ADVANCES IN MATHEMATICS 18, 243–244 (1975)

The Differentiability of Pólya's Function

R. T. Bumby

Department of Mathematics, Rutgers University, New Brunswick, NJ 08903

In [1] Pólya has defined a one parameter family of functions P on the unit interval whose values fill a right triangle with acute angle θ. In [2] Peter Lax has shown that

(a) If $30° < \theta < 45°$, P is nowhere differentiable.

(b) If $15° < \theta < 30°$, P is not differentiable on a set of measure 1, but has derivative zero on a nondenumerable set.

(c) If $\theta < 15°$, $P' = 0$ on a set of measure 1.

In this brief note, we settle the case $\theta = 30°$ and $\theta = 15°$. Notation and formulas are to be found in [2].

Lemma. *If P is differentiable at t, then $P'(t) = 0$.*

Proof. $P'(t) \neq 0$ implies that $P'(t)$ has a well-defined direction. However, from the definition of P one can show that the direction of $P(t') - P(t)$ fills at least a quadrant as t' runs through a neighborhood of t. In particular, if some $2^k t$ is an integer, then $P(\{t' : | t' - t | \leqslant 2^{-n}\})$ fills a rectangle with $P(t)$ at one corner if $n > k$. Otherwise $P(\{t' : t$ and t' have first n digits in common$\})$ fills a triangle for which $P(t)$ is not a vertex.

Corollary 1. *If $\theta = 30°$, P is nowhere differentiable.*

Proof. The lower bound (10) shows that $P'(t) \neq 0$ for all t.

Corollary 2. *If $\theta = 15°$, then P is not differentiable on a set of measure* 1.

Proof. The lower bound (15) gives $| P(t_N) - P(t)|/| t - t_N | >$ const$(s/c)^{D_N}$. Existence of the derivative would require $D_N \to +\infty$ which happens only on a set of measure zero.

One could also ask about the possibility that

$$D(t', t) = | P(t') - P(t)|/| t - t' |$$

remain bounded without having limit zero.

PROPOSITION. *If $\theta = 30°$, $D(t', t)$ is unbounded for all t.*

Proof. From (15) $D(t_N, t) >$ const $3^{N/4 - D_N/2}$. If not unbounded, $V_N = N/2 - D_N$ is bounded above. Thus t must have a terminating binary expansion. But then it also has an expansion in which Z_N is bounded, and for the t_N defined by this expansion $D(t_N, t)$ is unbounded.

PROPOSITION. *If $\theta < 30°$, there are uncountably many t for which $D(t', t)$ is bounded but does not approach 0 as $t' \to t$.*

Proof. If t has no k consecutive digits the same, then $M - N$ is bounded and the upper bound (23) is of the same order of magnitude as the lower bound (15), both being of the form $(2C)^{V_N}(2S)^{Z_N}$. Since $2C > 1 > 2S$, there is a quantity $\rho = -\log(2S)/\log(2C)$ such that lim sup$(V_N - \rho Z_N)$ finite gives the desired result. There are uncountably many sequences with this property.

REFERENCES

1. G. PÓLYA, Über eine Peanosche Kurve, *Bull. Acad. Sci. Cracovie, Ser. A* (1913), 305–313.
2. P. LAX, The Differentiability of Polya's Function, *Advances Math.* **10** (1973), 456–464.