Fragmentability and cardinal invariants

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Abstract

Several cardinal inequalities and identities concerning fragmentable and sigma-fragmentable spaces are proven. A cardinal function whose countable value in Banach space corresponds to the notion of a WCD space is studied. Some related cardinal functions and classes of Banach spaces are considered. © 2000 Elsevier Science B.V. All rights reserved.

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The notion spaces of countable separation (CS) was introduced by Kenderov and Moors [7,8]. In [7] the authors also introduced and studied the cardinal invariant “separation index” (see Definition 1), whose countable value is equivalent to the fact that the corresponding space has CS.

Let \( X \) be a topological space. A family \( \mathcal{A} \) of subsets of \( X \) is called “open in \( X \)” if it consists of open subsets of \( X \), and “closed in \( X \)” if it consists of closed subsets of \( X \).

**Definition 1** [7]. Let \( X \) be a subspace of the compact Hausdorff space \( Y \). The open in \( Y \) family \( \mathcal{G} \) is called a separating family for \( X \) if for every \( x \in X \) and for every \( y \in Y \setminus X \), there exists \( G \in \mathcal{G} \) such that \( |G \cap \{x, y\}| = 1 \).

The separation index of \( X \) in \( Y \) is the cardinal number

\[
s_i(X) = \mathfrak{c} \min \{|\mathcal{G}| : \mathcal{G} \text{ is an open separating family for } X \text{ in } Y\}.
\]

In [7] it is shown that (when infinite) the separation index of \( X \) does not depend on the space \( Y \), which allows to denote the separation index of \( X \) by \( si(X) \) (which equals \( s_i(X) \) for any \( Y \)).
This cardinal invariant appeared in [7] under the notation $s(X)$. It was used there to prove several theorems; for example, the fact that if the natural embedding of the closed unit ball $B$ of a Banach space $E$ has countable separation index in the second dual ball $(B^{**}, \text{weak}^*)$, then $(E, \text{weak})$ is fragmentable by a metric which majorizes the norm topology, or, equivalently, $E$ is sigma-fragmentable by the norm (see Definitions 10 and 12).

To avoid ambiguity with the spread of $X$, here we preferred the notation $si(X)$ instead of $s(X)$. Furthermore, in [7] the authors allowed finite values for the separation index. It seems practical here to adopt (as it is often done for the cardinal invariants) the convention that $si(X) \geq \aleph_0$ at least because only in this case one can omit the index $Y$ in $si_Y(X)$.

**Definition 2.** Let $X$ be a subspace of the compact Hausdorff space $Y$. The closed in $Y$ family $F$ is called a determining family for $X$ if for every $x \in X$ and for every $y \in Y \setminus X$, there exists $F \in F$ such that $x \in F$ and $y \notin F$.

We define the determination index of $X$ in $Y$ as the cardinal number

$$di_Y(X) = \aleph_0 \min \{|F| : F \text{ is a closed determining family for } X \text{ in } Y\}.$$ 

Thus the weakly countably determined (WCD) Banach spaces (see [12]) are exactly the Banach spaces $E$ whose unit balls, endowed with the weak topology, have countable determination index in the corresponding second dual balls (endowed with the weak* topology).

**Proposition 1.** If $Y$ is a compact Hausdorff space and $X \subset Y$, then

$$di_Y(X) = di_{\overline{Y}}(X),$$

where $\overline{X}$ is the closure of $X$ in $Y$.

**Proof.** Evidently, $di_Y(X) \geq di_{\overline{Y}}(X)$. At the same time, if $F$ is a closed determining family of $X$ in $\overline{Y}$ then the family $F \cup \{\overline{X}\}$ is closed and determining in $Y$, as $\overline{X}$ is closed in $Y$. \qed

**Theorem 1.** Let $Y$ and $Y'$ be compact Hausdorff spaces and let $f : Y \to Y'$ be a continuous surjective mapping. Let $X, X'$ be subspaces of $Y, Y'$, correspondingly, with $f^{-1}(X') = X$. Then $di_Y(X) = di_{Y'}(X')$.

**Proof.** We have a similar approach as that in Theorem 4.2 from [7]. Let $F'$ be a closed determining family of $X'$ in $Y'$. Then

$$F := \{f^{-1}(F') : F' \in F\}$$

is clearly a closed determining family of $X$ in $Y$, so $di_Y(X) \leq di_{Y'}(X')$.

Now let $F$ be a closed determining family of $X$ in $Y$. After adding all the finite intersections of members of $F$ to $F$ (which won’t increase its cardinality) we can consider $F$ closed under taking finite intersections. Now let $F' := \{f(F) : F \in F\}$. The sets $f(F)$ are compact, so it remains to prove that this family is determining. Let $a \in X'$ and
Let $b \in Y' \setminus X'$. Put $A = f^{-1}(a)$ and $B = f^{-1}(b)$. Assume that every set from $\mathcal{F}'$ which contains $a$ contains also $b$. (*)

Let $A = \{F \in \mathcal{F}: F \cap A \neq \emptyset\}$ and $B = \{F \in \mathcal{F}: F \cap B \neq \emptyset\}$. Then $A \subset B$, by our assumption (*). Furthermore, $A \neq \emptyset$. Indeed, if $x \in A \subset X$ and $y \in B \subset Y \setminus X$ then there exists $F \in \mathcal{F}$ containing $x$ and not $y$, so $F \in A$.

Denote by $\Delta$ the set of all subfamilies $C \subset A$ for which $A \cap \bigcap C \neq \emptyset$. For any $C \in \Delta$ and any finite $D \subset C$ the set $\bigcap D$ is in $\mathcal{F}$ and intersects $A$, hence also $B$. The compactness of $B$ implies that $B \cap \bigcap C \neq \emptyset$.

We order $\Delta$ by inclusion. Compactness of $A$ shows that the union of an increasing chain of elements of $\Delta$ is in $\Delta$. Let $\mathcal{E}$ be a maximal element of $\Delta$ guaranteed by the Zorn’s Lemma. Let $u \in A \cap \bigcap E \subset X$ and $v \in B \cap \bigcap E \subset Y \setminus X$. As $\mathcal{F}$ is a determining family, there exists $F \in \mathcal{F}$ containing $u$ and not $v$, so $\{F\} \cup E \in \Delta$. Thus $F \in \mathcal{E}$, a contradiction, because $v \in \bigcap E$ and $v \notin F$. Thus (*) is false. The proof is complete. $\square$

**Remark.** The same proof can be applied under a definition of $\text{di}_Y(X)$ allowing finite values for it. In this case the conclusion is that $\text{di}_Y(X)$ is finite iff $\text{di}_Y(X')$ is so. In fact, it is easy to see that the determination index of $X$ is finite if and only if $X$ is compact, in which case $\text{di}_X(X) = 0$ and $\text{di}_Y(X) = 1$ for $Y \neq X$. Indeed, suppose that $X$ is not compact and that $2 \leq \text{di}_Y(X) < \aleph_0$. As in Proposition 1 we see that $1 \leq \text{di}_X(X) < \aleph_0$. Put $n = \text{di}_X(X)$. Let $F$ be a closed determining family for $X$ in $\overline{X}$ having cardinality $n$. Take $y \in \overline{X} \setminus X$ and a net $(x_\alpha)_\alpha \in A \subset X$ converging to $y$. For every $x_\alpha$ there is some $F \in \mathcal{F}$ containing $x_\alpha$ and not $y$. Then $\{x_\alpha: \alpha \in A\} \subset \bigcup \mathcal{F}$ and the latter is a closed set which does not contain $y$—a contradiction.

Let’s mention, that unlike the determination index, the separation index (when finite values are allowed) may equal any given positive integer (see [9]).

**Corollary 1.** $\text{di}_c(X)$ does not depend on the compactification $cX$ of $X$. $\text{di}_Y(X)$ does not depend on the (compact Hausdorff) space $Y \supset X$.

This corollary allows the following definition.

**Definition 3.** The determination index of a completely regular topological space $X$ is $\text{di}(X) = \text{di}_Y(X)$ for some (hence any) compact Hausdorff $Y \supset X$.

**Remark.** Evidently $\text{si}(X) \leq \text{di}(X)$. Of course, we can speak about $\text{si}(X)$ and $\text{di}(X)$ only when $X$ is completely regular. After this remark, when speaking about them, we are not always going to require the complete regularity explicitly, although we will mean it.

**Corollary 2.** Let $g : X \rightarrow Z$ be a perfect (surjective) mapping. Then $\text{di}(X) = \text{di}(Z)$.

**Proof.** We follow the proof of Corollary 4.1 from [7]. Consider the natural extension $f : \beta X \rightarrow \beta Z$ to the Čech–Stone compactification of $X$ and see that $f^{-1}(Z) = X$. Indeed, let $x \in f^{-1}(z)$, $z \in Z$. Take a net $(x_F)_F$ in $X$ converging to $x$ in $\beta X$. Then $g(x_F) = f(x_F)$.
of course converges to $z$. But $g$ is perfect so $(x_y)_y$ has a cluster point $x_0 \in g^{-1}(z)$, and it necessarily coincides with $x$. Thus $f^{-1}(Z) = X$, so the last theorem applies. \hfill \square

**Definition 4** (see, e.g., [1, 3.1.17]). A family $\mathcal{A}$ of subsets of $X$ is called a net if for every open $U \subset X$ and every $x \in U$ one can find $A \in \mathcal{A}$ with $x \in A \subset U$.

The net weight of $X$ is the cardinal number
$$nw(X) = \aleph_0 \min \{ |A| : \mathcal{A} \text{ is a net in } X \}.$$  

**Proposition 2.** For any space $X$, $di(X) \leq nw(X)$.

**Proof.** Let $\mathcal{A}$ be a net in $X$ and let $Y$ be a compact Hausdorff space containing $X$ as a subspace. Put $\mathcal{F} = \{ \overline{A} : A \in \mathcal{A}\}$, where the closure is taken in $Y$. This is a determining family for $X$: indeed, if $x \in X$ and $y \in Y \setminus X$, let $U \ni x$ and $V \ni y$ be disjoint open (in $Y$) sets. Let $A \in \mathcal{A}$: $x \in A \subset U$. Then $\overline{A} \cap V = \emptyset$ and $\overline{A} \in \mathcal{F}$ contains $x$ and not $y$. Furthermore, $\mathcal{F}$ has the cardinality of $\mathcal{A}$. \hfill \square

The following version of the last result can be considered as a generalization of Proposition 4.1 in [7] in three senses: first, it considers arbitrary cardinalities instead of $\aleph_0$. Second, it contains on the left-hand side a greater cardinal function (namely, $di(X)$ instead of $si(X)$). Third, it uses a much weaker assumption about the space $Z$ ($Z$ being there a separable metric space). The theorem can be also obtained from the last proposition, using the fact that the net weight does not increase under continuous mappings.

**Theorem 2.** Let the space $X$ be the continuous image of some space $Z$. Then $di(X) \leq nw(Z)$.

**Proof.** Let $nw(Z) = \kappa$ and let $\mathcal{A}$ be a net in $Z$ of cardinality $\kappa$. Take $Y$ to be a compact Hausdorff space containing $X$ as a subspace, and let $f : Z \to X$ be a continuous surjection. Put $\mathcal{F} = \{ f(A) : A \in \mathcal{A}\}$, where the closure is taken in $Y$. We prove that $\mathcal{F}$ is a determining family for $X$ in $Y$ (of course $|\mathcal{F}| = \kappa$).

Indeed, if $x \in X$ and $y \in Y \setminus X$, let $U \ni x$ be an open (in $Y$) set with $y \notin \overline{U}$. Take $z \in f^{-1}(x)$ and let $W \ni z$ be an open subset of $Z$ with $f(W) \subset U$. Take $A \in \mathcal{A}$: $z \in A \subset W$. Then $x \in f(A) \subset \overline{U}$, so $y \notin f(A)$ and $f(A) \in \mathcal{F}$ contains $x$ and not $y$. \hfill \square

**Definition 5** (see [1, 3.9.E]). For any topological space $X$,
$$g(X) = \aleph_0 \min \{ \kappa : \text{there exist compact } Y \ni X \text{ and open family } \mathcal{U} \text{ in } Y, |\mathcal{U}| \leq \kappa,$$
$$\text{such that if } x \in X \text{ and } y \in Y \setminus X \text{ then}$$
$$\text{for some } U \in \mathcal{U}: x \in U, \ y \notin U \}.$$  

For convenience we will call a family $\mathcal{U}$ with the given property a $g$-separating family for $X$ in $Y$.

**Proposition 3.** For any space $X$, $si(X) \leq g(X)$. 

Proof. Right from the definitions. \(\square\)

It is natural to ask whether the definition of \(g(X)\) depends on the compact space \(Y\) (we will use only Hausdorff compacts \(Y\) and completely regular spaces \(X\)). To answer this question we introduce the following cardinal function.

**Definition 6.** For any compact Hausdorff space \(Y\) and any \(X \subseteq Y\),

\[
g_Y(X) = \aleph_0 \min \{ \kappa : \text{there exists an open family } \mathcal{U} \text{ in } Y, |\mathcal{U}| \leq \kappa, \text{ such that if } x \in X \text{ and } y \in Y \setminus X \text{ then for some } U \in \mathcal{U}: x \in U, y \notin U \}. \]

It may happen that \(g_Y(X) \neq g_Z(X)\): take \(Z\) to be any compact of uncountable pseudo-character (e.g., \([0, \omega_1]\) and \(X = Y = \{x\}\), where \(x\) is any non-\(G_\delta\) point of \(Z\). It is also obvious that if \(Y\) is a compact Hausdorff space and \(X \subseteq Y\), then

\[
g_Y(X) \geq g_X(Y) X. \]

where \(\overline{X}\) is the closure of \(X\) in \(Y\). Thus in the case when \(X\) is completely regular,

\[
g(X) = \aleph_0 \min \{ \kappa : \text{there exist a compactification } cX \text{ of } X \}
\]

and an open family \(\mathcal{U} \) in \(cX\), \(|\mathcal{U}| \leq \kappa\),

such that if \(x \in X\) and \(y \in cX \setminus X\) then for some \(U \in \mathcal{U}: x \in U, y \notin U \}.

Now we can use the proof of Theorem 1 to prove the following:

**Proposition 4.** Let \(Y\) and \(Y'\) be compact Hausdorff spaces and let \(f: Y \to Y'\) be a continuous surjective mapping. Let \(X, X'\) be subspaces of \(Y, Y'\), correspondingly, with \(f^{-1}(X') = X\). Then \(g_Y(X) = g_{Y'}(X')\).

**Proof.** Let \(\mathcal{U}'\) be an open \(g\)-separating family of \(X'\) in \(Y'\). Then \(\mathcal{U} := \{ f^{-1}(U'): U' \in \mathcal{U}' \}\) is clearly an open \(g\)-separating family of \(X\) in \(Y\), so \(g_Y(X) \leq g_{Y'}(X')\).

Now let \(\mathcal{U}\) be an open \(g\)-separating family of \(X\) in \(Y\). Define \(\mathcal{F} = \{ Y \setminus U : U \in \mathcal{U} \}\), which is a closed determining family for \(Y \setminus X\) in \(Y\). Now the argument of Theorem 1 shows that

\[
di_{Y'}(Y' \setminus X) \leq |\mathcal{F}| = |\mathcal{U}| \]

and thus \(g_Y(X) \geq g_{Y'}(X')\). \(\square\)

**Remark.** The same proof can be applied under a definition of \(g_Y(X)\) allowing finite values for it. In this case the conclusion is that \(g_Y(X)\) is finite iff \(g_{Y'}(X')\) is so.

**Corollary 3.** \(g_{cX}(X)\) does not depend on the compactification \(cX\) of \(X\).

This corollary shows that if \(cX\) is any compactification of the completely regular space \(X\), we have
\[ g(X) = \aleph_0 \min \{ \kappa : \text{there exists an open family } \mathcal{U} \text{ in } cX, |\mathcal{U}| \leq \kappa, \]

such that if \( x \in X \) and \( y \in cX \setminus X \) then

for some \( U \in \mathcal{U} : x \in U, \ y \notin U \].

We can use the argument from Corollary 2 to prove the following:

**Corollary 4.** Let \( f : X \to Z \) be a perfect mapping. Then \( g(X) = g(Z) \).

**Definition 7** (see [1, 3.8.12.], or [5]). The Lindelöf number of \( X \) is the cardinal invariant

\[ L(X) = \aleph_0 \min \{ \kappa : \text{every open cover of } X \text{ has a subcover of cardinality } \leq \kappa \} \]

The hereditary Lindelöf number of \( X \) is

\[ hL(X) = \sup \{ L(Y) : Y \subset X \} \]

Sometimes \( hL(X) \) is denoted by \( h(X) \) and called “height of \( X \).”

A space is Lindelöf if \( L(X) = \aleph_0 \) and hereditary Lindelöf if \( hL(X) = \aleph_0 \).

**Theorem 3.** For any space \( X \), \( \text{di}(X) \leq \text{si}(X) \cdot hL(X) \).

**Proof.** Let \( \text{si}(X) \cdot hL(X) = \kappa \) and let \( \mathcal{G} \) be a separating family (for \( X \) in some compact Hausdorff \( Y \)) of cardinality \( \leq \kappa \). From now on the open sets and the closures will be considered with respect to \( Y \). Let \( G \in \mathcal{G} \). Then for every \( x \in X \cap G \) there exists some open \( U_x \) such that \( x \in U_x \subset \overline{U_x} \subset G \). As \( hL(X) \leq \kappa \) we can choose \( I_G \) with \( |I_G| \leq \kappa \) with

\[ X \cap G \subset \bigcup_{i \in I_G} U_{x_i} \subset \bigcup_{i \in I_G} \overline{U_{x_i}} \subset G. \]

Let now

\[ \mathcal{F}_G := \{ \overline{U_{x_i}} : i \in I_G \} \quad \text{and} \quad \mathcal{F} := \{ Y \setminus G : G \in \mathcal{G} \} \cup \bigcup_{G \in \mathcal{G}} \mathcal{F}_G. \]

This is a closed family of cardinality \( \leq \kappa \). Now if \( x \in X \) and \( y \in Y \setminus X \) then \( \exists G \in \mathcal{G} \) such that \( |G \cap \{ x, y \}| = 1 \). Then, if \( y \in G, \ Y \setminus G \in \mathcal{F} \) contains \( x \) and not \( y \); inversely, if \( x \in G \) then

\[ x \in X \cap G \subset \bigcup_{i \in I_G} \overline{U_{x_i}} \subset G \not\ni y \]

that is, \( \exists U_{x_i} \in \mathcal{F}_G \subset \mathcal{F} \) containing \( x \) and not \( y \). \( \square \)

**Corollary 5.** In a hereditary Lindelöf space \( X \), \( \text{si}(X) = \text{di}(X) \).

**Remark.** If \( X \) is a Banach space with the weak topology, then one might be tempted to infer the equivalence of several conditions about \( X \) if it is hereditary Lindelöf. Namely, the following are equivalent:

(a) The unit ball of \( X \) has countable separation index in its second dual ball with the weak*-topology.
(b) $X$ is weakly countably determined.
(c) $X$ admits an equivalent LUR norm.
(d) $X$ admits an equivalent Kadec norm.
(e) $X$ is weakly Čech analytic.

The equivalence follows from the last corollary and from the known implications about these notions. But in fact, as we shall see later, this is a consequence from another stronger assertion (see the last statement in Corollary 12), which shows also the equivalence of these conditions (under the hypothesis that $X$ is hereditary Lindelöf) with each of the following:

(f) $X$ is separable.
(g) $X$ is WCG.
(h) $X$ is weakly $k$-analytic.
(i) $X$ is sigma-fragmentable.

**Proposition 5.** For any topological space $X$, $L(X) \leq \text{di}(X)$.

**Proof.** Suppose $\text{di}(X) = \kappa$. Let $Y$ be a compact space containing $X$ as a subspace, and let $\mathcal{F} = \{F_\alpha\}_{\alpha \leq \kappa}$ be a determining family for $X$ in $Y$. After adding some new sets (without increasing the cardinality of $\mathcal{F}$) we can assume that $\mathcal{F}$ has the finite intersection property. Take some open (in $Y$) cover $U = \{U_\beta\}_{\beta \leq \mu}$ of $X$. Take any $x \in X$. For every $y \in Y \setminus X$, there is some $\alpha_y$ such that $x \in F_{\alpha_y}$ and $y \notin F_{\alpha_y}$. Then

$$x \in \bigcap_{y \in Y \setminus X} F_{\alpha_y} \subset X \subset \bigcup U$$

so ($\mathcal{F}$ being centered) there is some $F_{\alpha_x} \in \mathcal{F}$ containing $x$ and included in $\bigcup U$. As $F_{\alpha_x}$ is compact, let $U_{\alpha_x}$ be the finite subfamily of $U$ whose union covers $F_{\alpha_x}$. Then, of course, $|\{F \in \mathcal{F} : F = F_{\alpha_x} \setminus x \in X\}| \leq \kappa$, so $|\bigcup U_{\alpha_x} : x \in X| \leq \kappa$ and thus $U$ has a subcover of $X$ having cardinality $\leq \kappa$. We conclude that $L(X) \leq \text{di}(X)$.

We remind that a topological space is called $T_\kappa$-space, if it is normal $T_1$ space which is also perfect in the sense that each open set in it is $F_\sigma$. It is known that in a $T_\kappa$-space, $hL(X) = L(X)$ (see, e.g., [5] for even a more general result). Now we have the following:

**Corollary 6.** If $X$ is a $T_\kappa$-space, then $\text{di}(X) = \text{si}(X) \cdot L(X)$.

**Proof.** We have $L(X) = hL(X)$, hence $\text{di}(X) \leq \text{si}(X) \cdot hL(X) = \text{si}(X) \cdot L(X) \leq \text{di}(X)$.

**Definition 8.** Let $X$ be a completely regular topological space. We introduce the hereditary versions of $\text{si}(X)$ and $\text{di}(X)$ in the usual manner. Namely, the hereditary separation index of $X$ is

$$h\text{si}(X) = \sup \{\text{si}(Y) : Y \subset X\},$$

and the hereditary determination index of $X$ is

$$h\text{di}(X) = \sup \{\text{di}(Y) : Y \subset X\}.$$
Corollary 7. For any completely regular topological space $X$, 
\[
\text{hdi}(X) = \text{hsi}(X) \cdot \text{hL}(X).
\]

Proof. By Theorem 3 and the last proposition, we have 
\[
\text{hL}(X) \leq \text{hdi}(X) \leq \text{hsi}(X) \cdot \text{hL}(X),
\]
and of course $\text{hsi}(X) \leq \text{hdi}(X)$. □

Example 1. The discrete space $D$ of cardinality $\mu$ has $\text{si}(D) = \text{hsi}(D) = g(D) = \aleph_\mu$ ($D$ is locally compact), but $\text{di}(D) = \text{hdi}(D) = L(D) = \text{hL}(D) = \mu$.

Definition 9 [1]. The Souslin number (or cellularity) of $X$ is the cardinal invariant 
\[
c(X) = \aleph_\omega \sup \{ |U| : U \text{ is a disjoint open family in } X \}.
\]

Corollary 8. If the space $X$ is hereditary paracompact or if its topology is generated by a linear order, then $\text{di}(X) \leq \text{si}(X) \cdot c(X)$.

Proof. In a hereditary paracompact space $X$, $\text{hL}(X) = c(X)$ [5]; the same conclusion is true in a linearly ordered space (see [1, 3.12.4]). □

Proposition 6. For any set $\{X_\alpha : \alpha \in A\}$ of completely regular spaces, 
\[
\text{si}\left( \prod_{\alpha \in A} X_\alpha \right) \leq |\alpha \in A : X_\alpha \text{ non-compact}| \cdot \sup \{ \text{si}(X_\alpha) : \alpha \in A \}.
\]

Proof. Put $B = \{ \alpha \in A : X_\alpha \text{ non-compact} \}$. Let $\kappa = |B| \cdot \sup \{ \text{si}(X_\alpha) : \alpha \in A \}$. For any $\alpha \in B$, suppose $Y_\alpha \supset X_\alpha$ be compact and let $\mathcal{U}_\alpha$ be separating open family for $X_\alpha$ in $Y_\alpha$ having cardinality $\leq \kappa$. For $\alpha \in A \setminus B$, let $Y_\alpha = X_\alpha$. Now $\prod_{\alpha \in A} Y_\alpha$ is a compactification of $\prod_{\alpha \in A} X_\alpha$. Pose 
\[
\mathcal{U} = \{ p_\alpha^{-1}(U) : U \in \mathcal{U}_\alpha, \alpha \in B \}
\]
(here $p_\alpha$ is the projection onto the $\alpha$th factor of $\prod_{\alpha \in A} Y_\alpha$). The cardinality of $\mathcal{U}$ is $\leq \kappa \cdot \kappa = \kappa$. If $\alpha \in \prod_{\alpha \in A} X_\alpha$ and $y \in \prod_{\alpha \in A} Y_\alpha \setminus \prod_{\alpha \in A} X_\alpha$, let $\beta \in A$: $y_\beta \in Y_\beta \setminus X_\beta$. Of course, $\beta \in B$. Then there is $U \in \mathcal{U}_\beta$ which separates $x_\beta$ from $y_\beta$. Then $p_\alpha^{-1}(U) \in \mathcal{U}$ separates $x$ and $y$. Thus $\mathcal{U}$ is a separating family for $\prod_{\alpha \in A} X_\alpha$ in $\prod_{\alpha \in A} Y_\alpha$. The proof is completed. □

Proposition 7. For any set $\{X_\alpha : \alpha \in A\}$ of completely regular spaces, 
\[
\text{di}\left( \prod_{\alpha \in A} X_\alpha \right) \leq |\alpha \in A : X_\alpha \text{ non-compact}| \cdot \sup \{ \text{di}(X_\alpha) : \alpha \in A \}.
\]

Proof. Let again $B \subseteq A$ be the set of non-compact factors. Like before, 
\[
\kappa = |B| \cdot \sup \{ \text{si}(X_\alpha) : \alpha \in A \},
\]

$Y_{\alpha} \supset X_{\alpha}$ is compact and for any $\alpha \in B$, let $\mathcal{F}_{\alpha}$ is a determining closed family for $X_{\alpha}$ in $Y_{\alpha}$ having cardinality $\leq \kappa$. For $\alpha \in A \setminus B$, let $Y_{\alpha} = X_{\alpha}$. Now $\prod_{\alpha \in A} Y_{\alpha}$ is a compactification of $\prod_{\alpha \in A} X_{\alpha}$. Pose

$$\mathcal{F} = \{ p_{\alpha}^{-1}(F) : F \in \mathcal{F}_{\alpha}, \alpha \in B \}$$

(here $p_{\alpha}$ is again the projection onto the $\alpha$th factor of $\prod_{\alpha \in A} Y_{\alpha}$). The cardinality of $\mathcal{F}$ is $\leq \kappa$. If $x \in \prod_{\alpha \in A} X_{\alpha}$ and $y \in \prod_{\alpha \in A} Y_{\alpha} \setminus \prod_{\alpha \in A} X_{\alpha}$, let $\beta \in B$: $y_{\beta} \in Y_{\beta} \setminus X_{\beta}$. Then there is $F \in \mathcal{F}_{\beta}$ containing $x_{\beta}$ and not $y_{\beta}$. Then $p_{\beta}^{-1}(F) \in \mathcal{F}$ contains $x$ and not $y$. Thus $\mathcal{F}$ is a determining family for $\prod_{\alpha \in A} X_{\alpha}$ in $\prod_{\alpha \in A} Y_{\alpha}$.

**Remark 1.** In view of Proposition 5, the last proposition gives an upper bound of the Lindelöf number of a product by the determination index of its factors.

**Example 2.** Let $K$ be the Sorgenfrey line and $D$ be its second diagonal (a closed subspace of $K^2$). We have $\exp(\mathfrak{N}_{\alpha}) = |D| = L(D) \leq L(K^2) \leq |K^2| = \exp(\mathfrak{N}_{\alpha})$ so $di(K) = \exp(\mathfrak{N}_{\alpha})$ and as $K$ is hereditary Lindelöf, $st(K) = \exp(\mathfrak{N}_{\alpha})$.

**Remark 2.** We know that both $hL(X)$ and $di(X)$ lie in the interval $[L(X), mw(X)]$ and it may be asked whether some of the functions $hL(X)$ and $di(X)$ majorizes the other one. The last example together with the example $[0, \mu]$ (for some ordinal $\mu$) shows that this is not so.

**Definition 10** [2]. The topological space $X$ is called fragmentable by a metric $d$ if for every $\varepsilon > 0$, every subset of $X$ has a relatively open subset of $\rho$-diameter $< \varepsilon$.

**Theorem 4.** If $X$ is fragmentable then $|X| \leq hL(X)^{\mathfrak{N}_{\alpha}}$.

**Proof.** Let $X$ be fragmented by the metric $d$ and let $hL(X) = \chi$. We will construct a sequence $(U_{\alpha})_{\alpha \geq 1}$ of covers of $X$,

$$U_{\alpha} = \left\{ U_{\alpha}^{\xi} : 0 \leq \xi < \mu_{\alpha} \right\},$$

and a sequence $R_{n}$ of sets such that

1. $\bigcup\{ U_{\alpha}^{\xi} : 0 \leq \xi < \beta \}$ is open in $X$ for each $\beta \leq \mu_{n}$ and each $\xi \in \mathbb{N}$,
2. $\rho\text{-diam}(U_{\alpha}^{\xi}) \leq n^{-1}$ for each $\alpha \in \mathbb{N}$ and each $\xi < \mu_{n}$,
3. $R_{n} = \{ r_{n}^{\xi} : 0 \leq \xi < \mu_{n} \}$,
4. $r_{n}^{\xi} \in U_{\alpha}^{\xi}$,
5. $U_{n}^{\xi} \cap U_{n}^{\eta} = \emptyset$ for $0 \leq \xi < \eta < \mu_{n}$.

Let $n \in \mathbb{N}$ be arbitrary. We use transfinite induction on the ordinal $\beta$. Suppose that $U_{\alpha}^{\xi}$ are already constructed for $0 \leq \xi < \beta$ for some ordinal $\beta$ in such a way that the properties (1), (2), (4) and (5) are satisfied for $\xi < \beta$.

(a) If $\bigcup\{ U_{\alpha}^{\xi} : 0 \leq \xi < \beta \} = X$ we put $\mu_{n} = \beta$,

$$U_{\alpha} = \left\{ U_{\alpha}^{\xi} : 0 \leq \xi < \mu_{n} \right\},$$

$$R_{n} = \{ r_{n}^{\xi} : 0 \leq \xi < \mu_{n} \}$$

and finish the construction for $n$ (thus the properties (1)–(5) are satisfied).
We assert that the set $U_j \cap n.x/n$ is some fragmentable by any metric. They have cardinality $2^n$ having for a base $f_{\sup n.x/n}$ some open set $V$ containing $x$. We obtain an injective mapping $q: X \to \mathcal{X}$ by means of $q(x) = (\xi_n(x))_{n \geq 1}$. Thus $|X| \leq \mathcal{X}$.

**Example.** The Tychonoff cube $[0, 1]^{\omega_1}$, the Cantor cube $[0, 1]^{\omega_1}$ and $\beta \mathbb{N}$ are not fragmentable by any metric. They have cardinality $2^{\omega_1}$ and for them $hL(X) \leq w(X) = 2^{\omega_1}$, so $|X| > hL(X)^{\omega_1}$.

**Remark.** The cardinality of a fragmentable space cannot be majorized in terms of the Lindelöf number instead of the hereditary version. Indeed, look at any scattered compact segment of ordinals $[O, \mu]$.

**Theorem 5.** Let $X$ be a set and $t_1, t_2$ be two (not necessarily distinct) topologies on it. If the topological space $(X, t_1)$ is fragmented by a metric $d$ whose topology majorizes $t_2$, then $nw(X, t_2) \leq hL(X, t_1)$.

**Proof.** Let $hL(X, t_1) = \chi$. We again construct a sequence $(U_n)_{n \geq 1}$ of covers of $X$,

$$U_n = \{U^\xi_n: 0 \leq \xi < \mu_n\},$$

and a sequence $R_n$ of sets such that

1. $\bigcup\{U^\xi_n: 0 \leq \xi < \beta\}$ is open in $X$ for each $\beta \leq \mu_n$ and each $n \in \mathbb{N}$,
2. $d$-diam$(U^\xi_n) \leq n^{-1}$ for each $n \in \mathbb{N}$ and each $\xi < \mu_n$,
3. $R_n = \{U^\xi_n: 0 \leq \xi < \mu_n\}$,
4. $r^\xi_n \in U^\xi_n$,
5. $U^\xi_n \cap U^\eta_n = \emptyset$ for $0 \leq \eta < \xi < \mu_n$.

The construction is the same as in the last theorem and for the same reasons $\mu_n \leq \chi$.

We assert that the set $\mathcal{U} = \bigcup_{n \in \mathbb{N}} U_n$ is a net for the $t_2$-topology in $X$. Take some $x \in X$ and some $t_2$-open set $V$ containing $x$. Then as $d$ defines a topology majorizing $t_2$, there is some $n$ such the ball $B_d(x, n^{-1}) \subset V$. Let $x \in U_n \in \mathcal{U}_n$. Then by the property (2), $U_n \subset B_d(x, n^{-1}) \subset V$. But $|\mathcal{U}| \leq \chi$ which finishes the proof.

**Remark.** The assumption that $d$ majorizes $t_2$ in the theorem is important, as seen from the next example.

**Example 3.** Take $X$ to be the real line and both $t_1$ and $t_2$ to be the Hausdorff topology on the real line, but the corresponding topology is strictly coarser than $t$. 

If $A = X \setminus \bigcup\{U^\xi_n: 0 \leq \xi < \beta\} \neq \emptyset$ we use the fragmentability of $X$ to take a non-empty relatively open subset $U^\beta_n$ of $A$ which has $d$-diameter less than $n^{-1}$. Then we fix $r^\beta_n \in U^\beta_n$. Thus the properties (1), (2), (4) and (5) are satisfied for $\xi \leq \beta$.

This finishes the induction. As $\bigcup\{U^\eta_n: \eta < \xi\}$ is an open subset of $X$ for every $\xi \leq \mu_n$, we conclude that the set $R_n$ is right-separated in the sense of [5, Definition 1.10]. According to [5], $hL(X) = \sup\{|R|: R$ is a right-separated subspace of $X\}$, so $\mu_n \leq \chi$.

Fix $x \in X$ and put $U^\xi_n(x)$ to be the element of $U_n$ containing $x$. By property (2) we have $\bigcap_{n \geq 0} U^\xi_n(x) = \{x\}$. We obtain a injective mapping $q: X \to \mathcal{X}$ by means of $q(x) = (\xi_n(x))_{n \geq 1}$. Thus $|X| \leq \mathcal{X}$.

□
**Corollary 9.** If the topological space $X$ is fragmentable by a metric stronger than its topology, then $\text{nw}(X) = hL(X)$. If, furthermore, $X$ is completely regular, then $\text{nw}(X) = hdi(X)$.

**Proof.** From Theorem 5, $\text{nw}(X) \leq hL(X)$, and $\text{nw}(X) \geq hL(X)$ is always true. If $X$ is completely regular, by Propositions 2 and 5, $L(X) \leq \text{di}(X) \leq \text{nw}(X)$, and the net weight of course does not increase in subspaces, so $hL(X) \leq hdi(X) \leq \text{nw}(X)$. □

**Definition 11** (see [1, 1.3.7 and 2.7.9], or [5]). The density of $X$ is the cardinal invariant
\[
d(X) = \aleph_0 \min \{|A| : A \text{ is dense in } X\}.
\]
The hereditary density of $X$ is $hd(X) = \sup\{d(Y) : Y \subset X\}$. Sometimes $hd(X)$ is denoted by $z(X)$ and called “width of $X$”.

**Corollary 10.** If the topological space $X$ is fragmentable by a metric stronger than its topology, and is either hereditary paracompact or its topology is generated by linear order, then
\[
\text{nw}(X) = hd(X) = d(X) = c(X) = hL(X).
\]

**Proof.** $\text{nw}(X) \geq hd(X) \geq d(X) \geq c(X)$ is always true. From the proof of Corollary 8, $c(X) = hL(X)$, and $hL(X) = \text{nw}(X)$ by Corollary 9. □

**Remark.** The additional assumptions in the last corollary cannot be both omitted. The Sorgenfrey line $K$ is hereditary paracompact (even hereditary Lindelöf) and its net weight equals the cardinality of the continuum. $K$ is fragmented by the usual metric $d$ on the real line, but the $d$-topology is strictly coarser than the topology of $K$.

The topology of the “Double arrow space” $D$ is defined by a linear order, $hL(D) = \aleph_0$ and $hdi(D) = hsi(D) = \text{nw}(D) = \exp(\aleph_0)$. By [4], $D$ is not fragmented by any metric.

Now we give some application for Banach spaces.

**Definition 12** [3]. The Banach space $X$ is called sigma-fragmentable if for every $\varepsilon > 0$, $X$ can be partitioned into countable family of subsets $(X_i)_{i \in \omega}$ such that every subset of $X_i$ has a relatively weakly open subset of norm-diameter $< \varepsilon$.

**Corollary 11.** If the Banach space $E$ is sigma-fragmentable, then
\[
\text{nw}(E, \text{weak}) = hL(E, \text{weak}) = hdi(E, \text{weak}).
\]

**Proof.** By Theorem 0.4 from [8], $E$ is sigma-fragmentable if and only if its weak topology can be fragmented by a metric, stronger than the weak topology. We apply Corollary 9 here with both $t_1$ and $t_2$ being the weak topology on $E$. □

We remind that the weight of a topological space $X$ is denoted by $w(X)$. 
Corollary 12. If the Banach space $E$ is sigma-fragmentable, then

$$w(E, \text{norm}) = hL(E, \text{weak}).$$

In particular, if in this case $(E, \text{weak})$ is hereditary Lindelöf, then $E$ is separable.

Proof. By Theorem 0.4 from [8], $E$ is sigma-fragmentable if and only if its weak topology can be fragmented by a metric, stronger than the norm topology. We apply Theorem 5 here with $t_1$ being the weak topology and $t_2$ the norm topology to get

$$nw(E, \text{norm}) \leq hL(E, \text{weak}).$$

By the last corollary, $hL(E, \text{weak}) = nw(E, \text{weak})$ and the latter is majorized by $nw(E, \text{norm})$ (by the comparison of the topologies). The norm topology is a metric one, so we get the desired result.

Remark. Another proof of the second statement in Corollary 12 can be found in [10]; it can as well be derived from the previous corollary. In fact, there is another proof of the whole corollary, which uses the approach in [10].

Second proof. We first prove that $d(E, \text{norm}) \leq hL(E, \text{weak})$. Let $hL(X, \text{weak}) = \chi$ and suppose that $d(E, \text{norm}) > \chi$. Then there is some $\varepsilon > O$ such that there is no $\varepsilon$-net in $E$ of cardinality $\leq \chi$. Using transfinite induction, construct subset $D$ of $E$ having cardinality $\chi^+$, such that every two distinct points in $D$ lie at a distance $\geq \varepsilon$. Let $(X_t)_{t \in \omega}$ be a partition of $X$ such that every subset of $X_t$ has a relatively weakly open subset of norm-diameter $< \varepsilon$. Put $D_t = D \cap X_t$. There is then some $D_n$ of cardinality $\chi^+$. If no topology is mentioned, in this proof we mean the weak topology. Let $A$ be the set of points

$$A = \{x \in D_n : x \text{ has a neighborhood in } D_n \text{ of cardinality } \leq \chi \}.$$ 

The points in $D_n \setminus A$ are not isolated: indeed, suppose $x \in D_n \setminus A$ and $W$ a neighborhood of $x$ in $D_n$ such that $W \setminus A = \{x\}$. Then for every $y \in W \setminus \{x\}$, there is some neighborhood $U_y$ in $W$ of cardinality $\leq \chi$. Now we use the fact that $hL(E, \text{weak}) \leq \chi$ to conclude that $W$ has cardinality $\leq \chi$, so we get the contradiction $x \in A$.

The points in $D_n \cap A$ are at most $\chi$: each of them has a neighborhood in $D_n$ of cardinality $\leq \chi$ and $hL(E, \text{weak}) \leq \chi$. Thus $D_n \setminus A \neq \emptyset$ has no weakly isolated points and by its nature it has no weakly open subset of norm-diameter $< \varepsilon$. This contradiction shows that $d(E, \text{norm}) \leq hL(E, \text{weak})$. Now $w(E, \text{norm}) \leq hL(E, \text{weak})$ is immediate. □

Remark. Corollary 12 can be obtained from Corollary 11 (and vice versa) by virtue of the following easy fact:

Proposition 8. For any Banach space $E$,

$$w(E, \text{norm}) = nw(E, \text{weak}) = hd(E, \text{weak}) = d(E, \text{weak}).$$

Proof. Any net for the norm topology is also a net for the weak one. Thus we have

$$d(E, \text{norm}) = w(E, \text{norm}) = nw(E, \text{norm})$$

$$\geq nw(E, \text{weak}) \geq hd(E, \text{weak}) \geq d(E, \text{weak}).$$
Now let $D$ be weakly dense subset of $E$ and let $C$ be the set of all finite rational linear combinations of the members of $D$. We have $|C| = |D|$ and we show that $C$ is norm-dense in $E$. Indeed, take the norm-closure $K$ of $C$ (it is convex). If $B$ is some ball outside $K$, then $K$ and $B$ can be separated by some functional from $E^*$ which contradicts the fact that $D$ is weakly dense.

Remark. Corollaries 11 and 12 cannot be proved (in ZFC) without the sigma-fragmentability assumption. Indeed, assume CH and use the example constructed in [10], Theorem 1.1, with uncountable $K$. The space $C(K)$ is then hereditary Lindelöf, but is not separable.

Definition 13. The spread of a topological space $X$ is the cardinal number

$$s(X) = \aleph_0 \cdot \sup \{|D|: \ D \subset X \text{ is discrete in the inherited topology}\}.$$ 

Proposition 9. For any Banach space $E$ having a Kadec norm,

$$w(E, \text{norm}) = s(E, \text{weak}) = hd(E, \text{weak}) = nw(E, \text{weak}).$$

Proof. Let $S$ be the unit sphere of the Kadec norm of $E$. We have

$$w(E, \text{norm}) = d(E, \text{norm}) \leq d(S, \text{norm}) = s(S, \text{norm})$$

$$= s(S, \text{weak}) \leq s(E, \text{weak}) \leq hd(E, \text{weak})$$

$$\leq nw(E, \text{weak}) \leq nw(E, \text{norm}) = w(E, \text{norm}).$$

Remark. This corollary cannot be proved (in ZFC) without the assumption of the Kadec renormability. This is seen again from the example mentioned in the last remark.

References