Nonlinear Feedback Systems and Weakly Stationary Stochastic Processes

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In this paper an $L^2$-stability condition is derived for a feedback system consisting of a nonlinear element $f$ and a linear element $g$ which involves the delta functional and its derivatives. An attempt is made to calculate the autocorrelation function of the output process when the input of the system is assumed to be a second order stationary stochastic process. Moreover, it is shown that the autocorrelation function of the output process is bounded when the input is an ergodic stationary stochastic process.

1. INTRODUCTION

Consider the feedback system illustrated in Fig. 1, where $f$ is a memoryless nonlinear element and $g$ is a convolution operator on integrable functions with bounded supports, defined by a function in $L^1_{(0,\infty)}$. In recent years $L^2$-stability conditions for such a system have been derived in terms of the frequency response $G(i\omega)$ of $g$. In this paper we shall show that a similar result on $L^2$-stability holds when $g$ involves, in addition, the delta functional and its derivatives.

The problem of $L^2$-stability has been treated by Sandberg (1964) and Zames (1964) as an extension of Popov's stability conditions (Popov, 1962). Although there are many papers related to the stability of nonlinear feedback
systems (e.g., Freedman, 1968; Zames and Falb, 1968), still it seems to me that no paper has dealt with the case in which the input is assumed to be a weakly stationary stochastic process, except a paper by Holtzman (1968).

In Section 2 we take up the problem in terms of the distribution theory. In Section 3 we give a lemma on the inversion of convolution operators using the distributional Fourier and Laplace transformations. In Section 4 we shall give the existence and uniqueness of the solution of the equation concerning the system, applying the results of Section 3. In Section 5 we consider the case in which the input is supposed to be a weakly stationary stochastic process. Moreover, we shall show that the autocorrelation function of the output is bounded if we assume the ergodicity of the input. Finally, we make some concluding remarks in Section 6.

2. STATEMENT OF THE PROBLEM

The purpose of this section is to give the conditions satisfied by \( f \) and \( g \), and to describe the feedback equations.

We first define some classes of functions and distributions.

A real-valued continuous function \( f(\cdot) \) is said to be of class \( \mathcal{N}(\alpha, \beta) \) if,

(i) \( f(0) = 0 \),

(ii) there are two real numbers \( \alpha, \beta \) such that

\[
\frac{f(\alpha) - f(\beta)}{\alpha - \beta} < \gamma, \quad (\alpha < \beta, \quad \gamma > 0),
\]

for all \( \alpha, \beta \) with \( \alpha \neq \beta \).

Let \( L^p_{loc}(\mathbb{R}) \), where \( p = 1, 2 \), be the class of Lebesgue measurable functions \( x(t) \) which vanish for negative arguments satisfying \( \int_{-T}^{T} |x(t)|^p \, dt < \infty \), for any finite \( T \).

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A real-valued distribution \( g \) is said to be of class \( \mathcal{F}_{n'} \) if it has the following form.

\[
g(t) = \sum_{k=1}^{n} a_k \delta^{(k-1)}(t) + g_1(t), \quad (1 \leq n < \infty),
\]

where \( a_k \in \mathbb{R}^1, \quad a_n \neq 0, \quad g_1(t) \in L^1(\mathbb{R}) \) and \( \delta^{(k-1)}(t) \) is the \((k-1)\)-th derivatives of the delta functional. Moreover, \( g \) is said to be of class \( \mathcal{F}_0' \) if \( n = 1 \) and \( a_1 = 0 \), that is \( g(t) = g_1(t) \).
Let $\mathcal{D}$ denote the space of testing functions and $\mathcal{D}'$ as its dual space. Furthermore, let $\mathcal{D}'(\mathbb{R})$ denote the space of distributions whose supports contained in the nonnegative real axis.

Let $f \ast g$ denote the convolution of distributions $f$ and $g$.

Let $\sigma$ be a real-valued regular distribution in $\mathcal{D}'(\mathbb{R})$. $\sigma$ is called an error of a nonlinear feedback system shown in Fig. 1, if and only if $\sigma$ satisfies the following relation.

$$
\langle \sigma, \phi \rangle = \langle l, \phi \rangle - \langle g \ast f(\sigma), \phi \rangle
$$

$$
= \langle l, \phi \rangle - \sum_{k=1}^{n} a_k \langle f(\sigma), (-1)^{k-1} \phi ^{(k-1)} \rangle - \langle g_1 \ast f(\sigma), \phi \rangle,
$$

for any $\phi \in \mathcal{D}$ and some $l \in \mathcal{D}'(\mathbb{R})$. Note that the assumption $\sigma \in \mathcal{D}'(\mathbb{R})$ implies $f(\sigma) \in \mathcal{D}'(\mathbb{R})$ and hence $g \ast f(\sigma)$ is well defined as a distribution.

### 3. Inversion of Convolution Operators

In this section we shall show that Eq. (1) can be transformed into a more convenient form. For this purpose, we need two basic lemmas. The first lemma will be stated without proof.

**Lemma 1.** If $f \in \mathcal{N}(\alpha, \beta)$, then for any $c \in \mathbb{R}$, there exists a real-valued continuous function $\tilde{f}(\sigma)$ such that

1. $f(\sigma) = c \sigma + \tilde{f}(\sigma)$,
2. $|\tilde{f}(\sigma_1) - \tilde{f}(\sigma_2)| \leq k |\sigma_1 - \sigma_2|$, $k = \max\{|\beta - c|, |c - \alpha|\}$, for real $\sigma_1, \sigma_2$ with $\sigma_1 \neq \sigma_2$.

Before going to the next lemma, we review some definitions of Fourier and Laplace transforms for distributions (Zemanian, 1965).

Let $\mathcal{S}$ be the space of testing functions of rapid descent and $\mathcal{S}'$ be its dual space. The Fourier transform of $k \in \mathcal{S}'$ is defined by

$$
\langle \mathcal{F}k, \phi \rangle = \langle k, \mathcal{F}\phi \rangle, \quad \phi \in \mathcal{S}.
$$

If $k(t) \in L^1_{(-\infty, \infty)} \cup L^2_{(-\infty, \infty)}$, then the Fourier transform $K(i\lambda)$ of $k(t)$ exists and since $\langle \mathcal{F}k, \phi \rangle = \langle K(i\lambda), \phi \rangle$ for any $\phi \in \mathcal{S}'$, we may identify $\mathcal{F}k$ with $K(i\lambda)$ and write $\mathcal{F}k = K(i\lambda)$.

If a distribution $k(t)$ belongs to $\mathcal{D}'(\mathbb{R})$ and if $e^{-\sigma t}k(t) \in \mathcal{S}'$ for $\sigma > \sigma'$, then the Laplace transform of $k(t)$ is given by $K(s) = \mathcal{L}k = \mathcal{F}\{e^{-\sigma t}k(t)\}$, where
σ > σ' and \( s = σ + iλ \). \( K(s) \) is an analytic function in its region of convergence \( \text{Re} \ s > σ' \).

Now we prove the following.

**Lemma 2.** Let \( g \) be in \( \mathcal{F}_{n} \) and suppose there exists a nonzero number \( c \) such that

\[
|1 + cG(s)| > 0, \quad \text{in} \quad \text{Re} \ s ≥ 0.
\]

Then we classify two cases.

(A) \( n ≥ 2 \).

In this case there exists a function \( h \in L^2_{(R)} \) such that

\[
(δ + cg) * h = δ.
\]

(B) \( n = 1 \) or \( n = 0 \).

In this case there exists a function \( h \in L^1_{(R)} \) and a number \( d ≠ 0 \) such that

\[
(δ + cg) * (dδ + h) = δ.
\]

**Proof of (A).** From (2), we see that \( H(s) = (1 + cG(s))^{-1} \) exists, which is analytic in \( \text{Re} \ s > 0 \) and is continuous in \( \text{Re} \ s ≥ 0 \). Since \( \lim_{|s|→∞} |G_1(s)| = 0 \), by the Riemann–Lebesgue lemma, we can show after some manipulations that, \( H(s) = H(σ + iλ) \) belongs to a Hardy class \( H^p \) in \( σ > 0 \). Thus, there exists a function \( H(iλ) \) in \( L^2_{(-∞,∞)} \) satisfying the following conditions. (Titchmarsh, 1948, pp. 125, 128)

(i) \( H(iλ) = \lim_{σ→-∞} H(σ + iλ) \).

(ii) The inverse Fourier transform \( \hat{h}(t) \) of \( H(iλ) \) is in \( L^2_{(R)} \).

(iii) \( \hat{H}(iλ) = \lim_{σ→0⁺} H(σ + iλ) \), a.e. \( λ \).

Now, since there exists a function \( h(t) \) in \( L^2_{(-∞,∞)} \) such that \( h(t) = \mathcal{F}^{-1}H(iλ) \) and since \( H(σ + iλ) \) is continuous in \( σ ≥ 0 \), we see that, \( H(iλ) = \hat{H}(iλ) \), a.e. \( λ \). Thus, we have \( h(t) = \hat{h}(t) \), a.e., and therefore \( h(t) \) belongs to \( L^2_{(R)} \).

**Proof of (B).** In order to obtain (B) it is sufficient to show that, if \( |1 + bG_1(s)| > 0 \), in \( \text{Re} \ s ≥ 0 \), then there exists a function \( h \) in \( L^1_{(R)} \) such that \( (δ + bg_1) * (δ + h) = δ \), where \( b = c(1 + a_4c)^{-1} \) and \( 1 + a_4c ≠ 0 \).

Now, noting the relation

\[
(1 + bG_1(iλ))^{-1} = 1 - bG_1(iλ)(1 + bG_1(iλ))^{-1},
\]
we can apply the result of Paley and Wiener (1934, p. 61) to show the existence of \( h(t) \) in \( L^1_{(R)} \).

We may restate the results of Lemma 2 by saying that an inverse convolution operator of \((\delta + cg)\) exists and unique in \( \mathcal{D}'_{(R)} \) and has one of the two forms shown in Lemma 2. We shall write this inverse by \((\delta + cg)^{-1}\) and write its Fourier transform by \((1 + cG(\lambda))^{-1}\).

As a consequence of Lemma 2 we may state the following corollary.

**Corollary 2.1.** Let \( g \in \mathcal{F}_{n}' \). Suppose there exists a nonzero number \( c \) such that \( |1 + cG(s)| > 0 \), in \( \text{Re } s \geq 0 \). Then

(i) if \( n \geq 2 \), there exists a nonzero number \( d \) and a function \( h \) in \( L^2_{(R)} \) such that

\[
(\delta + cg)^{-1} \ast g = d\delta + h;
\]

(ii) if \( n = 1 \), there exists a nonzero number \( d \) and a function \( h \) in \( L^1_{(R)} \) such that

\[
(\delta + cg)^{-1} \ast g = d\delta + h; \quad \text{and}
\]

(iii) if \( n = 0 \), there exists a function \( h \) in \( L^1_{(R)} \) such that

\[
(\delta + cg)^{-1} \ast g = h.
\]

**Proof.** If \( n \geq 2 \), we rewrite \( G(\lambda)(1 + cG(\lambda))^{-1} \) in the following form and apply Lemma 2(A).

\[
G(\lambda)(1 + cG(\lambda))^{1} = (1/c)(1 - (1 + cG(\lambda))^{-1}).
\]

If \( n = 1 \), we may assume \( 1 + a_1c \neq 0 \), without any loss of generality. From Lemma 2(B), there exists a function \( h \in L^1_{(R)} \) such that

\[
(\delta + cg)^{-1} \ast g = (1 + a_1c)^{-1}(\delta + h) \ast (a_1\delta + g_1).
\]

If \( n = 0 \), the corollary clearly holds from Lemma 2(B).

4. EXISTENCE AND UNIQUENESS

We are now ready to transform (1) to a convenient form and give a proof on the existence and uniqueness of \( \sigma \). We assume that \( f \) and \( g \) satisfy the conditions of Lemma 1 and Lemma 2, respectively.
If any $\sigma \in \mathcal{D}'(R)$ satisfies (1), then from Lemma 1, (1) can be rewritten by

$$\langle \sigma, \phi \rangle = \langle l, \phi \rangle - \langle g * (\sigma + \hat{f}(\sigma)), \phi \rangle,$$

for any $\phi \in \mathcal{D}$ and some $l \in \mathcal{D}'(R)$. We, thus, have

$$\langle (\delta + cg) * \sigma, \phi \rangle = \langle l, \phi \rangle - \langle g * \hat{f}(\sigma), \phi \rangle.$$

From Lemma 2, we see

$$\langle \sigma, \phi \rangle = \langle (\delta + cg)^{-1} * l, \phi \rangle - \langle \hat{g} * \hat{f}(\sigma), \phi \rangle \quad (3)$$
where $\hat{g} = (\delta + cg)^{-1} * g$. Suppose, in addition, $l(t)$ belongs to $L^2_{\text{loc}}(R)$.

Next we give a useful lemma for the later analysis.

**Lemma 3.** Let $x(t)$ be a function in $L^2_{\text{loc}}(R)$. Let $r$ be a distribution in $\mathcal{S}' \cap \mathcal{D}'(R)$ such that the Fourier transform $R(i\lambda)$ of $r$ is a function in $L^2_{\text{loc}}(-\infty, \infty)$ and satisfies

$$\text{ess sup}_{-\infty < \lambda < \infty} |R(i\lambda)| = \mu < \infty.$$

Then, for any $T < \infty$,

$$\int_0^T |r * x(t)|^2 dt \leq \mu^2 \int_0^T |x(t)|^2 dt.$$

**Proof.** For an arbitrary fixed $T$, define $\hat{x}(t)$ by

$$\hat{x}(t) = \begin{cases} x(t), & 0 \leq t \leq T, \\ 0, & \text{otherwise}. \end{cases}$$

Then $r * \hat{x}$ belongs to $\mathcal{S}' \cap \mathcal{D}'(R)$. Since $\hat{x}$ has a bounded support and $r$ is in $\mathcal{S}'$, we get $\mathcal{F}[r * \hat{x}] = R(i\lambda)\mathcal{F}\hat{x}$. From the assumptions on $R(i\lambda)$ and $\hat{x}$, we see that $\mathcal{F}[r * \hat{x}]$ is a measurable function and

$$\int_{-\infty}^{\infty} |\mathcal{F}[r * \hat{x}]|^2 d\lambda \leq \mu^2 \int_{-\infty}^{\infty} |\mathcal{F}\hat{x}|^2 d\lambda = 2\pi \mu^2 \int_0^{\infty} |\hat{x}(t)|^2 dt < \infty, \quad (6)$$

where $\hat{x}(t) = \begin{cases} x(t), & 0 \leq t \leq T, \\ 0, & \text{otherwise}. \end{cases}$

It is easy to see that if any regular distribution $\sigma$ in $\mathcal{D}'(R)$ satisfies (4), then this $\sigma$ satisfies (3) and also (1).
by the Parseval equality. This implies that $r \ast \mathcal{F}$ can be identified with a function in $L^2(\mathbb{R})$. Thus, we have from (6) and the Parseval equality,

$$\int_0^T |r \ast x(t)|^2 \, dt \leq \int_0^\infty |r \ast \mathcal{F}(t)|^2 \, dt \leq \mu^2 \int_0^T |x(t)|^2 \, dt.$$  

This completes the proof of Lemma 3.

Now we give the following theorem.

**Theorem 1.** Let $f \in \mathcal{N}(\alpha, \beta)$, $l \in L^2_{\text{loc}}(\mathbb{R})$ and $g \in \mathcal{F}'$. Suppose there exists a nonzero number $c$ such that

(i) $|1 + cG(s)| > 0$, \quad in $\Re s \geq 0$;

(ii) $\mu_1(c) = \sup_{-\infty < \lambda < \infty} \left| \frac{kG(i\lambda)}{1 + cG(i\lambda)} \right| < 1$, \quad $k = \max\{|\beta - c|, |c - \alpha|\}$.

Then there exists a unique solution $x$ of (1) which belongs to $L^2_{\text{loc}}(\mathbb{R})$.

**Proof.** Let $T$ be a fixed finite number. Define $\Phi_0(t)$ and $\Phi_n(t)$ by

$$\Phi_0(t) = \hat{l}(t) - \hat{g} \ast \hat{f}(\hat{l})(t), \quad t \in (0, T);$$  

$$\Phi_n(t) = \hat{l}(t) - \hat{g} \ast \hat{f}(\Phi_{n-1})(t), \quad n \geq 1, \quad t \in (0, T).$$  

(5)

It is easy to check that for any $n \geq 0$, $\Phi_n(t)$ belongs to $L^2_{(0,T)}$. From (ii) and Lemma 3, it is easy to show that (5) is a contraction mapping in $L^2_{(0,T)}$. Thus, there exists a function in $L^2_{(0,T)}$ which satisfies (4) and unique in $L^2_{(0,T)}$ (Kolmogorov and Fomin, 1970, p. 66).

Now, suppose that there are two inputs $\hat{l}_1(t) \in L^2_{\text{loc}}(\mathbb{R})$ and $\hat{l}_2(t) \in L^2_{\text{loc}}(\mathbb{R})$ such that for any fixed $T < \infty$,

$$\hat{l}_1(t) = \hat{l}_2(t), \quad t \in (0, T);$$  

$$\hat{l}_1(t) \neq \hat{l}_2(t), \quad t \in (T, T + A), \quad A > 0.$$  

Let $\sigma_1$ and $\sigma_2$ be the solution of (4) with respect to $\hat{l}_1$, $\hat{l}_2$. Since $\hat{g} \in \mathcal{D}'(\mathbb{R})$ and the solution of (4) is unique, we have

$$\sigma_1(t) = \sigma_2(t), \quad \text{a.e.,} \quad t, t \in (0, T).$$  

This completes the proof of Theorem 1.

Since $G(i\lambda)$ is a continuous function of $\lambda$, there are more than one $c$ which satisfy the conditions of Theorem 1 for some fixed $g$. In order to fix the constant $c$, Lemma 4 gives a good criterion.
Lemma 4. Define $\mu_4(c)$ and $D$, respectively, by

$$
\mu_4(c) = \left\{ \sup_{-\infty < \lambda < \infty} \left| \frac{kG(i\lambda)}{1 + cG(i\lambda)} \right| \right\}, \quad k = \max\{ |\beta - c|, |c - \alpha| \},
$$

and

$$
D = \{ c; \mid 1 + cG(s) \mid > 0, \text{ in Re } s \geq 0 \},
$$

for an arbitrary fixed $G(i\lambda)$. If there exists a number $c \in D$ such that $\mu_4(c) < 1$, then $c_0 \in D$ and

$$
\mu_4(c_0) \leq \mu_4(c).
$$

Proof. See, for example, Holtzman (1970, p. 49).

From now on we fix the constants $c, k$, respectively, by $c = (\alpha + \beta)/2$ and $k = (\beta - \alpha)/2$.

We next state a theorem on the weak boundedness of the solution $\sigma$. This theorem also gives a basis to calculate the time average of $|\sigma|^2$.

Theorem 2. Let $f \in \mathcal{N}(\alpha, \beta)$ and let $l(t) \in L^0_{\text{loc}}(\mathbb{R})$. Suppose $g \in \mathcal{F}$ satisfies the conditions of Theorem 1 for $c = (\alpha + \beta)/2$. Then the solution $\sigma$ of (4) satisfies the following inequality for any $T < \infty$.

$$
\int_0^T |\sigma(t)|^2 \, dt \leq (\mu_2(1 - \mu_1)^{-1}) \int_0^T |l(t)|^2 \, dt,
$$

where

$$
\mu_1 = \mu_4((\alpha + \beta)/2)
$$

and

$$
\mu_2 = \left\{ \sup_{-\infty < \lambda < \infty} |[1 + ((\alpha + \beta)/2) G(i\lambda)]^{-1}| \right\}.
$$

Proof. From (4), the Minkowski inequality, and Lemma 3, we have

$$
\| \sigma \|_T \leq \| l \|_T + \| g * f(\sigma) \|_T \\
\leq \mu_2 \| l \|_T + \mu_1 \| \sigma \|_T,
$$

where $\| z \|_T = \left\{ \int_0^T |z|^2 \, dt \right\}^{1/2}$. Thus we have

$$
\| \sigma \|_T \leq (\mu_2(1 - \mu_1)^{-1}) \| l \|_T,
$$

for any $T < \infty$. 
Remark 1. If \( l(t) \in L^2_{(0,\infty)} \), then \( \sigma(t) \in L^2_{(0,\infty)} \) from Theorem 2. In this case the system is called \( L^2 \)-stable (Freedman, 1968).

Remark 2. If \( n \geq 2 \), then condition (ii) of Theorem 2 implies \( |\alpha + \beta| > |\beta - \alpha| \) or \( \alpha > 0 \).

5. STATIONARY PROCESSES AND NONLINEAR FEEDBACK SYSTEMS

In this section we consider the case where \( l(t) \) is supposed to be a stationary stochastic process.

Let \( \Omega \) be the probability space with the generic element \( \omega \), a \( \sigma \)-field \( \mathcal{F} \) of subsets of \( \Omega \), and a probability measure \( P \) defined on \( \mathcal{F} \).

Let \( X(t) = X(t, \omega) \), \(-\infty < t < \infty\), be a real-valued weakly stationary stochastic process with
\[
E\{X(t)\} = \int_{\Omega} X(t, \omega) \, dP(\omega) = 0, \quad -\infty < t < \infty;
\]
\[
E|X(t)|^2 < \infty, \quad -\infty < t < \infty,
\]
and has the continuous covariance function
\[
\rho(\tau) = 2 \int_0^\infty \cos \lambda \tau \, dF(\lambda) = \int_{-\infty}^\infty e^{i\lambda \tau} \, dF(\lambda),
\]
where \( F(\lambda) \) is the spectral distribution function.

Since \( \rho(\tau) \) is continuous, \( X(t, \omega) \) is continuous in probability. Hence, there exists a random variable \( X_1(t, \omega) \) such that

(i) \( X_1(t, \omega) \) is measurable with respect to \( R^1 \times \Omega \);

(ii) \( P\{\omega; X_1(t, \omega) = X(t, \omega)\} = 1 \), for each \( t \).

(Gikhman and Skorokhod, 1969, p. 157). Therefore, we may assume that \( X(t, \omega) \) is measurable with respect to \( R^1 \times \Omega \). By Fubini's theorem,
\[
E \int_a^b |X(t, \omega)|^2 \, dt = \int_a^b E|X(t, \omega)|^2 \, dt = (b - a) \rho(0) < \infty,
\]
for arbitrary \( a, b \) with \(-\infty < a \leq b < \infty\). Hence, we have
\[
\int_a^b |X(t, \omega)|^2 \, dt < \infty, \quad \text{a.s.} \quad (-\infty < a \leq b < \infty).
\]
Now, let \( l(t, \omega) \) be a stochastic process defined by

\[
    l(t, \omega) = \begin{cases} 
        X(t, \omega), & t \geq 0, \\
        0, & t < 0, 
    \end{cases}
\]

where \( X(t, \omega) \) is a weakly stationary process with continuous covariance function. From this definition \( l(t, \omega) \) belongs to \( L^2_{100(\mathbb{R})} \), almost surely.

If \( f \) and \( g \) satisfy the conditions of Theorem 2, then we obtain

\[
    \left( \frac{1}{T} \right) \int_0^T |\sigma(t, \omega)|^2 \, dt \leq \gamma^2 \left( \frac{1}{T} \right) \int_0^T |l(t, \omega)|^2 \, dt, \quad \text{a.s.,}
\]

for any finite \( T \) and \( \gamma = \mu_2(1 - \mu_1)^{-1} \). We, thus, have

\[
    \limsup_{T \to \infty} \left( \frac{1}{T} \right) \int_0^T |\sigma(t, \omega)|^2 \, dt \leq \gamma^2 \limsup_{T \to \infty} \left( \frac{1}{T} \right) \int_0^T |l(t, \omega)|^2 \, dt, \quad \text{a.s.}
\]

If \( |l(t, \omega)|^2 \) is a process which individual ergodic theorem holds, then

\[
    \limsup_{T \to \infty} \left( \frac{1}{T} \right) \int_0^T |\sigma(t, \omega)|^2 \, dt \leq \gamma^2 \lim_{T \to \infty} \left( \frac{1}{T} \right) \int_0^T |l(t, \omega)|^2 \, dt = Y(\omega)
\]

where \( Y(\omega) \) is a second order random variable. This implies that

\[
    \limsup_{T \to \infty} \left( \frac{1}{T} \right) \int_0^T |\sigma(t, \omega)|^2 \, dt < \infty, \quad \text{a.s.}
\]

Furthermore, if \( |l(t, \omega)|^2 \) is ergodic, then we have

\[
    \limsup_{T \to \infty} \left( \frac{1}{T} \right) \int_0^T |\sigma(t, \omega)|^2 \, dt \leq \gamma^2 \rho(0), \quad \text{a.s.}
\]

Next, we shall show that the autocorrelation function of the output of the system is bounded.

If \( x = g * f(\sigma) \) is a regular distribution in \( \mathcal{D}'(\mathbb{R}) \) and satisfies the following relation (7), then \( x \) is called an output of the system.

\[
    \langle x, \phi \rangle = \langle g * f(\sigma), \phi \rangle = \langle l, \phi \rangle - \langle \sigma, \phi \rangle, \quad (7)
\]

where \( \sigma \in \mathcal{D}'(\mathbb{R}) \), \( l \in \mathcal{D}'(\mathbb{R}) \) and \( \phi \in \mathcal{D} \).

We assume that \( f \) and \( g \) satisfy the conditions of Theorem 2. In order to show the existence of an output \( x \), we follow the procedure of Section 4.
From Lemma 1 we obtain
\[
\langle x, \phi \rangle = \langle g * (c_0 \sigma + \hat{f}(\sigma)), \phi \rangle \\
= \langle c_0 g * (l - x), \phi \rangle + \langle g * \hat{f}(\sigma), \phi \rangle,
\]
where \( c_0 = (\alpha + \beta)/2 \). Hence, from Lemma 2, we have
\[
\langle x, \phi \rangle = \langle c_0 g * l, \phi \rangle - \langle \hat{g} * \hat{f}(\sigma), \phi \rangle,
\]
where \( \hat{g} = (\delta + c_0 g)^{-1} * g \).

If \( l(t) \) belongs to \( L^2_{\text{loc}}(\mathbb{R}) \), then from Theorem 1, \( \sigma \) and \( \hat{g} * \hat{f}(\sigma) \) also belong to \( L^2_{\text{loc}}(\mathbb{R}) \). Noting that \( c_0 \hat{g} * l \) belongs to \( L^2_{\text{loc}}(\mathbb{R}) \), \( x \) defined by (8) becomes a function in \( L^2_{\text{loc}}(\mathbb{R}) \). Therefore, (8) is equivalent to the following quantity (9).
\[
x(t) = c_0 \hat{g} * l(t) - \hat{g} * \hat{f}(\sigma)(t), \quad t \geq 0.
\]

Using the results of Lemma 3 and Theorem 2, we see that
\[
\| x \|_T \leq \| c_0 \hat{g} * l \|_T + \| \hat{g} * \hat{f}(\sigma) \|_T \\
\leq (\| c_0 \|_k_0) \mu_1 \| l \|_T + \mu_1 \| \sigma \|_T \\
\leq \mu_2 (\| c_0 \|_k_0 + \gamma) \| l \|_T ,
\]
where \( k_0 = (\beta - \alpha)/2 \). Consequently, we have
\[
\int_0^T | x(t) |^2 dt \leq \mu_1^2 (\| c_0 \|_k_0 + \gamma)^2 \int_0^T | l(t) |^2 dt.
\]

Moreover, if \( l(t, \omega) \) is defined by (6) and if \( | l(t, \omega) |^2 \) is ergodic, then
\[
\lim_{T \to \infty} \sup (1/T) \int_0^T | x(t, \omega) |^2 dt \leq \mu_2^2 (\| c_0 \|_k_0 + \gamma)^2 \rho(0) < \infty,
\]
holds almost surely.

Thus, we have proved the following theorem.

**Theorem 3.** Let \( l(t, \omega) \) be a stationary stochastic process defined by (6) and let \( | l(t, \omega) |^2 \) be ergodic. Let \( f \) and \( g \) satisfy the conditions of Theorem 2 and let \( x \) be the output of the system shown in Fig. 1. Then,
\[
\lim_{T \to \infty} \sup (1/T) \int_0^T | x(t, \omega) |^2 dt \leq \mu_1^2 \{(\alpha + \beta)(\beta - \alpha)^{-1} + \mu_2 (1 - \mu_2)^{-1}\} \rho(0),
\]
where \( \rho(0) = E | l(t, \omega) |^2 \).
6. Concluding Remarks

We have derived $L^2$-stability conditions for a class of feedback systems whose linear part involves the delta functional and its derivatives. The key point of the proof was to transform (1) into (3) by inverting the convolution operator.

The main result here is Theorem 3 which shows that the autocorrelation function of the output becomes bounded, when the input is an ergodic stationary stochastic process.

A natural extension of the problem we have considered here might be the following: is it possible to minimize

$$\lim_{T \to \infty} \sup_{\omega} \left( \frac{1}{T} \int_0^T |x(t, \omega)|^2 \, dt \right)$$

in some sense? This problem will be handled in a forthcoming paper.

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