $L_i \neq L_2 = L$. Hence $Q(x, y, z)$ acts transitively on $\Gamma(z) - L$ and $Q(x, y, z) > D$ contains an element inducing a $(q-1)$-cycle on $L$. □

From the proof of Lemma 2.2, we have the following result:

**LEMMA 3.1.** Let $F = \{w_1, w_2, w_3, \ldots, w_n\}$ be a frame of $P(y)$ and let $D$ be the subgroup of $\tilde{Q}_y$ fixing $F$ pointwise. For $2 \leq i \leq n$, let $\ell_i \in [y : w_1]$ with $w_i \in \ell_i$. Let $L_i = \varphi_{\psi_{y/1}}(\ell_i)$ for $2 \leq i \leq n$. Then $D$ fixes each $L_i$ and some $u_i \in L_i - \{y\}$, $D$ induces a $(q-1)$-cycle on $L_i - \{y, u_i\}$ for $2 \leq i \leq n$, and $F' = \{y, u_2, u_3, \ldots, u_n\}$ is a frame of $P(w_1)$.

Let $(w, x, y)$ be a 2-path. Let $\ell$ be the line in $P(x)$ containing $w$ and $y$, and let $L = \varphi_{xy}(\ell)$. By Lemma 2.2, $Q(w, x, y)$ is either fixed-point free on $\Gamma(y) - \{x\}$ or it fixes some $z \in L - \{x\}$, with $z$ unique if $q \neq 2$. We now separate the proof of Theorem 1.1 into two cases: the case $Q(w, x, y)$ is fixed-point free on $\Gamma(y) - \{x\}$ yields $n = 4$ and $q = 2$, while the case $Q(w, x, y)$ is not fixed-point free gives the remaining results of Theorem 1.1.

### 3. PROOF OF THEOREM 1.1: $Q(w, x, y)$ NOT FIXED-POINT FREE

Let $(w, x, y)$ be a 2-path in $\Gamma$. Let $\ell$ be the line in $P(x)$ containing $w$ and $y$, and $L = \varphi_{xy}(\ell) \in [y : x]$. Throughout this section, we assume $Q(w, x, y)$ fixes some $z \in L - \{x\}$. Suppose $Q(w, x, y)$ fixes a unique point in $L - \{x\}$, so $q \neq 2$. We can define a bijection $f_{xy}$ from $\Gamma(x)$ to $\Gamma(y)$ as follows. For each $u \in \Gamma(x)$, $v = f_{xy}(u)$ is the unique vertex in $\Gamma(y)$ such that $Q(u, x, y) = Q(x, y, v)$. (Note that $f_{xy}(y) = x$.) Suppose $q = 2$. We define a map $f_{xy}$ from $\Gamma(x)$ to $\Gamma(y)$ as follows. For each $t \in \Gamma(x) - \{y\}$, let $\ell_t$ be the line in $P(x)$ containing $t$ and $y$. Then $Q(t, x, y)$ fixes pointwise $\varphi_{xy}(\ell_t)$. If $\varphi_{xy}(\ell_t) - \{x\} = \{u_1, u_2\}$, then we have two choices for $f_{xy}(t)$, namely, $f_{xy}(t) = u_1$ or $f_{xy}(t) = u_2$. Suppose we choose $f_{xy}(t) = u_1$. Let $g_{xy}$ be another map from $\Gamma(x)$ to $\Gamma(y)$ and define $g_{xy}(t) = u_2$. For any $v \in \Gamma(y) - \{t, y\}$, there exists $h \in Q_{xy}$ such that $t^h = v$. We define $f_{xy}(v) = u_1^h = f_{xy}(t)^h$ and $g_{xy}(v) = u_2^h = g_{xy}(t)^h$. Note that $u_1^h \neq u_1$, otherwise $h$ is an element in $Q(x, y, u_1)$ acting nontrivially on $\ell_t$. Furthermore, this definition does not depend on the element $h$ since for any element $k \in Q_{xy}$ such that $t^k = v$, we have $u_1^k = u_1^h$ and $u_2^k = u_2^h$, otherwise $hk^{-1}$ is an element in $Q(t, x, y)$ acting nontrivially on $\varphi_{xy}(\ell_t)$. Lastly, we define $f_{xy}(y) = x$ and $g_{xy}(y) = x$.

**LEMMA 3.1.** Let $f_{xy}$ and $g_{xy}$ be defined as above. Then

(a) $f_{xy}$ is a collineation from $P(x)$ to $P(y)$, or $q = 2$, or $(n, q) = (3, 4)$ and $Q \cong PGL(3, 4)$. If $f_{xy}$ is not a collineation from $P(x)$ to $P(y)$ and $\ell$ is a line of $P(x)$ not containing $y$, then $f_{xy}(\ell) \cup \{x\}$ is a hyperoval of $P(y)$ — that is, a set of $q + 2$ points no three of which are collinear.

(b) If $f_{xy}$ is not a collineation and $q = 2$, then $g_{xy}$ is a collineation. Hence, a collineation map can always be defined in the case $q = 2$.

(c) If $f_{xy}$ is a collineation, then the map $k \mapsto k^{f_{xy}}$ is an isomorphism from $Q_x$ onto $Q_y$, where $d = f_{xy}(u)^d = f_{xy}(u^d)$ for $u \in \Gamma(x)$. Furthermore, $k^{f_{xy}} = k$ for $k \in Q_{xy}$, so the map $k \mapsto k^{f_{xy}}$ is the identity on $Q_{xy}$.
Proof. (a) Let $\ell$ be an arbitrary line in $P(x)$ containing $y$. Since $Q_{xy} \cap Q_{\{\ell\}}$ acts transitively on $\Gamma(x) - \ell$, $Q_{xy} \cap Q_{\{\ell\}} = Q_{xy} \cap Q_{f_{xy}(\ell)}$ acts transitively on $\Gamma(y) - f_{xy}(\ell)$. Since $Q_{xy} \cap Q_{\{\ell\}}$ fixes a unique line in $P(y)$, namely $\varphi_{xy}(\ell)$, $f_{xy}(\ell) = \varphi_{xy}(\ell)$ is a line in $P(y)$. Now let $\ell$ be a line in $P(x)$ not containing $y$. Suppose $f_{xy}(\ell)$ is not a line of $P(y)$. Then there exist $u_1, u_2, u_3 \in \ell$ such that $u_i = f_{xy}(u_i)$, $i = 1, 2, 3$, are not collinear in $P(y)$. Note that $\{\widetilde{u}_1, \widetilde{u}_2, \widetilde{u}_3, x\}$ is a quadrangle in $P(y)$ — i.e., a set of four points no three of which are collinear. Let $V = (u_1, u_2, u_3, y)$ be the smallest subspace of $P(x)$ containing $u_1, u_2, u_3$ and $y$. Then $V$ has dimension 2. Since $Q_{xy} \cap Q_{\{V\}}$ acts transitively on $\Gamma(x) - V$, $Q_{xy} \cap Q_{\{V\}}$ acts transitively on $\Gamma(y) - f_{xy}(V)$. Since $Q_{xy} \cap Q_{\{V\}}$ fixes a unique plane in $P(y)$, namely $\varphi_{xy}(V)$, $f_{xy}(V) = \varphi_{xy}(V)$ is a subspace of $P(y)$ of dimension 2. Let $H = Q_{xy} \cap Q_{\{\ell\}} = Q_{xy} \cap Q_{f_{xy}(\ell)}$. Then $H$ fixes pointwise the quadrangle $\{\widetilde{u}_1, \widetilde{u}_2, \widetilde{u}_3, x\}$ in $f_{xy}(V)$. Therefore $Hf_{xy}(V) = 1$, and so $H = Q_{xy} \cap Q_{f_{xy}(V)} = Q_{xy} \cap Q_{\{V\}}$. But $H$ induces on $V$ a group of order at least $(q - 1)(3, q - 1)$ since $u_1, u_2$ and $u_3$ are collinear. We conclude that $q = 2$ or $(n, q) = (3, 4)$ and $Q_x \not\leq PGL(3, 4)$. We show $f_{xy}(\ell) \cup \{x\}$ is a hyperoval. Since $Q_{xy} \cap Q_{\ell}$ induces on $\ell$ the group $S_n$ for $q = 2$, and the group $PSL(2, 4) \cong A_5$ for $(n, q) = (3, 4)$, $Q_{xy} \cap Q_{\ell}$ acts 3-transitively on $\ell$. Hence $Q_{xy} \cap Q_{\ell}$ acts 3-transitively on $f_{xy}(\ell)$. Since $u_1, u_2$ and $u_3$ are three noncollinear points in $f_{xy}(\ell)$, it follows that $f_{xy}(\ell)$ is an oval of $P(x)$ — that is, a set of $q + 1$ points no three of which are collinear. Therefore $f_{xy}(\ell) \cup \{x\}$ is a hyperoval of $P(y)$.

(b) It follows from the proof of part (a) that $g_{xy}$ maps lines of $P(x)$ containing $y$ to lines of $P(y)$ containing $x$. By part (a), if $f_{xy}$ is not a collineation and $\ell$ is a line of $P(x)$ not containing $y$, then $f_{xy}(\ell) \cup \{x\}$ is a hyperoval of $P(y)$. Note that if $V$ is the smallest subspace of $P(x)$ containing $\ell \cup \{y\}$, then $V$ has dimension 2 and $f_{xy}(V) = g_{xy}(V)$ is a subspace of $P(y)$ of dimension 2. Since $q = 2$, the complement of the hyperoval $f_{xy}(\ell) \cup \{x\}$ in the plane $g_{xy}(V)$ is a line, and this line is precisely $g_{xy}(\ell)$.

(c) By definition, the map $f_{xy}$ commutes with $k$ for $k \in Q_{xy}$, so $k^{f_{xy}} = k$ for $k \in Q_{xy}$. Let $Q_x$ denote the subgroup in $Q_x$ isomorphic to $PSL(n, q)$. Since $f_{xy}$ is a collineation, $Q_{xy}^{f_{xy}}$ is a collineation group on $P(y)$ and since $Q_{xy}^{f_{xy}} \cong PSL(n, q)$, $Q_{xy}^{f_{xy}} = \tilde{Q}_x$. Now let $a$ be an element in $Q_x$ not fixing $y$. Then $b = a^{f_{xy}}$ is an element in $Q_x$ not fixing $x$. Since $Q_x = \langle Q_{xy}, a \rangle$, $Q_x^{f_{xy}} = \langle Q_{xy}, b \rangle = Q_y$.

For the remainder of this section, we let the vertex $x$ represent the coset $P_1$ in $Q$ and let $y \in \Gamma(x)$ represent the coset $P_2$, so $Q_x = P_1$, $Q_y = P_2$ and $Q_{xy} = B$. By Lemma 3.1, $f_{xy}$ is a collineation from $P(x)$ to $P(y)$ or $n = 3, q = 4$ and $Q_x \not\leq PGL(3, 4)$. We assume, for the remainder of this section, that for each edge $(u, v)$ in $\Gamma$, $f_{uv}$ is a collineation from $P(u)$ to $P(v)$, and hence $f_{uv}$ induces an isomorphism from $Q_u$ to $Q_v$ which is trivial on $Q_{uv}$. In particular, we let $f$ denote the isomorphism from $Q_x$ to $Q_y$ induced by $f_{xy}$.

Lemma 3.2. Let $(w, x, y)$ be a 2-path in $\Gamma$ and $f_{xy}$ a collineation from $P(x)$ to $P(y)$. Let $a, a_1$ be elements in $Q_x$ switching $w$ and $y$. Let $b = a_1^{f}$, $b_1 = a_1^{f}$, and $h = ab^{-1}$. Then $[h, Q(w, x, y)] = 1$ and $Q(w, x, y)$ fixes pointwise the path $C = (\ldots, x^{a}, w, x, w^{b}, x^{h}, w^{b}, x^{h^{2}}, \ldots)$, where $w^{b} = y$. Furthermore, $h = a_1b_1^{-1}$, so $h$ is defined independent of the choice of $a$.

Proof. For $k \in Q(w, x, y)$, $k^{a} = (k^{a})^{f} = k^{b}$ since $f$ is the identity on $Q_{xy}$ by Lemma 3.1(c). Therefore $k^{ab_{1}} = k$, so $[h, Q(w, x, y)] = 1$. Since $Q(w, x, y)^{a} = Q(w, x, y), Q(w, x, y)$ fixes $C$ pointwise. Now $a^{-1}a_1 \in Q(w, x, y)$, so $a_1 = ag$ for some $g \in Q(w, x, y)$. Therefore $b_1 = a_1^{f} = bg$, so $a_1b_1^{-1} = agg^{-1}h^{-1} = ab^{-1} = h$. \[\square\]
Let \( W \) be a group freely generated by involutions \( d_1, \ldots, d_n \). \( W \) admits the symmetric group of degree \( v \), \( S_v \), as a group of automorphisms defined by \( d_i^v = d_i \) for \( g \in S_v \). For \( 2 \leq i \leq v \), let \( z_i = d_i d_1 \), and \( T = \langle z_2, z_3, \ldots, z_v \rangle \). Since \( z_i^{d_1} = z_i^{-1} \), \( z_i d_1 = d_i \) and \( T \cap \langle d_1 \rangle = 1 \), \( W = T : \langle d_1 \rangle \). We define an action of \( P_1 \) on \( W \) as follows: Let \( X \) be the set of right-cosets of \( B \) in \( P_1 \) and identify the elements of \( X \) with the elements \( d_1, d_2, \ldots, d_n \). Then there is an action of \( P_1 \) on \( \{ d_1, d_2, \ldots, d_n \} \) induced by the action of \( P_1 \) on \( X \) by right-multiplication. We may choose a bijection from \( X \) onto \( \{ d_1, d_2, \ldots, d_n \} \) so that \( B \) is identified with \( d_1 \). The elements of \( W : P_1 \) may be represented as ordered pairs \( (g, w) \), \( g \in P_1 \) and \( w \in W \), with group operation \( (g_1, w_1)(g_2, w_2) = (g_1 g_2, w_1^{g_2} w_2) \). Note that \( T : P_1 \) is a subgroup since \( g \in P_1 \) and \( z_k \in T \), \( z_k^T = (d_1 d_i)^g = d_i d_j \) for some \( i, j \), and \( d_i d_j = z_i^{-1} z_j \in T \). Furthermore, the \( z_i \)'s are all conjugate under \( T \).

Let \( \Gamma(x) = \{ y_1, y_2, \ldots, y_k \} \), with \( y_1 = y \). Let \( a \in Q_x \) be an element switching \( y \) and \( y_2 \). Let \( b = a^g \in Q_y \). Since \( Q_{xy} \) acts transitively on \( \Gamma(x) \setminus \{ y \} \), there exists \( g_i \in Q_{xy} \) such that \( y_i^g = y \) for \( 3 \leq i \leq v \). Let \( h_2 = ab^{-1} \) and for \( 3 \leq i \leq v \), let \( a_i = a^g, b_i = a_i^g = b_i^g \), and \( h_i = a_i b_i^{-1} \). By Lemma 3.2, \( h_2 \) is defined independent of the choice of \( a \), and if \( k_i \in Q_{xy} \), then there is an element such that \( y_i^{k_i} = y \), then \( g_i k_i^{-1} \in Q(\langle y \rangle, x, y) \). Hence, by Lemma 3.2, \( [h_2, g_i k_i^{-1}] = 1 \) and so \( h_i^2 = h_i^g \). Therefore \( h_i \) is defined independent of the element \( g_i \in Q_{xy} \).

Let \( V = \langle h_2, h_3, \ldots, h_n \rangle = \langle h_2 \rangle_{Q_{xy}} \). Note that for \( i \neq 1 \), \( y_i^{h_i} = y \) and \( y_i^{h_i^{-1}} = y_j \). Hence, for any pair \( y_i, y_j \) in \( \Gamma(x) \), there exists \( k \in V \) such that \( y_i^k = y_j \). We assume, for the moment, that \( J \) is a normal subgroup of \( Q \) such that \( V \leq J \) and \( J \cap Q_x = 1 \). We claim \( Q_x \) normalizes \( V \). Let \( g \in Q_x \) and \( h_i \in \langle h_2, h_3, \ldots, h_n \rangle \). Then \( y_i^g = y_j \) and \( y_i^g = y_j \). Therefore \( h_i^g \) sends \( y_i \) to \( y_j \), and there exists \( k \in V \) such that \( h_i^g k \) fixes \( y_i \). Therefore \( h_i^g \in Q_{xy} \). Since \( V \leq J \), \( h_i^g k \in J \). If \( h_i^g k \neq 1 \), then \( J \cap Q_{xy} = 1 \), contradicting the assumption. Hence \( h_i^g = k^{-1} \in V \), so \( Q_x \) normalizes \( V \). Since \( V \cap Q_x \neq 1 \), \( V : Q_x \) is a subgroup of \( Q \). Since \( b^{-1} = ah_2 \in V : Q_x \), \( V : Q_x \) contains \( Q_x = \langle Q_{xy} \rangle \). Therefore \( Q = \langle ab^{-1} \rangle Q \cap Q_x = \langle ab^{-1} \rangle B : P_1 \), assuming the existence of the subgroup \( J \).

Let \( T : P_1 \) be the group defined previously and define \( \phi : Q \to T : P_1 \) by \( \phi(g) = (g, 1) \) and \( \phi(g) = (g, d_i^g d_1) \), \( g \in P_1 \). For \( g \in B \), \( d_i^g = d_i \), so \( \phi(g^e) = (g, d_i^g d_1) = (g, 1) = \phi(g) \). Therefore \( \phi \) is well-defined and can be extended to a homomorphism from \( Q = P_1 ** P_2 \) into \( T : P_1 \). Note that \( \phi(g) \phi(b^{-1}) = \phi(a) \phi((a^{-1})g) = (a, 1)(a^{-1}, d_i^g d_1) = (1, d_i^g d_1), \) and \( d_i^g \) for some \( i \neq 1 \) since \( g \notin B \). Therefore \( \phi(a) \phi(b^{-1}) = (1, z_i) \) in \( T \). Since the \( z_i \)'s are all conjugate under \( B, \phi \) is a homomorphism onto \( T : P_1 \). Let \( V = \langle h_2, h_3, \ldots, h_v \rangle \) be defined as above. Then \( \phi \) maps \( V \) onto \( T \). Let \( J = \{ g \in Q \mid (\phi(g) \in T) \), the preimage of \( T \) in \( Q \). Then \( J \triangleleft Q \) and \( J \cap P_1 = 1 \). Since \( V \leq J \), our assumption above about the existence of the subgroup \( J \) is now justified, so \( Q = \langle ab^{-1} \rangle B : P_1 \).

4. PROOF OF THEOREM 1.1: \( Q(w, x, y) \) FIXED-POINT FREE

Let \((w, x, y)\) be a 2-path, \( \ell \) be the line in \( P(x) \) containing \( w \) and \( y \), and \( L = \varphi_{xy}(\ell) \).

Throughout this section, we assume \( Q(w, x, y) \) is fixed-point free on \( \Gamma(x) \setminus \{ x \} \). Hence, by Lemma 2.2, \( Q(w, x, y) \) acts 2-transitively on \( L \setminus \{ x \} \) and transitively on \( \Gamma(y) \setminus \{ L \} \). We show \( n \neq 3 \). Suppose \( n = 3 \). Let \( N = Q_x(\Gamma(x)) \cap Q_y \). Suppose \( N^B \neq 1 \). Let \( k \) be an element in \( N \) acting nontrivially on \( L \). Since the elements of \( Q_x(\Gamma(x)) \) are elations, \( k \) must fix pointwise a line in \( y : x \setminus \{ L \} \), say \( L_1 \). Let \( L_1, L_2, \ldots, L_q \) be the \( q \) lines in \( y : x \setminus \{ L \} \). Since \( Q(w, x, y) \) acts transitively on \( y : x \setminus \{ L \} \), there exist elements \( g_i \in Q(w, x, y) \) such that \( L_i^g = L_i \) for \( 2 \leq i \leq q \). Since \( N \triangleleft Q(w, x, y) \), \( k^{g_i} \in N \). Since \( |N - \{ 1 \}| = q - 1 \), \( k^{g_i} = k \) for some \( g_i \).
Therefore $k$ fixes pointwise $L_1$ and $L_j$. Since only the identity can fix pointwise two lines, we conclude that $N_L^L = 1$. Now let $A = O_p(Q_{u,v}) \cap Q_v$. Then $O_p(Q(w, x, y)) = A \times N$. If $O_p(Q(w, x, y))$ acts trivially on $L$, then $q$ does not divide $|Q(w, x, y) : O_p(Q(w, x, y))|$. But $Q(w, x, y)$ acts transitively on $L - \{x\}$, so $O_p(Q(w, x, y))^L \neq 1$. Since $N_L^L = 1$, we must have $A^L \neq 1$. Therefore $A$ acts transitively on $L - \{x\}$ since $A \triangleleft Q(w, x, y)$ and $Q(w, x, y)$ acts 2-transitively on $L - \{x\}$.

Let $u \in \Gamma(x) - \ell$, so $F = \{w, x, u\}$ is a frame of $P(x)$. Let $D$ be the subgroup of $Q_z$ fixing $F$ pointwise, and let $M$ be the line in $P(x)$ containing $w$ and $u$. Let $h$ be an element in $D \cap Q_M$. Then $[A, h]$ fixes pointwise $M$ and $\ell$, so $[A, h] = 1$. Now, each point $v \in M$ determines a unique line, $\ell_v$, in $[y : x]$, so $h$ fixes $\psi_{xy}(\ell_v) \in [y : x]$. Therefore $h$ fixes the $q + 1$ lines of $[y : x]$, and since $|h|$ divides $q - 1$, $h$ fixes some point $z_v \in \psi_{xy}(\ell_v) - \{x\}$ for each $\psi_{xy}(\ell_v)$ in $[y : x]$. But $A$ acts transitively on $\psi_{xy}(\ell) - \{x\} = L - \{x\}$ and $[A, h] = 1$. Therefore $h$ fixes pointwise $L$ and $q$ points $z_v$ not on $L$, so $h = 1$. Therefore $D \cap Q_M = 1$, so $q = 2$ or 4. Let $g \in A = O_2(Q_{u,v}) \cap Q_z$ be an involution acting nontrivially on $L$. Since $g$ fixes $\ell$ pointwise, $g$ must act nontrivially on $M$. Therefore $g$ acts nontrivially on $[y : x]$. Note that $g \in A < Q_x \cap Q_z$, so $g \in Q_z$ by Lemma 2.1. Thus, it suffices to show that $PSL(3, 2)$ and $PSL(3, 4)$ do not possess an involution $g$ with $g^{[y : x]} \neq 1$ and $g^L \neq 1$, $L \in [y : x]$. Let the points of $P(y)$ be represented by equivalence classes of coordinate triples $[z_1, z_2, z_3], z_i \in GF(q), q = 2$ or 4, and without loss of generality, let $x = [1, 0, 0]$ and $L = [[1, 0, 0], [0, 1, 0]]$. Since $g$ is a 2-element fixing a maximal flag, $(x, L)$, of $P(y)$, we may represent the action of $g$ on $P(y)$ by a lower triangular matrix with ones along the diagonal. It follows, using $g^2 = 1$, that either $g^{[y : x]} = 1$ or $g^L = 1$. We leave the details to the reader.

Now let $n \geq 4$. Let $(w, x, y)$ be a 2-path. Let $\ell$ be the line in $P(x)$ containing $w$ and $y$, and $L = \psi_{xy}(\ell)$. Let $P(x : w, y)$ denote the projective space of dimension $n - 3$ where the points of $P(x : w, y)$, denoted by $[x : w, y]$, are the subspaces of $P(x)$ of dimension 2 containing $w$ and $y$. The following properties hold:

1. $PSL(n - 2, q) \leq Q(w, x, y)^{[x : w, y]} \leq PGL(n - 2, q)$.
2. $O_p(Q(w, x, y))$ acts trivially on $[x : w, y], |O_p(Q(w, x, y))| = q^{2(n-2)}$, and $q$ does not divide $|Q(w, x, y) / (O_p(Q(w, x, y)))| \cdot |PSL(n - 2, q)|$.
3. If $H \triangleleft Q(w, x, y)$, then $O_p(H) = O_p(Q(w, x, y))$.

$Q(w, x, y)$ contains a subgroup $K \cong SL(n - 2, q)$ with $K^{[x : w, y]} \cong PSL(n - 2, q)$. Since $Q(w, x, y)$ acts on $L$, $K$ acts on $L$. Suppose $K$ acts trivially on $L - \{x\}$. Then $\ker \phi$, the kernel of the homomorphism $\phi : Q(w, x, y) \rightarrow Q(w, x, y)^L$, contains $K$. Since $\ker \phi \cong Q(w, x, y)$ and $\ker \phi^{[x : w, y]} \leq PSL(n - 2, q), O_p(\ker \phi) = O_p(Q(w, x, y))$, and so $q$ does not divide $|Q(w, x, y) : \ker \phi|$, a contradiction since $Q(w, x, y)$ acts transitively on $L - \{x\}$. Let $M$ be the largest (normal) subgroup of $K$ acting trivially on $L - \{x\}$. Then $M^{[x : w, y]} \triangleleft K^{[x : w, y]}$. Since $\ker \phi \geq M, M^{[x : w, y]} \geq K^{[x : w, y]}$, otherwise $\ker \phi^{[x : w, y]} \geq PSL(n - 2, q)$ and again, $q$ does not divide $|Q(w, x, y) : \ker \phi|$. If $M^{[x : w, y]} = 1$, then $|PSL(n - 2, q)|$ divides $|K : M|$. But $K : M = |K^L|$ divides $q(q - 1)$. Therefore $M^{[x : w, y]}$ is a proper normal subgroup of $K^{[x : w, y]} \cong PSL(n - 2, q)$. Hence $n = 4$ and $q = 2$ or 3.

**Lemma 4.1.** Let $q = 2$ or 3. Let $(w, x, y, \ell, L$ and $K$ be given as above. Let $z \in L - \{x\}$.

(a) If $g$ is a $q$-element in $K$, then $|g^{[x : w, y]}| = |g^L| = q$.
(b) $O_p(Q(w, x, y)) = O_p(Q(x, y, z))$.
(c) Let $F$ be a frame of $P(x)$ containing $w$ and $y$, and $D$ be the subgroup of $Q_z$ fixing $F$ pointwise. If $h \in D$, then $h^L = 1$ if and only if $h^L = 1$.
PROOF. (a) We have $K \cong SL(2,q)$ and $K^{[x:w,y]} \cong PSL(2,q)$. Since $g$ is a $q$-element, $[g]_{[x:w,y]} = q$. As we have shown above, if $M$ is the largest subgroup of $K$ acting trivially on $L - \{x\}$, then $M^{[x:w,y]}$ is a proper normal subgroup of $K^{[x:w,y]}$. Therefore $|M^{[x:w,y]}| = q + 1$ and hence $[g]_{[x:w]} = q$.

(b) Let $A = O_p(Q_{xy}) \cap Q_w$ and $N = O_p(Q_{xy}) \cap Q_y$. Then $O_p(Q(w,x,y)) = (A,N)$ and $A,N$ are both elementary abelian normal subgroups of $Q(w,x,y)$ of order $q^2$ and $Q(w,x,y)$ acts irreducibly on each subgroup. Since $Q(w,x,y)$ acts on $L_n$, we claim that $A^{L_n} \neq 1$. Then $1 \neq A \cap R \trianglelefteq Q(w,x,y)$, where $R$ is the largest (normal) subgroup of $Q(w,x,y)$ acting trivially on $L$. Since $Q(w,x,y)$ acts on $L$, we conclude that $A^{L_n} = 1$. Similarly, $N^{L_n} = 1$. Therefore $O_p(Q(w,x,y))$ acts trivially on $L$, so $O_p(Q(w,x,y)) \trianglelefteq Q(x,y,z)$. Since $O_p(Q(w,x,y)) = O_p(Q_x \cap Q_{[t]} \trianglelefteq Q(x,y,z)$, $O_p(Q(w,x,y)) = O_p(Q(x,y,z))$.

(c) If $h^t = 1$, then by Lemma 1.2(b), there exists $g \in O_p(Q_{xy})$ such that $[g,h] = 1$ and $g^t \neq 1$ if $g^t = 1$, then $g \in O_p(Q_{xy}) \cap Q_2 < O_p(Q(x,y,z)) = O_p(Q(w,x,y))$ by part (b). Since $O_p(Q(w,x,y)) \trianglelefteq Q_x \cap Q_{[t]}$ and $g^t \neq 1$, we conclude that $g^t \neq 1$. Since $[g,h] = 1$, $h^t = 1$. Similarly, if $h^t = 1$ then $h^t = 1$.

Let $q = 3$. Let $(w,x,y,\ell,$ and $L$ be defined as in the lemma above. Let $u_1,u_2 \in \Gamma(x) - \{w,y\}$ be two vertices such that $F = \{w,y,u_1,u_2\}$ is a frame of $P(x)$ and $O_3(Q_{u_1y}) \cap Q_{wy} < K$, where $K$ is a subgroup of $Q(w,x,y)$ such that $K \cong SL(2,3)$ and $K^{[x:w,y]} \cong PSL(2,3)$. Let $V = \{u_1,w,y\}$ and $1 \neq k \in O_3(Q_{u_1y}) \cap Q_{wy}$, $k$ fixes $V$ pointwise. Since $k$ is a $3$-element in $K$, $k^L = 1$ by Lemma 4.1(a). Let $D = Q_x \cap Q_{[t]}$. By Lemma 2.3, there exist $w',u'_1,u'_2 \in \Gamma(y)$ such that $F' = \{w',u'_1,u'_2,x\}$ is a frame of $P(y)$, $D = Q_y \cap Q_{[t]}$, $w' \in L$, $u'_i \in \varphi_{w'}(\ell_i)$ for $i = 1,2$, where $\ell_i$ is the line in $P(x)$ containing $u_i$ and $y$. Let $V' = \{w',u'_1,x\} = \varphi_{w'}(V)$. Applying Lemma 2.3 to $\Gamma(w')$, $w' \in \varphi_{w'}(\{x,u'_1,u'_2,y\}) = \varphi_{w'}(\{x,u'_1,y\}) = \varphi_{w'}(\{x,y\}) = \varphi_{w'}(V')$. Let $W = \{x,u'_1,u'_2\} \subset P(x)$. We claim $D \cap Q_{[t]} = 1$. Suppose $1 \neq h \in D \cap Q_{[t]}$. Then $h^t \neq 1$. By Lemma 4.1(b), $h^t \neq 1$. But $[k,h]$ fixes pointwise $W$ and $\ell$, so $[k,h] = 1$. Since $k^L = 1$, we conclude that $D \cap Q_{[t]} = 1$. Hence $Q_x \cong PSL(4,3)$.

Let $t$ be an element in $Q_{xy}$ switching $w$ and $u_1$. Then $t$ switches $L$ and $L_1 = \varphi_{w'}(\ell_1)$ and $g = k' \in O_3(Q_{w'}) \cap Q_{u_1y}$ acts nontrivially on $L_1$. By Lemma 4.1(b), $O_3(Q_{x}(w,y)) = O_3(Q_{w'}) \cap Q_{u_1y}$. Therefore $g \in Q_{w'}$. Since $g^t \neq 1$, $g$ acts nontrivially on the set of lines in $\{y \in w'\}$ that are contained in $V'$. Hence $g$ acts nontrivially on the set of lines in $\{w' : y\}$ that are contained in $V^w$. Therefore $g \not\in \varphi_{w'}(V^w)$. We now produce a contradiction by showing $g \not\in Q_{w'^w}$. Let $1 \neq d \in D$ be an element fixing pointwise the lines $\{w,u_2\}$ and $(y,u_1)$ in $P(x)$. Then $[d,g]$ fixes pointwise $(w,u_2), (y,u_1)$ and $\ell$, so $[d,g] = 1$. Since $g^t \neq 1$ and $[d,g] = 1$, $d$ fixes $L_1$ pointwise. Therefore $d$ acts nontrivially on each of the lines $(u'_1,w')$ and $L$ in $P(y)$, otherwise $d \in D \cap Q_{[t]} = 1$. Hence, by Lemma 4.1(c), $d$ acts nontrivially on each of the the lines $(u'_1,y)$ and $(y,x')$ in $P(w')$. Since $[d,g] = 1$, $g$ fixes pointwise $(u'_1,y)$ and $(y,x')$ in $P(w')$. Therefore $g \in Q_{w'^w}$, a contradiction. Hence $q \neq 3$.

5. PROOF OF THEOREM 1.2

Let $\Gamma$ and $G$ satisfy the hypothesis of Theorem 1.2. Let $(x,y)$ be an edge in $\Gamma$. Then all our previous lemmas hold for $\Gamma$ and $(G_x,G_y)$. Let $w \in \Gamma(x) - \{y\}$. By Lemma 2.2, $G(w,x,y)$ acts intransitively on $\Gamma(y) - \{x\}$. It follows that $\Gamma$ is $(G,2)$-transitive. If $G(w,x,y)$ is fixed-point free on $\Gamma(y) - \{x\}$, then $n = 4$ and $q = 2$ by Section 4. Now suppose $G(w,x,y)$ is not fixed-point free on $\Gamma(y) - \{x\}$. By Lemma 3.1, if $f_{xy}$ is not a collineation from $P(x)$ to $P(y)$, then we obtain part (b) of Theorem 1.2. Now suppose $f_{xy}$ is a collineation. Let $a_1$ be an element
in \( G_x \) switching \( w, y \) and let \( b_1 = a_1^{f(w)} \) and \( h = a_1b_1^{-1} \). By Lemma 3.2, \([h, G(w, x, y)] = 1 \) and \( G(w, x, y) \) fixes pointwise the path \( C = (\ldots, x^{h^{-1}}, w, x, w^{h}, x^{h}, w^{h^2}, x^{h^2}, \ldots) \), where \( w^h = y \). If \( C \) is finite, then \( C \) is a circuit of length, say, \( k \). Suppose \( k \) is odd. Let \( t = (k+1)/2 \). Then \((w, x, y)^{h^t} = (x, y, z) \), where \( z = x^h \), so \( O_p(G_{wyz} \cap G_y)^{h^t} = O_p(G_{xy}) \cap G_z \). By Lemma 3.2, \([h, O_p(G_{wyz} \cap G_y)] \leq [h, G(w, x, y)] = 1 \), so \( O_p(G_{wyz} \cap G_y) = O_p(G_{xy}) \cap G_z \). But \( O_p(G_{wyz}) \cap O_p(G_{xy}) = 1 \). Therefore \( k \) is even, so \( C \) is a \( 2m \)-circuit for some \( m \geq 2 \). Hence \( h^m \in G(w, x, y) \), and \([h^m, G(w, x, y)] = 1 \). Therefore \( h^m = 1 \) or \((n, q) = (3, 2) \). Suppose \((n, q) = (3, 2) \) and \( h^m \neq 1 \). Then \([h^m] = 2 \) since \( G(w, x, y) = O_2(G_{wyz}) \cap G_y \times O_2(G_{xy}) \cap G_w \cong Z_2^2 \). We have \( a_1^n \in G(w, x, y) \) and since \( f_{xy} \) induces the identity on \( G_{xy} \), \( a_1^n = (a_1^n)^{f(w)} = b_2^{f(w)} = b_1^{-f(w)}a_1^{-1} = b_1^{-1}a_1 = a_1b_1^{-2} \) since \( a_1b_1^{-1}, b_1^{-1}a_1 = a_1b_2^{-1}a_1 = 1 \). Therefore \((h^m)^{a_1} = h^{-m} = h^m \), so \([h, a_1] = 1 \). If \((d) = O_2(G_{wyz}) \cap G_y \) and \((u) = O_2(G_{xy}) \cap G_w \), then \( d^m = u, a_1^m = d, \) and \((du)^m = ud = du \). Hence \( c_1 = du \) is the unique nontrivial element in \( G(w, x, y) \) centralized by \( a_1 \), so \((a_1b_1^{-1})^m = c_1 \).

6. PROOF OF THEOREM 1.4

Let \( \Gamma \) and \( G \) satisfy the hypothesis of Theorem 1.3(d). Suppose \( \Gamma \) is not bipartite. Let \( \Gamma' \) be the bipartite double of \( \Gamma \), the graph with vertex set \( \Pi_0 = \{(x, 0) \mid x \in V(\Gamma)\} \) and \( \Pi_1 = \{(x, 1) \mid x \in V(\Gamma)\} \), where two vertices \((x, i)\) and \((y, j)\) are adjacent if and only if \( x \) and \( y \) are adjacent in \( \Gamma \) and \( i \neq j \). There is an induced action of \( G \) on \( \Gamma' \) defined as follows: For each vertex \((x, i)\) in \( \Gamma' \) and each \( g \in G \), \((x, i)^g = (x^g, i) \). Now define an automorphism \( h \) of \( \Gamma' \) by \((x, i)^h = (x, j), i \neq j \), for all vertices \((x, i)\) of \( \Gamma' \). Then \( \Gamma' \) and \( G = (G, h) \cong G \times Z_2 \) satisfy the hypothesis of Theorem 1.3(d). Therefore we may assume from now on that \( \Gamma \) is bipartite and treat the non-bipartite case in the end. By [4], the following holds for any connected, undirected, bipartite graph \( \Gamma \):

LEMMA 6.1. Let \((x, y)\) be an edge in \( \Gamma \) and \( K \) a group of automorphisms of \( \Gamma \) acting edge-by-edge not vertex-transitively on \( \Gamma \). Define \( \Gamma' = \Gamma(K, K_x, K_y) \) to be the bipartite graph whose vertices are the right-cosets of \( K_x \) and \( K_y \) in \( K \), where two right-cosets, \( K_x h \) and \( K_y d \), are adjacent if \( K_x h \cap K_y d \neq \emptyset \). Then the map \( K_g \mapsto u^g, u = x, y, g \in K \), defines an isomorphism from \( \Gamma \) to \( \Gamma' \) and the action of \( K \) on \( \Gamma' \) by right-multiplication is equivalent to the action of \( K \) on \( \Gamma \).

We divide the proof of Theorem 1.4 into three cases; these correspond to the parameter \( k \) as listed in Theorem 1.3(d).

7. PROOF OF THEOREM 1.4: THE CASE \( k = q + 1 \)

In this case, \( n \geq 4 \) and \( \Gamma(x) \cap \Gamma(z) \) is a line in \( P(x) \) (or \( P(z) \)) whenever \( d(x, z) = 2 \). Let \((w, x, y)\) be a 2-path in \( \Gamma \). Let \( \ell \) be the line in \( P(x) \) containing \( w \) and \( y \), and let \( L = \psi_{\ell x}(\ell) \in \{x : x \} \). Then \( L = \Gamma(w) \cap \Gamma(y) \) since \( G(w, x, y) \) acts transitively on \( \Gamma(y) \) - \( \{L\} \) by Lemma 2.2. Let \( A = G_x \cap G_{\{\ell\}} \).

Assume \( A^L = 1 \). We consider, first, the case \( n = 4 \). Let \( V \) be a two-dimensional subspace of \( P(x) \) containing \( \ell \). Let \( N \) be the connected component of the subgraph of \( \Gamma \) containing \( x \) left pointwise fixed by \( G_x \cap G_{\{\ell\}} \). The assumption \( A^L = 1 \) implies that \( G_x \cap G_{\{\ell\}} \) fixes pointwise \( \psi_{\ell x}(V) \) for all \( u \in V \). Let \( N \) be the normalizer of \( G_x \cap G_{\{\ell\}} \) in \( G \) and \( D \leq N \) the subgroup fixing \( \Delta \) setwise. Then \( D \cap G_x = G_x \cap G_{\{\ell\}} \cap PSL(3, q) \leq (D \cap G_x)_\Delta \leq PGL(3, q) \). D acts transitively on the set of 2-paths in \( \Delta \) and for every pair of vertices \( u, v \) in \( \Delta \) at distance 2, \( \Delta(u) \cap \Delta(v) \) is a line in \( P(u) \). Thus, by Theorem 1.3, \( \Delta \) is the incidence graph
of points and hyperplanes in $PG(3, q)$. Since $D \cap G_x$ acts faithfully on $\Delta(x)$, by [7], $q = 2$ and $(D \cap G_x, D \cap G_y)$ induces on $\Delta$ the group $A_7$. Let $M = (D \cap G_x, D \cap G_y)$ and $K = G_{[\lambda]} = G_x \cap G_y$. Since $G_x \cong PSL(4, 2)$, $K$ is elementary abelian of order 2. Let $C$ be the centralizer of $K$ in $M$. Then $M/C \cong Aut(K) \cong GL(3, 2)$ and $C \neq K$ since $M/K \cong A_7 \neq GL(3, 2)$. Therefore $C = M$ since $C/K \cong M/K$. But $[M, K] \geq [D \cap G_x, K] \neq 1$. With this contradiction, we conclude that $\Gamma$ does not exist.

Now let $n \geq 5$. Let $\Delta$ and $D$ be similarly defined as before except now we take $V$ to be a three-dimensional subspace of $P(x)$ containing $\ell$. Then $\Delta$ is a 2-transitive graph of girth 4, and for every pair of vertices $u, v$ in $\Delta$ at distance 2, $\Delta(u) \cap \Delta(v)$ is a line in $P(u)$. Furthermore, $PSL(4, q) \leq (D \cap G_x)^{M(x)} \leq PTL(4, q)$, $D \cap G_x$ acts faithfully on $\Delta(x)$ and $(D \Delta)^{[\lambda]} = 1$. By our previous result, such graphs do not exist. Hence $\Gamma$ does not exist. We may therefore assume for the rest of this section that $A^x \neq 1$. This implies that $G(w, x, y)$ acts fixed-point freely on $\Gamma(y) - \{x\}$. By Theorem 1.2, $n = 4$ and $q = 2$, and the results of Section 4 now apply.

**Lemma 7.1.**

(a) Let $(y, z)$ be an edge in $\Gamma$. Let $L$ be an arbitrary line in $P(x)$ not containing $z$. For each $v \in L$, let $\ell_v = \Gamma(v) \cap \Gamma(z)$, which is a line in $P(z)$. Let $R$ be the plane in $P(y)$ containing $L$ and $z$, and let $T$ be the smallest subspace in $P(z)$ containing all the $\ell_v$s. Then $T = \{z\}$ is a line in $P(z)$.

(b) Let $(x, y, z, u)$ be a 3-path in $\Gamma$ with $d(x, u) = 3$. Let $t \in \Gamma(y) \cap \Gamma(u)$, $t \neq z$, so $(y, z, u, t)$ is a 4-circuit in $\Gamma$. Then $G(x, y, z, u)$ contains an element $g$ inducing a 2-cycle on $[u : z, t]$, where $[u : z, t]$ denotes the set of planes in $P(u)$ containing $z$ and $t$.

**Proof.** (a) Since $G_{xz} \cap G_{zu}$ fixes $R$ and no other plane in $P(y)$, $T = \{z\}$. (b) Let $w \in \Gamma(x) \cap \Gamma(z)$, $w \neq y$, so $(x, y, z, w, x)$ is a 4-circuit in $\Gamma$. Choose an element $g$ in $O_2(G(y, z, w)) \cap G_w$ so that $g$ induces a 2-cycle on $[z : y, u] - \{R\}$, where $R$ is the plane in $P(z)$ containing $w$, $y$, and $u$. Then $g$ induces a 2-cycle on $[u : z, t] - \{q_{xy}(R)\}$. By Lemma 4.1(b), $O_2(G(y, z, w)) \cap G_{ux} < G(x, y, z, u)$. □

We show $\Gamma$ does not exist by considering its suborbit diagram. Let $(x, y, z)$ be a 2-path in $\Gamma$. For each vertex $v$, denote by $[v]$ the set $[vG_x]$. Let $u \in [\Gamma(z)]$, with $u \neq \Gamma(x)$. Since $\Gamma$ is bipartite, $u \not\in \Gamma_0(x)$, so $u \in \Gamma(z) \cap \Gamma(x)$. Let $R$ be the plane in $P(z)$ containing $u$ and $\Gamma(x) \cap \Gamma(z)$. Let $T = \{z\}$. By Lemma 7.1(a), applied to the edge $(z, u)$ and the line $\Gamma(x) \cap \Gamma(z)$ in $P(z)$, we have $T \subseteq \Gamma(u) \cap \Gamma_2(x)$.

Suppose $T = \Gamma(u) \cap \Gamma_2(x)$. Then $[u] = (15 \cdot 14 \cdot 12) / (3 \cdot 7) = 120$. Therefore $|G_{uv}| = |PSL(3, 2)|$. Let $h$ be a 7-element in $G_{uv}$. Then $(h)$ has precisely two orbits in $\Gamma(u) - T$: one orbit of length 1 and one of length 7. Let $t \in \Gamma(u) \cap \Gamma(y)$, $t \neq z$. By Lemma 7.1(b), $G_{uv}$ contains an element $g$ inducing a 2-cycle on $[u : z, t] - \{T\}$, so $G_{uv}$ is fixed point free on $\Gamma(u) - T$. Therefore $G_{uv} \geq (h, g)$ acts transitively on $\Gamma(u) - T$. Now let $v \in \Gamma(u) - T$. Then $v \in \Gamma_4(x)$ and $[\Gamma(u) \cap \Gamma_4(x)] = 2^3$. Applying Lemma 7.1(a) to the edge $(u, v)$ and the seven lines in $T$ shows $\Gamma(v) \cap \Gamma_3(x)$ is a subgroup of index $2^6$ [3], we conclude that $T \subseteq \Gamma(u) \cap \Gamma_2(x)$.

Let $v \in \Gamma(u) \cap \Gamma_2(x), v \neq T$. Let $W$ be the plane in $P(u)$ containing $t, z$ and $v$. By Lemma 4.1(b), $O_2(G(t, y, z)) \cap G_x = O_2(G(y, z, u)) \cap G_x = O_2(G(z, u, t)) \cap G_x$. Let $K = O_2(G(z, u, t)) \cap G_z$. Then $|K| = 2^2$ and $K$ acts on each $U$ in $[u : z, t]$. Suppose $K$ fixes $v$. Then $K$ fixes $W$. By Lemma 7.1(b), $G(x, y, z, u)$ contains an element $g$ inducing a 2-cycle on $[u : z, t] - \{T\}$, so $G(x, y, z, u)$ acts transitively on $[u : z, t] - \{T\}$. Since $K \leq
and we have $H^d = 1$. Let $\mathcal{H} = \langle G_x, G_y \rangle$. Then $\langle P, c \rangle$ is abelian and acts transitively, hence regularly, on $X$. Therefore $c \in P$, and so $C(P) = P$. Let $N(P)$ be the normalizer of $P$ in $H$. Then $N(P)/C(P) \leq \text{Aut}(P) \cong \mathbb{Z}_7$. Therefore $\langle N(P) \rangle$ divides $70 \cdot 71$. Thus $\text{Syl}_{71}(H) = \langle H : N(P) \rangle = 2^b \cdot 3^2 \cdot 5 \cdot 7 / m, m \mid 70$. Trying all possible values of $m$ yield $\text{Syl}_{71}(H) \not\equiv 1 \mod 71$. Therefore $\Gamma$ does not exist.

8. **Proof of Theorem 1.4: The Case $k = 2$**

In this case, every 2-path $(w, x, y)$ is contained in a unique 4-circuit. Therefore $G(w, x, y, z) = G(w, y, y, z)$, where $(w, x, y, z)$ is a 4-circuit, and hence the results of Section 3 apply. We claim for each edge $(x, y)$, the map $f_{xy}$ is a collineation from $P(x)$ to $P(y)$, and in the case $q = 2$, we choose $f_{xy}(u)$, for each $u \in \Gamma(x) \setminus \{y\}$, to be the unique vertex in $\Gamma(y) \setminus \{x\}$ such that $(u, x, y, f_{xy}(u), u)$ is a 4-circuit. Suppose $f_{xy}$ is not a collineation. By Lemma 3.1(a), $(n, q) = (3, 4)$ or $q = 2$. Let $(x, y, z, w, x)$ be a 4-circuit in $\Gamma$. Let $\ell$ be a line in $P(x)$ containing $y$ but not $w$. Then $L = f_{xy}(\ell)$ is a line in $P(y)$ containing $x$ but not $z$. Therefore, by Lemma 3.1(a), $C = f_{xy}(L) \cup \{y\}$ is a hyperoval in $P(z)$ containing $w = f_{xy}(x)$. Furthermore, letting dim $(\ell, w)$ denote the dimension of the smallest subspace containing $\ell$, $w$, we have dim $(\ell, w) = \text{dim}(L, z) = \text{dim} V = 2$, where $V$ is the smallest subspace of $P(z)$ containing $C$. For $q = 2$, $(G_x \cap G_C) \mathcal{H} \cong S_4$, and for $(n, q) = (3, 4)$, $G_x \cap G_C \cong A_4$. Therefore there is an $a \in G_x \cap G_C$ such that $\phi$ switches $w$ and $y$. Since $k = 2$, $a$ fixes $x$, and we have $\ell^a \neq \ell$ since $y \in \ell$ but $w \notin \ell$. Since $a$ normalizes $G(x, y, z, w) \cap G_C = G(x, y, z, w) \cap G_x = G(x, y, z, w) \cap G_y = G(x, y, z, w) \cap G_x \cap G_y = G(x, y, w) \cap G_x \cap G_y = G(x, y, w) \cap G_x \cap G_y \cong G_x \cap G_y \cong A_4$. Therefore $G(x, y, w) \cap G_x \cap G_y \cong A_4$. But $G(x, y, w) \cap G_x \cap G_y$ is fixed-point-free on $\Gamma(x) \setminus \{w, y\}$ and $|\ell \cap \ell^a| = 1$ since $a$ fixes $\ell$, $\ell$. With this contradiction, we conclude that for each edge $(x, y)$ in $\Gamma$, $f_{xy}$ is a collineation from $P(x)$ to $P(y)$ and hence induces an isomorphism from $G_x$ onto $G_y$, trivial on $G_{xy}$. By Theorem 1.2(a), $H = \langle G_x, G_y \rangle$ is isomorphic to a quotient of $Y_{n,q,2}$ or $W_2$.

Suppose $H$ is isomorphic to a quotient of $Y_{n,q,2}$. We show $Y_{n,q,2} \cong \mathbb{Z}_2^{2^{n-1}} : G_x$. Let $D$ be generated by involutions $d_i$, for $1 \leq i \leq u$, such that for all pairs $i, j$, $[d_i, d_j] = 1$. Let $G_x$ act, by conjugation, on $d_1, d_2, \ldots, d_u$ as points of $P(x)$. Let $\Sigma$ be the Cayley graph on $\{d_1, d_2, \ldots, d_u\}$, whose vertices are the elements of $D$ such that two vertices $u, v$ are adjacent if $uv^{-1} \in [d_1, d_2, \ldots, d_u]$. Then $\Sigma$ is isomorphic to the $u$-cube and is $(D : G_x)$-transitive. We may identify $d_i$ with the vertex $y$ in $\Gamma(x)$. Let $T$ be the subgroup of $D$ generated by $d_1 + d_2$, for $2 \leq i \leq v$. Then $T \cong \mathbb{Z}_2^{v-1}$ and $T^G = T$. The elements of $T : G_x$ may be represented by ordered pairs $(g, t)$, $g \in G_x$, $t \in T$, with group operation $(g_1, t_1)(g_2, t_2) = (g_1g_2, t_1^{g_2} + t_2)$. We may let $Q = G_x *_{G_y} G_y$ and define a map $\phi : Q \rightarrow T : G_x \phi(g) = (g, 0)$ and $\phi(gf_{xy}) = (g, d_1 + d_2^g)$, $g \in G_x$. The map $\phi$ is well-defined and can be extend to a homomorphism from $Q$ into $T : G_x$. Let $a \in G_x$, $a \not\in G_y$ but $a^2 \in G_{xy}$, and let $b = a^{-1}$.

We have $\phi(a) \phi(b^{-1}) = (a, 0)(a^{-1}, d_1 + d_2^{-1}) = (1, d_1 + d_2^{-1}) = (d_1, d_1^{-1})$ for some $d_1 \neq d_1$ as $a \not\in G_{xy}$. Both $G_x$ acts 2-transitively on $d_1, d_2, \ldots, d_u$, the image of $\phi$ is isomorphic to $T : G_x$. Furthermore, $(ab)^{-1} = a^2$ is in the kernel of $\phi$, so $T : G_x$ is isomorphic to a quotient of $Y_{n,q,2}$. It suffices now to show $\{Y_{n,q,2} \leq 2^{n-1} : G_x\}$. Let $\Gamma = \Gamma(Y_{n,q,2}, G_x, G_y)$ be defined as in Lemma 6.1. Since $(ab)^{-1} = a^2$, $(G_x, G_y, G_xb^{-1}, G_ya^{-1}, G_x)$ is a 4-circuit in $\Gamma$. Therefore
\[ \hat{\Gamma} \text{ has girth } 4 \text{ and hence } |\hat{\Gamma}| \leq 2^v. \] Since \( Y_{n,q,2} \) acts faithfully and transitively on the set of edges of \( \hat{\Gamma}, |Y_{n,q,2}| \leq 2^{v-1} \cdot |G_x| \). Therefore \( Y_{n,q,2} \cong T : G_x \cong Z_2^{v-1} : G_x \) and \( \hat{\Gamma} \) is isomorphic to the \( v \)-cube.

We mention several results in [8] and [10] concerning quotients of \( Y_{n,q,2} \). Let \( J = (d_1 + d_2 + \cdots + d_0) \). Then \( J, G_x \cong 1 \) and \( J < T \) if \( v \) is even. By [10], if \( q \) is odd then there are no proper \( G_x \)-submodules in \( T \) except \( J \) in the case \( J < T \). Now suppose \( q \) is even.

Since \( G_x \) induces a projective space structure on \( \{d_1, d_2, \ldots, d_0\} \), we may consider the \( d_i s \) as points of a projective space \( P \). Let \( S_i \) denote the set of \( i \)-dimensional subspaces of \( P \). For \( 1 \leq i \leq n-2 \), let \( R_i = \{a_{i_1} + a_{i_2} + \cdots + a_{i_j} : \{a_{i_1}, a_{i_2}, \ldots, a_{i_j}\} = P - V, V \in S_i\} \), i.e., \( R_i \) is generated by the complements of all \( i \)-dimensional subspaces. Then \( R_i^{G_x} = R_i \) and by [8], \( R_{n-2} \leq \cdots \leq R_2 \leq R_1 \leq T, R_{n-2} \leq M \) for every \( G_x \)-submodule \( M \) in \( T \), and \( |R_{n-2}| = 2^a \) for \( a = 2^r \). We conclude with a remark concerning the group \( G = H.2 \). Let \( \Gamma \) be the \( v \)-cube and \( G_1 = D : G_x \), with \( G_x \cong PSL(n, q) \), \( g = p^{2l} \). Let \( \sigma \) be an involutory automorphism of \( G_x \) induced by an automorphism of \( G(\Gamma) \). We may assume \( \sigma \) fixes the vertex \( y \) in \( \Gamma(\chi) \)

that's identified with the element \( d_1 \in D \), so \( [d_1, \sigma] = 1 \). Let \( f = d_1 \sigma \) and \( G_2 = (H, f) \), \( H : T = G_x \). Then \( \Gamma \) is \((G_1, 2)\)-transitive, \( i = 1, 2 \). Since \( H^f = H \) and \( f^2 = 1 \), \( H \) and \( f \) are the only right-cosets of \( H \) in \( G_2 \). Therefore \( H \) is a subgroup of index 2 in \( G_1, i = 1, 2 \), but \( G_1 \neq G_2 \) since \( G_2 \) contains no element centralizing \( G_y \).

Now suppose \( H = (G_x, G_y) \) is isomorphic to a quotient of \( W \). We show \( H \cong W_2 \cong U_5(3) \) and \( G \cong G(2) \). Let \( P_1 \) be the group with generators \( e_0, e_1, \ldots, e_4 \) and relations \( R = \{e_0^2 = e_1^2 = \cdots = e_2^2 = 1, [e_0, e_1] = 1, [e_0, e_2] = 1, r = e_0 e_2 e_0, s = e_1 e_4 e_1, r^2 = s^2 = 1, (rs)^2 = 1, e_0 = e_2, e_0' = e_3, e_1' = e_4, e_2' = e_3\} \). The set \( R \) is the set of Steinberg relations for \( PSL(3, 2) \), so \( P_1 = \langle e_0, e_1, \ldots, e_4 | R \rangle \cong PSL(3, 2) \). From the relations for \( P_1, B = \langle e_0, e_1, e_2, e_3 \rangle \cong S_4 \), so \( B \) is a subgroup of index 7 in \( P_1 \). Let \( f \) be an involution such that \([f, B] = 1\). Let \( a = r s r, b = a^2 \) and \( P_2 = P_1 f = \langle B, b \rangle \). Note that \( a \neq B = P_1 \cap P_2 \), but \( a^2 = 1 \in B \). Furthermore, \( P_1 \cap P_2 \cap P_2^a = \langle e_0, e_1 \rangle \cong Z_2^2 \) and \( e_0^a = e_1 \) so \( (e_0 e_1)^a = e_0 e_1 \). Therefore \( W_2 = (P_1 *_{B} P_2 | (ab)^2 = e_0 e_1) \) and using the software program GAP, we find that \( W_2 \) is a rank-4 extension of \( P_1 \) of degree 1 + 7 + 7 + 14. Hence, by [9], \( W_2 \cong U_5(3) \). Therefore \( W_2, f \cong Aut(U_5(3)) \cong G_2(3) \). It suffices now to show \( G \) contains an involution \( f \) with the above properties. Let \( y_j, i = 1, 2 \), be two points in \( \Gamma(\chi) - \{y\} \) such that \( F = \{y, y_1, y_2, y_3\} \) is a frame of \( P(\chi) \). Then \( F^f = \{x, f_x(y_1), f_x(y_2)\} \) is a frame of \( P(\chi) \). Since \( G_x \cap G_F \) acts 3-transitively on \( F^f \), there exists \( f \in G \) such that \( f \) switches \( x \) and \( y \), and \( y_f = f_x(y_1), i = 1, 2 \). Since \( f \) fixes the unique 4-circuit containing \( y, x, y_1, f_x(y) \), we have \( f_x(y_1)^f = y_1 \). Thus, \( f \) switches \( v \) and \( f_x(v) \) for any point \( v \) in the line containing \( y, y_1 \) or \( y_1, y \). Hence \( f \) switches \( v \) and \( f_x(v) \) for any \( v \in \Gamma(\chi) - \{y\} \).

Therefore \([f, G_x] = 1 \) and \( f^2 \in Z(G_x) = 1 \), so \( G \cong G(2) \). Let \( \hat{\Gamma} = \Gamma(W, P_1, P_2) \) be defined as in Lemma 6.1. Since \( (ab)^{-1} = (ab)^2 = e_0 e_1 \in P_1 \cap P_2, (P_1, P_2, P_1 b, P_2 a, P_1) \) is a 4-circuit in \( \Gamma \). Therefore \( \hat{\Gamma} \) has girth 4 and \( \Gamma \cong \hat{\Gamma} \).

9. PROOF OF THEOREM 1.4: THE CASE \( k = q \)

Let \((w, x, y)\) be a 2-path and \( \ell \) be the line in \( P(\chi) \) containing \( w \) and \( y \). In this case, \( \Gamma(w) \cap \Gamma(y) \) is a line of \( P(\chi) \) with a point removed. Since \((w, x, y)\) acts transitively on \( P(\chi) - \varphi_{xy}(\ell), \Gamma(w) \cap \Gamma(y) \subseteq \varphi_{xy}(\ell) \) and \( G(w, x, y, z) = G(w, x, y, z) \), where \( z \) is the unique vertex in \( \varphi_{xy}(\ell) \) at distance 3 from \( w \). Hence the results of Section 3 apply. We may assume \( q \neq 2 \) since the case \( k = 2 \) has already been dealt with. Let \( \Delta(\chi, \ell) \) be the connected component of the subgraph of \( \Gamma \) containing \( x \) left pointwise fixed by \( G_x \cap G(\ell) \). For each \( u \in \ell, G_x \cap G(\ell) = G_x \cap G[\varphi_u(\ell)] = G_x \cap G[\varphi_u(\ell)] \). Thus, it follows that \( \Delta = \Delta(\chi, \ell) \) is a
graph of valency \( q + 1 \) and girth 4. Furthermore, \( G_\Delta \) acts transitively on the set of 2-paths in \( \Delta \) and \( |\Delta(u) \cap \Delta(v)| = k = q \) for every pair of vertices \( u, v \) in \( \Delta \) at distance 2. Let \( (x, y, z, u) \) be a 3-path in \( \Delta \) with \( d(x, u) = 3 \). We have \( |\Delta_2(x)| = q + 1 \). Therefore \( |\Delta(u) \cap \Delta_2(x)| \) divides \( q + 1 \). Since \( |\Delta(u) \cap \Delta_2(x)| \geq |\Delta(u) \cap \Delta(y)| = q \), \( \Delta(u) = \Delta_2(x) \) and so \( \Delta \) is distance-transitive with intersection array \( \{q + 1, q, 1; 1, q, q + 1\} \).

Note that every 2-path \((u, v, w)\) in \( \Gamma \) determines a unique subgraph \( \Delta(v, L) \), where \( L \) is the line in \( P(v) \) containing \( u \) and \( w \). Therefore two subgraphs \( \Delta(v_1, L_1) \) and \( \Delta(v_2, L_2) \) have a 2-path in common if and only if \( \Delta(v_1, L_1) = \Delta(v_2, L_2) \). We will also write \( \Delta(u, v, w) \) to denote the subgraph determined by the 2-path \((u, v, w)\).

We claim \((n, q) = (3, 4)\) and \( G_x \not\cong PGL(3, 4) \). By Lemma 3.1(a), it suffices to consider the case when \( f_{xy} \) is a collineation from \( P(x) \) to \( P(y) \) for every edge \((x, y)\) in \( \Gamma \). Let \((w, x, y, t, u)\) be a 4-circuit in \( \Delta = \Delta(x, \ell) \), where \( \ell \) is the line in \( \Delta(x) \) containing \( y \) and \( w \). Let \( L \) be a line in \( \Delta(x) \) containing \( y \) but not \( w \). Then \( L_1 = f_{xw}(L) \) is a line in \( \Delta(x) \) containing \( x \). Therefore \( L_2 = f_{xw}(L_1) \) is a line in \( \Delta(x) \) containing \( x' = f_{xw}(x) \), which is the unique vertex in \( \Delta \) at distance 3 from \( x \). Hence \( L_3 = f_{wv}(L_2) \) is a line in \( P(w) \) containing \( x = f_{wv}(x') \). Therefore \( L_4 = f_{wv}(L_3) \) is a line in \( P(w) \) containing \( w \), so \( L_4 \neq L \). If \( R \) is the plane in \( P(x) \) containing \( L \) and \( w \), then \( R = f_{xw}f_{wv}f_{xw}f_{xw}(R) \) since \( G_\Delta \cap G_R \) fixes pointwise \( f_{xw}f_{wv}f_{xw}(x) \). Therefore \( L_4 \subset R \) and so \( |L_4 \cap L| = 1 \). Let \( A = G_x \cap G_{\ell} = G_{\Delta(x)} \). Since \( A \cap G_L \) fixes \( L_4 \), \( A \cap G_L \) fixes \( L \). But \( A \cap G_L \) is fixed-point free on \( \Gamma(x) \). With this contradiction, we conclude that \( f_{xy} \) is not a collineation. Hence \( n = 3, q = 4 \) and \( G_x \not\cong PGL(3, 4) \).

**LEMMA 9.1.**

(a) Let \( u \) be a vertex in \( \Delta \). Then \( G_{|\Delta} \) acts transitively on \( \Gamma(u) - \Delta(u) \).

(b) Let \((u, v)\) be an edge in \( \Gamma \). Let \( L \) be a line in \( P(u) \) not containing \( v \). If \( K \) is a subgroup of \( G_{uv} \cap G_L \) acting transitively on the points of \( L \), then \( K \) is fixed-point free on \( \Gamma(v) - \{u\} \).

**Proof.** (a) Since \( \Delta(u) \) is a line in \( P(u) \), \( G_u \cap G_{\Delta(u)} = G_{|\Delta} \) acts transitively on \( \Gamma(u) - \Delta(u) \).

(b) Let \( L = \{w_1, w_2, \ldots, w_5\} \). Then \( \ell_i = \langle v, w_i \rangle \), for \( 1 \leq i \leq 5 \), are five distinct lines in \( P(u) \). Therefore \( \Delta(u, \ell_i) \), for \( 1 \leq i \leq 5 \), are five distinct subgraphs of \( \Gamma \). If \( K \leq G_{u} \cap G_L \) acts transitively on \( \{w_1, w_2, \ldots, w_5\} \), then \( K \) acts transitively on \( \{\Delta(x, \ell_1), \ldots, \Delta(x, \ell_5)\} \). Hence \( K \) is fixed-point free on \( \Gamma(v) - \{u\} \).

We will use the suborbit diagram for \( \Delta = \Delta(x, \ell) \) to obtain possible suborbit diagrams for \( \Gamma \). Let \((x_1, x_2, x_3, x_4)\) be a 3-path in \( \Delta \) with \( d(x_1, x_3) = 3 \). Let \([v]\) denote the set \([v G^2] \). We have \([x_1] = 21 \), \([x_2] = 21 \cdot 20/4 = 105 \) and \( \Gamma(x_2) \cap [x_3] = 1 \) or \( 1 + 16 \) since \( G_{|\Delta} \) acts transitively on \( \Gamma(x_2) = \Delta(x) \) by Lemma 9.1(a). Since \( x_3 \) is the unique vertex in \( \Delta \) at distance 3 from \( x \), for each \( g \in G_x \), \( x_3^g \) is the unique vertex in \( \Delta^g = \Delta(x, \ell^g) \) at distance 3 from \( x \).

Since \( P(x) \) contains 21 lines and \( G_x \) acts 2-transitively on these lines, \([x_3] = 21 \) or \( 1 \). Since \([x_2] = 105 \) and \([x_3] \) with \( \Gamma(x_2) \cap [x_3] = [x_3] \), we conclude that \([x_3] = 21 \), \( \Gamma(x_2) \cap [x_3] = 1 \) and \( |\Gamma(x_3) \cap [x_2]| = 5 \). Let \( u \in \Gamma(x_3) - \Delta(x_2) \). Since \( \Gamma \) is bipartite, \( u \in \Gamma(x_1) \) and \( |\Gamma(x_2) \cap [u]| = 16 \). Let \( h = \Gamma(u) \cap [x_2] \). Then \([x_1, a, b, c] = \Delta(x_2) \cap [x_3] \). The graphs \( \Delta(x_1, x_2, u), \Delta(a, x_2, u), \Delta(b, x_2, u) \) and \( \Delta(c, x_2, u) \) are all distinct and therefore do not have a 2-path in common. Since every pair of vertices at distance 2 are connected by four 2-paths, we conclude that \( h \geq 1 + 4 \cdot 3 = 13 \). Let \( w \in \Gamma(x_3) - \Delta(x_3) \). Then \( w \in \Gamma(x_3) \) and \( |\Gamma(x_3) \cap [w]| = 16 \). The 2-path \((x_2, x_3)\) determines a subgraph \( \Delta(u, x_2, x_3) \) which is distinct from \( \Delta \), so \( \Delta(u, x_2, x_3) \) and \( \Delta \) cannot have a 2-path in common. Hence \( x_2 \) is the only vertex in \([x_2]\) adjacent to both \( u \) and \( x_3 \). Since \( \Gamma(u) \cap \Gamma(x_3) = 4 \), we conclude
Let \( Q \) and \( T \) be two conjugacy classes of subgroups isomorphic to \( S_3 \). Since each \( v \in \Delta(x_3) \cap [x_2] \) determines a distinct subgraph \( \Delta(v, x_3, w) \) and \( |\Gamma(v) \cap \Gamma(w)| = 4 \), it follows that \( m \geq 5 \cdot 3 = 15 \). Let \( d \) be a 5-element in \( G_{s_{x_1}} \). Since \( |\Gamma(x_3) \cap [w]| = 16 \), \( (d) \) fixes some \( y \in \Gamma(x_3) \cap [w] \). By Lemma 9.1(b), \( (d) \) is fixed-point free on \( \Gamma(y) - \{x_3\} \). Therefore \( r \equiv 1 \) mod 5 and \( 5 \mid m \). Since \( m \geq 15 \), either \( m = 15 \) and \( r = 1 \) or 6, or \( m = 20 \) and \( r = 1 \). We claim \( h = 14 \). Suppose \( h = 15 \). Then \( 3 \leq t \leq 6 \) and \( |[u]| \cdot t = 7 \cdot 16 \cdot t = |[w]| \cdot m = |[x_3]| \cdot 16 \cdot m/r = 21 \cdot 16 \cdot m/r \). Therefore \( t \cdot r = 3 \cdot 6 \). If \( r = 1 \) then \( t = 3 \cdot 6 > 6 \). Therefore \( r = 6 \) and \( m = 15 \). Since \( 6 \not\mid 3 \cdot 15 \), we conclude that \( h \neq 15 \). Suppose \( h = 16 \). Then \( 3 \leq t \leq 5 \) and \( |[u]| \cdot t = 5 \cdot 21 \cdot t = 21 \cdot 16 \cdot m/r \). Therefore \( 5 \cdot t \cdot r = 16 \cdot m \). Since \( r = 1 \) or 6, we have \( 8 \mid t \). Since \( t \leq 5 \), we conclude that \( h = 14 \), \( 3 \leq t \leq 7 \) and \( |[u]| \cdot t = 3 \cdot 5 \cdot 8 \cdot t = 21 \cdot 16 \cdot m/r \). Hence \( 5 \cdot t \cdot r = 2 \cdot 7 \cdot m \). If \( m = 20 \) then \( r = 1 \) and \( t = 56 \). Therefore \( m = 15 \), \( r = 6 \) and \( t = 7 \). Thus \( \Gamma \) has the following suborbit diagram:

Let \((x, y)\) be an edge in \( \Gamma \) and \( H = \langle G_x, G_y \rangle \). Then \( H \) is a rank-3 extension of \( G_x \geq PSL(3, 4) \) of degree 1 + 105 + 56 = 162. By [11], \( U_4(3) \) is the unique rank-3 extension of \( PSL(3, 4) \) of degree 162, so \( H \geq U_4(3) \). Therefore \( U_4(3) \leq G \leq \text{Aut}(U_4(3)) \). Note that from the suborbit diagram for \( \Gamma \), \( G \cong H \times Z_2 \). It was shown in [6] that the group \( U_4(3) \) has two conjugacy classes of subgroups isomorphic to \( PSL(3, 4) \) and the two classes are fused by an involutory outer automorphism \( r \). Furthermore, there exist subgroups \( R_i \cong PSL(3, 4) \), \( i = 1, 2 \), belonging in different classes such that \( R_i^r = R_2 \) and \( |R_1 : R_1 \cap R_2| = 21 \). Let \( \tilde{\Gamma} = \Gamma(U_4(3), R_1, R_2) \) be defined as in Lemma 6.1. If \( \tilde{\Gamma} \) has girth greater than 4, then \( |\tilde{\Gamma}| \geq 1 + 21 + 21 + 20 = 442 \). Therefore \( \tilde{\Gamma} \) has girth 4 and hence \( \tilde{\Gamma} \cong \Gamma \). By [3], \( G \cong U_4(3).2_2 \) or \( U_4(3).2_2 \).

10. Proof of Theorem 1.4: The Non-bipartite Case

Suppose \( \Gamma \) is non-bipartite. Then \( \Gamma' \), the bipartite double of \( \Gamma \) defined in Section 6, is bipartite and so \( G \times Z_2 \) must be one of the groups in Section 8 or Section 9. Therefore \( G \) is isomorphic to a quotient of \( Y_{n,q,2} \).

References


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