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Remez-type inequalities and their applications *

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Abstract

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The Remez inequality gives a sharp uniform bound on $[-1, 1]$ for real algebraic polynomials p of degree at most n if the Lebesgue measure of the subset of $[-1, 1]$, where $|p|$ is at most 1, is known. Remez-type inequalities give bounds for classes of functions on a line segment, on a curve or on a region of the complex plane, given that the modulus of the functions is bounded by 1 on some subset of prescribed measure. This paper offers a survey of the extensive recent research on Remez-type inequalities for polynomials, generalized nonnegative polynomials, exponentials of logarithmic potentials and Müntz polynomials. Remez-type inequalities play a central role in proving other important inequalities for the above classes. The paper illustrates the power of Remez-type inequalities by giving a number of applications.

Keywords: Bernstein-, Markov-, Nikolskii- and Remez-type inequalities; generalized polynomials; exponentials of logarithmic potentials; Müntz polynomials; generalized Jacobi weight functions.

1. Introduction

Remez-type inequalities give bounds for classes of functions on a line segment, on a curve or on a region of the complex plane, given that the modulus of the functions is bounded by 1 on some subset of prescribed measure. In Section 2 we define the classes of functions (generalized algebraic and trigonometric polynomials, exponentials of logarithmic potentials and Müntz polynomials) for which Remez-type inequalities will be established. The Remez inequality is stated in Section 3, and its trigonometric and pointwise algebraic analogues are discussed as well. Generalized nonnegative algebraic polynomials (the terminology will be explained in Section 2) of the form

$$f(z) = |\omega| \prod_{j=1}^{k} |z - z_j|^{r_j}, \quad 0 \neq \omega,\ z_j \in \mathbb{C},\ 0 < r_j \in \mathbb{R},\ j = 1, 2, \ldots, k, \tag{1.1}$$

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were studied in a sequence of recent papers [6,14,18,20,22,23,25,26]. A number of well-known inequalities in approximation theory were extended to them by utilizing the generalized degree

$$N = \sum_{j=1}^{k} r_j, \tag{1.2}$$

in place of the ordinary one. Our motivation was to find tools to examine systems of orthogonal polynomials simultaneously, associated with generalized Jacobi, or at least generalized nonnegative polynomial weight functions of degree at most Γ. In a recent paper we gave sharp estimates, in this spirit, for the Christoffel function on $[-1, 1]$ and for the distances of the consecutive zeros of orthogonal polynomials associated with generalized nonnegative polynomial weight functions; these results are stated in Section 12.

The theory of orthogonal polynomials assoicated with Jacobi or generalized Jacobi weight functions of the form (1.1) obviously calls for a profound study of generalized polynomials of the form (1.1). As an example, the inequality

$$|q(y)|^2 \leqslant c \min\left\{n^2, \frac{n}{\sqrt{1-y^2}}\right\} \int_{-1}^{1} |q(z)|^2 \, \mathrm{d}z, \quad -1 \leqslant y \leqslant 1, \tag{1.3}$$

for every polynomial $q \in \mathscr{P}_n^{\mathrm{r}}$ ($\mathscr{P}_n^{\mathrm{r}}$ denotes the set of all algebraic polynomials of degree at most n with real coefficients), with an absolute constant c, is well known. To give a lower bound for the Christoffel function

$$\lambda_{n+1}(\alpha, 2, y) := \inf_{\substack{q \in \mathscr{P}_n^{\mathrm{r}} \\ |q(y)|=1}} \int_{-1}^{1} |q(z)|^2 w(z) \, \mathrm{d}z \tag{1.4}$$

associated with the measure α, where $\mathrm{d}\alpha/\mathrm{d}z = w(z)$ is of the form (1.1), we may have the following idea. We consider $|q| w^{1/2}$ as a (generalized) polynomial of (generalized) degree at most $n + \frac{1}{2}N$, and apply (1.3) to it to obtain

$$\lambda_{n+1}(\alpha, 2, y) \geqslant c^{-1} w(y) \max\left\{\left(n + \tfrac{1}{2}N\right)^{-2}, \frac{\sqrt{1-y^2}}{n + \frac{1}{2}N}\right\}, \quad -1 \leqslant y \leqslant 1. \tag{1.5}$$

Of course this philosophy is justified only if we show that generalized polynomials are very much like ordinary polynomials, namely only if many of the inequalities for ordinary polynomials can be extended to generalized polynomials by utilizing the generalized degree in place of the ordinary degree. We will justify (1.5) in Section 12.

Another motivating factor is the unpleasant fact that the pth power of a polynomial is not necessarily a polynomial. Without having a package of inequalities for generalized polynomials, proofs of inequalities for ordinary polynomials in L_p-norm may include boring, inconvenient, technical parts covering the essential ideas.

How can one prove that there is an absolute constant c such that

$$\max_{-1 \leqslant x \leqslant 1} |q'(x)| \leqslant \frac{cn^2}{\gamma} \max_{-1 \leqslant x \leqslant 1} |q(x)|, \tag{1.6}$$

for every $q \in \mathscr{P}_n^{\mathrm{r}}$ such that every zero of q has multiplicity at least $\gamma \geqslant 1$? To start, we write $|q|$ as $|q| = |f|^{\gamma}$, where f is of the form (1.1) with $N = n/\gamma$ and each $r_j \geqslant 1$. Unfortunately, $|f|$

is typically not a polynomial. If it were a polynomial, (1.6) would follow immediately from Markov's inequality. However, though f is not a polynomial, it is very much like that, and a Markov-type inequality for all f of the form (1.1) with each $r_j \geqslant 1$ can be proved, cf. Theorem 9.3, by utilizing the generalized degree in place of the ordinary degree. Therefore even (1.6) (a statement for *ordinary* polynomials) appeals to inequalities for generalized polynomials.

To prove our results in Section 12, we needed various inequalities such as Remez-type inequalities (Section 4), Nikolskii-type inequalities (Section 5) and weighted Bernstein- and Markov-type inequalities (Sections 9 and 10) for generalized nonnegative algebraic polynomials. Weighted Markov–Bernstein-type inequalities in L_p-norms play a significant role in the proof of inverse theorems of approximation with the Ditzian–Totik modulus of smoothness. Section 10 offers more general weighted Markov–Bernstein-type inequalities in L_p-norms than those of [12].

Typically, the extension of a polynomial inequality to generalized nonnegative polynomials is not trivial at all, and the proof is far from a simple density argument. However, our Remez-type inequalities of Section 3 can be extended quite simply (Section 4) and these play a central role in the proof of other important inequalities for generalized nonnegative polynomials, which do not follow from the corresponding polynomial inequalities by a density argument.

Since (1.1) implies

$$\log f(z) = \sum_{j=1}^{k} r_j \log |z - z_j| + \log |\omega|, \tag{1.7}$$

a generalized nonnegative polynomial can be considered as the exponential of a logarithmic potential with respect to a finite Borel measure on $\mathbb{C}$ that is supported in finitely many points (the measure has mass $r_j > 0$ at each z_j, $j = 1, 2, \ldots, k$). This suggests that some of the inequalities holding for generalized nonnegative polynomials may be true for exponentials of logarithmic potentials of the form

$$Q_{\mu,c}(z) = \exp\left(\int_{\mathbb{C}} \log |z - t| \, \mathrm{d}\mu(t) + c\right), \tag{1.8}$$

where μ is a finite nonnegative Borel measure on $\mathbb{C}$ having compact support, and $c \in \mathbb{R}$. The quantity $\mu(\mathbb{C})$ plays the role of the generalized degree N defined by (1.2). In a recent paper [24] Remez-type inequalities are established for exponentials of logarithmic potentials on line segments and circles of the complex plane (Section 6) and on bounded domains of $\mathbb{C}$ with C^2 boundary (Section 7). We present a "standard" method to prove such an extension of the corresponding polynomial inequality, using Fekete polynomials. Each of these Remez-type inequalities implies a Nikolskii-type inequality (Section 8) and to demonstrate the power of Remez-type inequalities, we prove one of them.

Each of our Remez-type inequalities has an L_p, $0 < p < \infty$, analogue which is discussed in the corresponding section.

A sharp Remez-type inequality on $[-1, 1]$ was established in [6] for real algebraic polynomials of degree at most n having at most k zeros in the open unit disk. A numerical version of this Remez-type inequality (unlike other important inequalities) has a straightforward extension to generalized nonnegative polynomials f of degree at most N with

$$\sum_{\{j: |z_j| < 1\}} r_j \leqslant K, \quad 0 \leqslant K \leqslant N, \tag{1.9}$$

in (1.1). This opened a way to prove sharp Markov- and Nikolskii-type inequalities for generalized nonnegative polynomials under the same constraints. These results are presented in Section 11. Some of the results of Sections 5 and 9 are recaptured here as special cases, by the choice $K = N$.

The well-known Müntz–Szász Theorem [10] asserts that

$$\text{span}\{1, x^{\lambda_1}, x^{\lambda_2}, \ldots\}, \quad 0 < \lambda_1 < \lambda_2 < \cdots, \tag{1.10}$$

is dense in $C[0, 1]$ in the uniform norm if and only if $\sum_{j=1}^{\infty} \lambda_j^{-1} = \infty$. Does the same characterization hold when the interval $[0, 1]$ is replaced by an arbitrary closed subset of $[0, 1]$ with positive measure? In [7] we proved a bounded "left-hand side" Remez-type inequality for Müntz spaces (1.10) with $\lambda_j \geqslant q^j$, $q > 1$, which gives a partial answer. This is the central part of Section 13 of this paper. The same bounded left-hand side Remez-type inequality is conjectured whenever $\sum_{j=1}^{\infty} \lambda_j^{-1} < \infty$. In the case $\lambda_j = j^2$, $j = 1, 2, \ldots$, this would answer an open problem of Newman negatively, concerning the denseness of the set

$$\left\{q = p_1 p_2 \colon p_1, p_2 \in \text{span}\left\{x^{j^2} \colon j = 0, 1, 2, \ldots\right\}\right\}$$

in $C[0, 1]$. Some other related results are discussed in Section 13 as well.

2. Classes of functions; notations

Denote by $\mathscr{P}_n^{\mathrm{r}}$ the set of all algebraic polynomials of degree at most n with real coefficients, and let $\mathscr{P}_n^{\mathrm{c}}$ be the set of all algebraic polynomials of degree at most n with complex coefficients.

The function

$$f = \prod_{j=1}^{k} P_j^{r_j}, \quad P_j \in \mathscr{P}_{n_j}^{\mathrm{r}} \backslash \mathscr{P}_{n_j - 1}^{\mathrm{r}}, \ 0 < r_j \in \mathbb{R}, \ j = 1, 2, \ldots, k, \tag{2.1}$$

will be called a *generalized real algebraic polynomial* of (generalized) degree

$$N = \sum_{j=1}^{k} r_j n_j. \tag{2.2}$$

To be precise, in this paper we will use the definition

$$z^r = \exp(r \log|z| + ir \arg z), \quad z \in \mathbb{C}, \ 0 < r \in \mathbb{R}, \ -\pi \leqslant \arg z < \pi. \tag{2.3}$$

Obviously,

$$|f| = \prod_{j=1}^{k} |P_j|^{r_j}, \tag{2.4}$$

for every f defined by (2.1). We denote by GRAP_N the set of all generalized real algebraic polynomials of degree at most N. We introduce the class $|\text{GRAP}|_N = \{|f| \colon f \in \text{GRAP}_N\}$. The function

$$f(z) = \omega \prod_{j=1}^{k} (z - z_j)^{r_j}, \quad 0 \neq \omega, \ z_j \in \mathbb{C}, \ 0 < r_j \in \mathbb{R}, \ j = 1, 2, \ldots, k, \tag{2.5}$$

will be called a *generalized complex algebraic polynomial* of (generalized) degree

$$N=\sum_{j=1}^{k} r_j. \tag{2.6}$$

We have

$$|f(z)|=|\omega|\prod_{j=1}^{k}|z-z_j|^{r_j}, \tag{2.7}$$

for every f defined by (2.5). Denote by GCAP_N the set of all generalized complex algebraic polynomials of degree at most N. The set $\{|f|: f\in \mathrm{GCAP}_N\}$ is denoted by $|\mathrm{GCAP}|_N$.

In the trigonometric case we denote by $\mathscr{T}_n^{\mathrm{r}}$ the set of all trigonometric polynomials of degree at most n with real coefficients, and let $\mathscr{T}_n^{\mathrm{c}}$ be the set of all trigonometric polynomials of degree at most n with complex coefficients.

The function

$$f=\prod_{j=1}^{k} P_j^{r_j}, \quad P_j\in \mathscr{T}_{n_j}^{\mathrm{r}}\setminus \mathscr{T}_{n_j-1}^{\mathrm{r}},\ 0<r_j\in\mathbb{R},\ j=1,2,\dots,k, \tag{2.8}$$

will be called a *generalized real trigonometric polynomial* of (generalized) degree N defined by (2.2). Obviously (2.4) holds for every f defined by (2.8). We will denote by GRTP_N the set of all generalized real trigonometric polynomials of degree at most N. The set $\{|f|: f\in\mathrm{GRTP}_N\}$ will be denoted by $|\mathrm{GRTP}|_N$. We say that the function

$$f(z)=\omega\prod_{j=1}^{k}\left(\sin\left(\tfrac{1}{2}(z-z_j)\right)\right)^{r_j}, \quad 0\neq\omega\in\mathbb{C},\ z_j\in\mathbb{C},\ r_j>0,\ j=1,2,\dots,k, \tag{2.9}$$

is a *generalized complex trigonometric polynomial* of (generalized) degree

$$N=\tfrac{1}{2}\sum_{j=1}^{k} r_j. \tag{2.10}$$

We have

$$|f(z)|=|\omega|\prod_{j=1}^{k}\left|\sin\left(\tfrac{1}{2}(z-z_j)\right)\right|^{r_j}, \tag{2.11}$$

for every f defined by (2.9). Denote by GCTP_N the set of all generalized complex trigonometric polynomials of degree at most N. The set $\{|f|: f\in\mathrm{GCTP}_N\}$ is denoted by $|\mathrm{GCTP}|_N$.

We remark that if $f\in|\mathrm{GCAP}|_N$, then restricted to the real line we have $f\in\mathrm{GRAP}_N$. Similarly, if $f\in|\mathrm{GCTP}|_N$, then restricted to the real line we have $f\in|\mathrm{GRTP}|_N$. These follow from the observations

$$|z-z_j|=\left((z-z_j)(z-\bar{z}_j)\right)^{1/2}, \quad z\in\mathbb{R}, \tag{2.12}$$

and

$$\begin{aligned}\left|\sin\left(\tfrac{1}{2}(z-z_j)\right)\right| &= \left(\sin\left(\tfrac{1}{2}(z-z_j)\right)\sin\left(\tfrac{1}{2}(z-\bar{z}_j)\right)\right)^{1/2}\\ &= \frac{1}{\sqrt{2}}\left(\cosh(\operatorname{Im} z_j)-\cos(z-\operatorname{Re} z_j)\right)^{1/2}, \quad z\in\mathbb{R}.\end{aligned} \tag{2.13}$$

Using (2.12) and (2.13), one can easily check that, restricted to the real line, we have

$$|\mathrm{GCAP}|_N = \left\{ f \in \prod_{j=1}^{k} P_j^{r_j/2} : 0 \leqslant P_j \in \mathscr{P}_2^{\mathrm{r}},\ 0 < r_j \in \mathbb{R},\ j = 1, 2, \ldots, k;\ \sum_{j=1}^{k} r_j \leqslant N \right\}$$

and

$$|\mathrm{GCTP}|_N = \left\{ f \in \prod_{j=1}^{k} P_j^{r_j/2} : 0 \leqslant P_j \in \mathscr{T}_1^{\mathrm{r}},\ 0 < r_j \in \mathbb{R},\ j = 1, 2, \ldots, k;\ \sum_{j=1}^{k} r_j \leqslant 2N \right\}.$$

Therefore, restricted to the real line, $|\mathrm{GCAP}|_N = \{f \in \mathrm{GRAP}_N : f \geqslant 0\}$ and $|\mathrm{GCTP}|_N = \{f \in \mathrm{GRTP}_N : f \geqslant 0\}$.

In Section 11 we will work with the following classes of constrained (generalized) polynomials. Denote by $\mathscr{P}_{n,k}^{\mathrm{r}}$, $0 \leqslant k \leqslant n$, the set of all $p \in \mathscr{P}_n^{\mathrm{r}}$ having at most k zeros (by counting multiplicities) in the open unit disk. Let $|\mathrm{GCAP}|_{N,K}$, $0 \leqslant K \leqslant N$, be the set of all $f \in |\mathrm{GCAP}|_N$ of the form (2.7) for which

$$\sum_{\{j: |z_j| < 1\}} r_j \leqslant K. \tag{2.14}$$

Let $\mathscr{M}$ denote the set of all finite Borel measures μ on $\mathbb{C}$ with compact support and $\mu(\mathbb{C}) > 0$. For $\mu \in \mathscr{M}$ and $c \in \mathbb{R}$ we define

$$P_{\mu,c}(z) = \int_{\mathbb{C}} \log|z - t| \,\mathrm{d}\mu(t) + c, \quad z \in \mathbb{C}, \tag{2.15}$$

and

$$Q_{\mu,c}(z) = \exp\left(P_{\mu,c}(z)\right). \tag{2.16}$$

Let $\Lambda = \{\lambda_j\}_{j=0}^{\infty}$, $0 \leqslant \lambda_0 < \lambda_1 < \cdots$. The set of all Müntz polynomials of the form $p(z) = \sum_{j=0}^{n} a_j x^{\lambda_j}$ with real coefficients a_j is denoted by $H_n(\Lambda)$. Let $H(\Lambda) = \bigcup_{n=0}^{\infty} H_n(\Lambda)$.

3. Remez-type inequalities for algebraic polynomials on $[-1, 1]$ and for trigonometric polynomials on $[-\pi, \pi]$

Assume that the absolute value of a real algebraic polynomial of degree at most n is less than 1 on a subset $A \subset [-1, 1]$, and $m(A) \geqslant 2 - s$, $0 < s < 2$, is known, where $m(A)$ denotes the Lebesgue measure of A. The question that arises is how large the polynomial can be on $[-1, 1]$ in terms of n and s. This was answered by Chebyshev when this subset $A \subset [-1, 1]$ is an interval; however, his elegant "zero counting" method fails to work when we do not have this additional piece of information. The solution, due to Remez [53], and its application in the theory of orthogonal polynomials can be found in [32, pp. 119–121], a simpler proof is given in [14]. To formulate the Remez inequality we need the Chebyshev polynomials $T_n \in \mathscr{P}_n^{\mathrm{r}}$ defined by

$$T_n(x) = \cos(n \arccos x), \quad -1 \leqslant x \leqslant 1. \tag{3.1}$$

Since $T_n \in \mathscr{P}_n^{\mathrm{r}}$, (3.1) defines T_n on the whole complex plane. The following explicit formula for T_n is well known [32, p.34]. We have

$$T_n(x) = \tfrac{1}{2}\left(\left(x + \sqrt{x^2 - 1}\right)^n + \left(x - \sqrt{x^2 - 1}\right)^n\right), \tag{3.2}$$

for every $x \in \mathbb{R} \setminus (-1, 1)$.

Theorem 3.1 (Remez). *Let* $0 < s < 2$. *We have*

$$\max_{-1 \leqslant x \leqslant 1} |p(x)| \leqslant T_n\left(\frac{2+s}{2-s}\right), \tag{3.3}$$

for every $p \in \mathscr{P}_n^{\mathrm{r}}$ *satisfying*

$$m(\{x \in [-1, 1] : |p(x)| \leqslant 1\}) \geqslant 2 - s. \tag{3.4}$$

Equality holds in (3.3) *if and only if*

$$p = \pm T_n\left(\frac{\pm 2x}{2-s} + \frac{s}{2-s}\right)$$

(*these are the Chebyshev polynomials* $\pm T_n$ *transformed linearly from* $[-1, 1]$ *to either* $[-1, 1-s]$ *or* $[-1+s, 1]$).

Theorem 3.1 and formula (3.2) give the following immediately.

Corollary 3.2. *We have*

$$\max_{-1 \leqslant x \leqslant 1} |p(x)| \leqslant \exp(5n\sqrt{s}), \quad 0 < s \leqslant 1, \tag{3.5}$$

and

$$\max_{-1 \leqslant x \leqslant 1} |p(x)| \leqslant \left(\frac{8}{2-s}\right)^n, \quad 1 < s < 2, \tag{3.6}$$

for every $p \in \mathscr{P}_n^{\mathrm{r}}$ *satisfying* (3.4).

The constants 5 and 8 in the above corollary are not optimal, slightly better constants are obtained in Section 6.

A natural question that arises now is the following. How large can $|p(x)|$ be if $x \in [-1, 1]$ is fixed and $p \in \mathscr{P}_n^{\mathrm{r}}$ satisfies (3.4)? An obvious bound for $|p(x)|$ comes immediately from the Remez inequality, but it turns out that for any fixed $x \in (-1, 1)$ this can be essentially improved. The following theorem is proved in [22, Theorem 4].

Theorem 3.3. *There is an absolute constant* $c_1 > 0$ *such that*

$$|p(x)| \leqslant \exp\left(c_1 n \min\left\{\frac{s}{\sqrt{1-x^2}}, \sqrt{s}\right\}\right), \quad 0 < s \leqslant 1, \ -1 \leqslant x \leqslant 1, \tag{3.7}$$

for every $p \in \mathscr{P}_n^{\mathrm{r}}$ *satisfying* (3.4).

The sharpness (up to the constant c_1) is also shown in [22, Section 12]. We remark that in case of $1 < s < 2$ there is no essentially better pointwise Remez-type bound than the uniform one given by (3.6). This explains why we deal only with the case $0 < s \leqslant 1$ in Theorem 3.3.

The proof of Theorem 3.3 is based on an essentially sharp Remez-type inequality for trigonometric polynomials; however, it is not a simple "substitution $x = \cos t$" argument. This trigonometric Remez-type inequality [22, Theorem 2] states the following.

Theorem 3.4. *There is an absolute constant $c_2 > 0$ such that*

$$\max_{-\pi \leqslant t \leqslant \pi} |p(t)| \leqslant \exp(c_2 n s), \quad 0 < s \leqslant \tfrac{1}{2}\pi, \tag{3.8}$$

for every $p \in \mathcal{T}_n^{\mathrm{r}}$ satisfying

$$m(\{t \in [-\pi, \pi] : |p(t)| \leqslant 1\}) \geqslant 2\pi - s. \tag{3.9}$$

In fact, $c_2 ns$ in (3.8) can be replaced by $n(s + 1.75\, s^2)$.

The last statement of Theorem 3.4 is not stated in [22, Theorem 2]; however, as von Golitschek and Lorentz pointed it out, one can easily verify it by a straightforward calculation, analyzing the proof given in [22]. Though Theorem 3.4 with its unnatural restriction $0 < s \leqslant \frac{1}{2}\pi$ turned out to be completely satisfactory in several applications, it is still a natural question what happens when $\frac{1}{2}\pi < s < 2\pi$. It can be proved that there is an absolute constant $c_3 > 0$ such that

$$\max_{-\pi \leqslant t \leqslant \pi} |p(t)| \leqslant \left(\frac{c_3}{2\pi - s}\right)^n, \quad \tfrac{1}{2}\pi < s < 2\pi, \tag{3.10}$$

for every $p \in \mathcal{T}_n^{\mathrm{r}}$ satisfying (3.9). This result has never been published.

If we know that the absolute value of a trigonometric polynomial of degree at most n is not greater than 1 on an interval of prescribed length, we can give the exact pointwise bound for the polynomial outside this interval To be able to use Chebyshev's classical method based on zero counting, it is very important to define a trigonometric polynomial of degree n, which equioscillates $2n + 1$ times on the interval $[-\omega, \omega]$, $0 < \omega \leqslant \pi$. This is possible by taking

$$Q_{n,\omega}(t) = T_{2n}\left(\frac{\sin(\frac{1}{2}t)}{\sin(\frac{1}{2}\omega)}\right) \in \mathcal{T}_n^{\mathrm{r}}, \tag{3.11}$$

where $T_{2n}(x) = \cos(2n \arccos x)$, $-1 \leqslant x \leqslant 1$, is the Chebyshev polynomials of degree $2n$. This polynomial was used in [62] to establish sharp Markov- and Bernstein-type inequalities for the derivatives of trigonometric polynomials on an interval $[-\omega, \omega]$ shorter than the period. With the help of the trigonometric polynomial $Q_{n,\omega}$, we can formulate the following theorem [22, Lemma 3].

Theorem 3.5. *Let $p \in \mathcal{T}_n^{\mathrm{r}}$ and $|p(t)| \leqslant 1$ for every $t \in [-\omega, \omega]$, where $0 < \omega < \pi$. Then $|p(y)| \leqslant Q_{n,\omega}(y)$ for every $y \in [-\pi, \pi) \setminus (-\omega, \omega)$.*

From (3.11) and (3.2) we easily deduce that there are absolute constants $0 < c_4 < c_5$ such that

$$\exp(c_4 n(\pi - \omega)) \leqslant Q_{n,\omega}(\pi) \leqslant \exp(c_5 n(\pi - \omega)), \tag{3.12}$$

for every $\frac{1}{2}\pi < \omega < \pi$. This shows that Theorem 3.4 is sharp up to the constant c_3. I have the following conjecture.

Conjecture 3.6. *Let* $0 < s < 2\pi$ *and* $\omega = \pi - \frac{1}{2}s$. *Then*

$$\max_{-\pi \leqslant t \leqslant \pi} |p(t)| \leqslant Q_{n,\omega}(\pi), \tag{3.13}$$

for every $p \in \mathcal{T}_n^{\mathrm{r}}$ *satisfying* (3.9).

4. Remez-type inequalities for $|\mathrm{GCAP}|_N$ on $[-1, 1]$ and for $|\mathrm{GCTP}|_N$ on $[-\pi, \pi]$

The Remez-type inequalities of Section 3 can be easily extended to generalized nonnegative algebraic and trigonometric polynomials by utilizing the generalized degree N in place of the ordinary degree. To illustrate this, we give the proof of our Remez-type inequality for generalized nonnegative trigonometric polynomials using a simple density argument. One would expect to extend a number of polynomial inequalities for generalized nonnegative polynomials in this spirit. However, a serious problem arises from the fact that density arguments do not seem to work in most of the cases. To obtain extensions of polynomial inequalities to the classes $|\mathrm{GCAP}|_N$ and $|\mathrm{GCTP}|_N$ (see the definitions in Section 2), the basic idea is the following. The Remez-type inequalities for $|\mathrm{GCAP}|_N$ and $|\mathrm{GCTP}|_N$ are easy to obtain from the corresponding polynomial inequalities; however, they seem general and deep enough to try to deduce other inequalities (such as Nikolskii-, Markov-, Bernstein- and Schur-type inequalities) for $|\mathrm{GCAP}|_N$ and $|\mathrm{GCTP}|_N$ from them. It turns out that Remez-type inequalities for $|\mathrm{GCAP}|_N$ and $|\mathrm{GCTP}|_N$ play a significant role in the proof of our results in Sections 5, 9, 11 and 12. This is illustrated in Section 5 by giving a short proof of a Nikolskii-type inequality for $|\mathrm{GCAP}|_N$ based on the corresponding Remez-type inequality from this section. After explaining their importance we present the results.

To establish the analogue of the Remez inequality (Theorem 3.1) for $|\mathrm{GCAP}|_N$ it may seem hard to tell what we should put in place of the Chebyshev polynomial T_n in (3.3), since T_N does not make any sense. Nevertheless, as a consequence of our sharp Remez-type inequality for exponentials of potentials (Theorem 6.1), we obtained the following corollary.

Corollary 4.1. *Let* $0 < s < 2$. *Then*

$$\max_{-1 \leqslant x \leqslant 1} f(x) \leqslant \left(\frac{\sqrt{2} + \sqrt{s}}{\sqrt{2} - \sqrt{s}} \right)^N \tag{4.1}$$

for every $f \in |\mathrm{GCAP}|_N$ *satisfying*

$$m(\{x \in [-1, 1]: f(x) \leqslant 1\}) \geqslant 2 - s. \tag{4.2}$$

The sharpness of the above inequalities in a "limit sense" can be seen by taking the generalized nonnegative algebraic polynomials

$$f = |T_n|^{N/n} \in |\mathrm{GCAP}|_N, \quad n = 1, 2, \ldots, \tag{4.3}$$

where T_n is the Chebyshev polynomial of degree n defined by (3.1). In [22, Theorem 1] the following less sharp result was proved as the extension of Corollary 3.2 to the class $|\mathrm{GCAP}|_N$.

Theorem 4.2. *We have*

$$\max_{-1 \leqslant x \leqslant 1} f(x) \leqslant \exp(5N\sqrt{s}), \quad 0 < s \leqslant 1, \tag{4.4}$$

and

$$\max_{-1 \leqslant x \leqslant 1} f(x) \leqslant \left(\frac{8}{2-s}\right)^N, \quad 1 < s < 2, \tag{4.5}$$

for every $f \in |\mathrm{GCAP}|_N$ *satisfying* (4.2).

The extension of Theorem 3.3 to the class $|\mathrm{GCAP}|_N$ is given in [22, Theorem 4].

Theorem 4.3. *We have*

$$f(x) \leqslant \exp\left(c_1 N \min\left\{\frac{s}{\sqrt{1-x^2}}, \sqrt{s}\right\}\right), \tag{4.6}$$

for every $f \in |\mathrm{GCAP}|_N$ *satisfying* (4.2). *Here* c_1 *is the same as in Theorem* 3.3.

The case $1 < s < 2$ is not examined here, see the comment right after Theorem 3.3. The extension of Theorem 3.4 to the class $|\mathrm{GCTP}|_N$ [22, Theorem 2] is formulated in the following theorem.

Theorem 4.4. *We have*

$$\max_{-\pi \leqslant t \leqslant \pi} f(t) \leqslant \exp(c_2 N s), \quad 0 < s \leqslant \tfrac{1}{2}\pi, \tag{4.7}$$

for every $f \in |\mathrm{GCTP}|_N$ *satisfying*

$$m(\{t \in [-\pi, \pi] : p(t) \leqslant 1\}) \geqslant 2\pi - s. \tag{4.8}$$

Here c_2 *is the same as in Theorem* 3.4.

For $\frac{1}{2}\pi < s < 2\pi$, the extension of (3.10) asserts that

$$\max_{-\pi < t \leqslant \pi} f(t) \leqslant \left(\frac{c_3}{2\pi - s}\right)^N \tag{4.9}$$

for every $f \in |\mathrm{GCTP}|_N$ satisfying (4.8).

To convince the reader that the results of this section are straightforward consequences of the corresponding polynomial inequalities of Section 3, we show how Theorem 4.4 follows from Theorem 3.4.

Proof of Theorem 4.4. Let $f \in |\mathrm{GCTP}|_N$ satisfy (4.8) and first assume that each r_j is rational in the representation (2.11), thus let $r_j = q_j/q$, $j = 1, 2, \ldots, k$, with some positive integers q_j and

q. Observe that the trigonometric polynomial

$$p(z) = |\omega|^{2q} \prod_{j=1}^{k} \left(\sin\left(\tfrac{1}{2}(z - z_j)\right) \sin\left(\tfrac{1}{2}(z - \bar{z}_j)\right)\right)^{q_j} \in T_{2Nq}$$

satisfies (3.9); hence by Theorem 3.4 we obtain

$$\max_{-\pi \leqslant t \leqslant \pi} f(t) = \max_{-\pi \leqslant t \leqslant \pi} |p(t)|^{1/(2q)} \leqslant (\exp(c_2 2Nqs))^{1/(2q)} = \exp(c_2 Ns),$$

which gives the desired result. In general, when the exponents r_j are arbitrary positive real numbers in the representation (2.11) of f, we obtain the theorem from the already proved rational case by a limiting argument. □

The following pair of theorems [25, Theorems 8 and 9] gives the L_p, $0 < p < \infty$, analogues of Theorems 4.2 and 4.4.

Theorem 4.5. *Let χ be a nonnegative, nondecreasing function defined on* $[0, \infty)$ *such that* $\chi(x)/x$ *is nonincreasing on* $[0, \infty)$. *There is an absolute constant* $c_6 > 0$ *such that*

$$\int_{-1}^{1} (\chi(f(x)))^p \, \mathrm{d}x \leqslant \left(1 + \exp(c_6 pN\sqrt{s})\right) \int_A (\chi(f(x)))^p \, \mathrm{d}x, \tag{4.10}$$

for every $f \in |\mathrm{GCAP}|_N$, $0 < p < \infty$, $0 < s \leqslant \frac{1}{2}$ *and* $A \subset [-1, 1]$ *with* $m(A) \geqslant 2 - s$. *Here* $c_6 = 5\sqrt{2}$ *is a suitable choice.*

Theorem 4.6. *Let χ be the same as in Theorem* 4.5. *There is an absolute constant* $c_7 > 0$ *such that*

$$\int_{-\pi}^{\pi} (\chi(f(t)))^p \, \mathrm{d}t \leqslant (1 + \exp(c_7 pNs)) \int_A (\chi(f(t)))^p \, \mathrm{d}t, \tag{4.11}$$

for every $f \in |\mathrm{GCTP}|_N$, $0 < p < \infty$, $0 < s < \frac{1}{4}\pi$ *and* $A \subset [-\pi, \pi)$ *with* $m(A) \geqslant 2\pi - s$. *If* $0 < s \leqslant \frac{1}{4}$, *then* $c_7 = 4$ *is a suitable choice.*

Theorems 4.5 and 4.6 are simple consequences of our Theorems 4.2 and 4.4, respectively; we demonstrate it by giving the proof of Theorem 4.6, following the method used in [25].

Proof of Theorem 4.6. For an $f \in |\mathrm{GCTP}|_N$ we define the sets

$$I_1(f) = \left\{t \in [-\pi, \pi): (\chi(f(t)))^p \geqslant \exp(-2c_2 pNs) \max_{-\pi \leqslant \tau \leqslant \pi} (\chi(f(\tau)))^p\right\}, \tag{4.12}$$

$$I_2(f) = \left\{t \in [-\pi, \pi): f(t) \geqslant \exp(-2c_2 Ns) \max_{-\pi \leqslant \tau \leqslant \pi} f(\tau)\right\}, \tag{4.13}$$

where $c_2 > 0$ is defined in Theorem 4.4. By the conditions prescribed for χ, one can easily deduce that

$$I_2(f) \subset I_1(f). \tag{4.14}$$

Further, Theorem 4.4 implies that

$$m(I_2(f)) \geqslant 2s, \quad 0 < s \leqslant \tfrac{1}{4}\pi, \tag{4.15}$$

which, together with (4.14), yields

$$m(I_1(f)) \geqslant 2s, \quad 0 < s \leqslant \tfrac{1}{4}\pi. \tag{4.16}$$

Now let $I = A \cap I_1(f)$. Since $m(A) \geqslant 2 - s$, we have $m(I) \geqslant s$. Therefore, by (4.12) we obtain

$$\begin{aligned}\int_{[-\pi,\pi]\setminus A} (\chi(f(t)))^p \,\mathrm{d}t &\leqslant \int_{[-\pi,\pi]\setminus A} \max_{-\pi\leqslant\tau\leqslant\pi} (\chi(f(\tau)))^p \,\mathrm{d}t \\ &\leqslant \exp(2c_2 pNs) \int_I (\chi(f(t)))^p \,\mathrm{d}t \\ &\leqslant \exp(2c_2 pNs) \int_A (\chi(f(t)))^p \,\mathrm{d}t,\end{aligned}$$

and the theorem is proved. □

5. Nikolskii-type inequalities for $|\mathrm{GCAP}|_N$ on $[-1, 1]$ and for $|\mathrm{GCTP}|_N$ on $[-\pi, \pi]$

In this section we give sharp upper bounds for the $L_p(-1, 1)$-norm of functions from $|\mathrm{GCAP}|_N$ if their $L_q(-1, 1)$-norm is bounded by 1 and $0 < q < p \leqslant \infty$. This is the content of Theorem 5.1. The trigonometric analogue of this result is established by Theorem 5.2. In Theorems 5.3–5.5 we offer some weighted analogues of Theorems 5.1 and 5.2 for wide families of weight functions. Such inequalities are called Nikolskii-type inequalities. The proofs of these Nikolskii-type inequalities are based heavily on the Remez-type inequalities of Section 4, and we demonstrate this by proving Theorem 5.2 following a method from [25, Theorem 5]. Schur-type inequalities (which can be considered as special Nikolskii-type inequalities) for $|\mathrm{GCAP}|_N$ with close to sharp constant will be given by Theorem 5.6. Applications of Nikolskii-type inequalities in the theory of orthogonal polynomials will be given in Section 12.

The following pair of theorems was proved in [25, Theorems 5 and 6].

Theorem 5.1. *Let χ be a nonnegative, nondecreasing function defined on* $[0, \infty)$ *such that $\chi(x)/x$ is nonincreasing on* $[0, \infty)$. *There is an absolute constant $c_8 > 0$ such that*

$$\|\chi(f)\|_{L_p(-1,1)} \leqslant \left(c_8(2 + qN)\right)^{2/q-2/p} \|\chi(f)\|_{L_q(-1,1)}, \tag{5.1}$$

for every $f \in |\mathrm{GCAP}|_N$ and $0 < q < p \leqslant \infty$. If $\chi(x) = x$, then $c_8 = \mathrm{e}^2(2\pi)^{-1}$ is a suitable choice.

Theorem 5.2. *Let χ be the same as in Theorem* 5.1. *There is an absolute constant $c_9 > 0$ such that*

$$\|\chi(f)\|_{L_p(-\pi,\pi)} \leqslant \left(c_9(1 + qN)\right)^{1/q-1/p} \|\chi(f)\|_{L_q(-\pi,\pi)}, \tag{5.2}$$

for every $f \in |\mathrm{GCTP}|_N$ and $0 < q < p \leqslant \infty$. If $\chi(x) = x$, then $c_9 = \mathrm{e}(4\pi)^{-1}$ is a suitable choice.

Following [25, Theorem 5], we present the proof of the first statement of Theorem 5.2, which gives a greater constant than $c_9 = \mathrm{e}(4\pi)^{-1}$ in the case $\chi(x) = x$.

Proof of Theorem 5.2. For the sake of brevity we use the short notation $\|\cdot\|_p = \|\cdot\|_{L_p(-\pi,\pi)}$. It is sufficient to prove the theorem when $p=\infty$, and then a simple argument gives the desired result for arbitrary $0<q<p\leqslant\infty$. To see this, assume that

$$\|\chi(f)\|_\infty \leqslant K^{1/q}\|\chi(f)\|_q,$$

for every $f\in|\mathrm{GCTP}|_N$ and $0<q<\infty$, with some constant K. Then

$$\|\chi(f)\|_p^p \leqslant \|(\chi(f))^{p-q+q}\|_1 \leqslant \|\chi(f)\|_\infty^{p-q}\|\chi(f)\|_q^q$$
$$\leqslant K^{p/q-1}\|\chi(f)\|_q^{p-q}\|\chi(f)\|_q^q,$$

for every $f\in|\mathrm{GCTP}|_N$ and $0<q<\infty$, and therefore,

$$\|\chi(f)\|_p \leqslant K^{1/q-1/p}\|\chi(f)\|_q,$$

for every $f\in|\mathrm{GCTP}|_N$ and $0<q<\infty$.

Now let $p=\infty$. We show that

$$m\Big(\big\{t\in[-\pi,\pi) = (\chi(f(t)))^q \geqslant \exp(-qNs)\|\chi(f)\|_\infty^q\big\}\Big) \geqslant c_{10}s, \tag{5.3}$$

for every $f\in|\mathrm{GCTP}|_N$ and $s\in(0,2\pi)$, where $c_{10}:=\min\{1/c_2, \frac{1}{4}\}$. Indeed, since χ is nonnegative and nondecreasing and $(\chi(x)/x)^q$ is nonincreasing on $[0,\infty)$, we have

$$(\chi(f(t)))^q \geqslant \exp(-qNs)\|\chi(f)\|_\infty^q, \tag{5.4}$$

for every real t satisfying

$$f(t)\geqslant \exp(-Ns)\|f\|_\infty, \tag{5.5}$$

so it is sufficient to prove (5.3) only when $\chi(x)=x$ and $q=1$. If (5.3) were not true in this case, then

$$g:=f\exp(Ns)\|f\|_\infty^{-1} \tag{5.6}$$

would satisfy

$$m(\{t\in[-\pi,\pi):\ g(t)\leqslant 1\}) > 2\pi - c_{10}s, \tag{5.7}$$

and Theorem 4.4 would imply

$$\|g\|_\infty < \exp(c_2c_{10}Ns) \leqslant \exp(Ns), \tag{5.8}$$

which would contradict (5.6). Thus (5.3) holds indeed. Choosing $s=(1+qN)^{-1}$ in (5.3), we obtain

$$m\Big(\big\{t\in[-\pi,\pi) = (\chi(f(t)))^q \geqslant \mathrm{e}^{-1}\|\chi(f)\|_\infty^q\big\}\Big) \geqslant c_{10}(1+qN)^{-1}, \tag{5.9}$$

for every $f\in|\mathrm{GCTP}|_N$. Now integrating only on the subset I of $[-\pi,\pi)$, where

$$(\chi(f(t)))^q \geqslant \mathrm{e}^{-1}\|\chi(f)\|_\infty^q, \tag{5.10}$$

and using (5.9), we get

$$\|\chi(f)\|_\infty^q \leqslant \mathrm{e}c_{10}^{-1}(1+qN)\int_I(\chi(f(t)))^q\,\mathrm{d}t \leqslant \mathrm{e}c_{10}^{-1}(1+qN)\|\chi(f)\|_q^q, \tag{5.11}$$

for every $f\in|\mathrm{GCTP}|_N$; thus the first statement of Theorem 5.2 is proved. □

If g is a measurable function on the interval $[a, b]$ and for every $\lambda > 0$ there is a constant $K = K(g)$ depending only on g such that

$$m(\{x \in [a, b] = |g(x)| \geqslant \lambda\}) \leqslant K(g)\lambda^{-p}, \tag{5.12}$$

then we say that g is in weak $L_p(a, b)$ and we use the notation $g \in \mathrm{WL}_p(a, b)$. It is obvious that if g is in $L_p(a, b)$, then g is in $\mathrm{WL}_p(a, b)$. In the rest of this section w denotes a nonnegative weight function from $L_1(-1, 1)$. Let $\log^-(x) := \min\{\log x, 0\}$. In [20, Theorems 3–5] the following Nikolskii-type inequalities are proved.

Theorem 5.3. *Let* $0 < \alpha < 1$, $p = 2/\alpha - 2$ *and* $\log^-(w) \in \mathrm{WL}_p(-1, 1)$. *There is a constant* $c(\alpha, K)$ *depending only on* α *and* $K = K(\log^-(w))$ (*see* (5.12)) *such that*

$$\max_{-1 \leqslant x \leqslant 1} (f(x))^q \leqslant \exp\left(c(\alpha, K)(1 + qN)^\alpha\right) \int_{-1}^1 (f(x))^q w(x)\, \mathrm{d}x \tag{5.13}$$

and

$$\left(\int_{-1}^1 (f(x))^p w(x)\, \mathrm{d}x\right)^{1/p} \leqslant \left(\exp(c(\alpha, K)(1 + qN^\alpha))\right)^{1/q - 1/p} \left(\int_{-1}^1 (f(x))^q w(x)\, \mathrm{d}x\right)^{1/q} \tag{5.14}$$

for every $f \in |\mathrm{GCAP}|_N$ *and* $0 < q < p < \infty$.

In our next theorem we take only half as large p as in Theorem 5.3, but we assume that $\log^-(w(\cos\theta)) \in \mathrm{WL}_p(-\pi, \pi)$ and we obtain the same conclusion.

Theorem 5.4. *Let* $0 < \alpha < 1$, $p = 1/\alpha - 1$ *and* $\log^-(w(\cos\theta)) \in \mathrm{WL}_p(-\pi, \pi)$. *There is a constant* $c(\alpha, K)$ *depending only on* α *and* $K = K(\log^-(w(\cos\theta)))$ (*see* (5.12)) *such that* (5.13) *and* (5.14) *hold for every* $f \in |\mathrm{GCAP}|_N$ *and* $0 < q < p < \infty$.

We remark that if $\frac{1}{2} \leqslant \alpha < 1$, $0 < p \leqslant 1$, then the Szegő class

$$\{f \in L_1(-1, 1): f \geqslant 0, \log(f(\cos\theta)) \in L_1(-\pi, \pi)\}$$

is properly contained in the classes of Theorem 5.4. The Nikolskii-type inequalities of our next theorem give better upper bounds for less wide classes.

Theorem 5.5. *Let* $w^{-\epsilon} \in \mathrm{WL}_1(-1, 1)$ *for some* $\epsilon > 0$ *and let* $M = 2/\epsilon + 2$. *There is a constant* $c(\epsilon, K)$ *depending only on* ϵ *and* $K = K(w^{-\epsilon})$ (*see* (5.12)) *such that*

$$\max_{-1 \leqslant x \leqslant 1} (f(x))^q \leqslant c(\epsilon, K)(1 + qN)^M \int_{-1}^1 (f(x))^q w(x)\, \mathrm{d}x \tag{5.15}$$

and

$$\left(\int_{-1}^1 (f(x))^p w(x)\, \mathrm{d}x\right)^{1/p} \leqslant \left(c(\epsilon, K)(1 + qN)^M\right)^{1/q - 1/p} \left(\int_{-1}^1 (f(x))^q w(x)\, \mathrm{d}x\right)^{1/q}, \tag{5.16}$$

for every $f \in |\mathrm{GCAP}|_N$ *and* $0 < q < p < \infty$.

The Remez-type inequalities of Section 4 play a central role in the proof of Theorems 5.3–5.5. Similarly to the proof of Theorem 5.2, it is sufficient to prove only the first inequality of each theorem, and then the second one follows immediately. To prove the first inequality of each theorem, the basic idea is to integrate only on a sufficiently large subset of $[-1, 1]$, where both $f\|f\|_{L_\infty(-1,1)}^{-1}$ and the weight function w are "sufficiently large", and to balance in an optimal way.

We close this section with a Schur-type inequality [25, Theorem 7] for the class $|\mathrm{GCAP}|_N$ with an almost sharp constant.

Theorem 5.6. *We have*

$$\max_{-1\leqslant x\leqslant 1}(f(x))^q \leqslant \mathrm{e}(1+qN)\max_{-1\leqslant x\leqslant 1}\left((f(x))^q\sqrt{1-x^2}\right), \tag{5.17}$$

for every $f\in|\mathrm{GCAP}|_N$ *and* $0<q<\infty$.

According to the well-known Schur inequality [55], if f is an ordinary polynomial and q is a positive integer, then the constant e in the above inequality can be replaced by 1. For ordinary algebraic polynomials and for arbitrary $0<q<\infty$, Theorem 5.6 was also proved in [33, Remark 3, p.18]. We remark that with an absolute constant c instead of e, Theorem 5.6 is a simple consequence of the Remez-type inequality (Theorem 4.2) for $|\mathrm{GCAP}|_N$.

6. Remez-type inequalities for exponentials of logarithmic potentials on $[-1, 1]$ and on the unit circle

In this section we state our main results concerning Remez-type inequalities for exponentials of logarithmic potentials on $[-1, 1]$ and on the unit circle. It turns out that each inequality from Section 4 has an analogue for exponentials of logarithmic potentials, by utilizing $\mu(\mathbb{C})$ in place of the generalized degree N. Our first theorem [24, Theorem 2.1] establishes a sharp Remez-type inequality for exponentials of logarithmic potentials on $[-1, 1]$.

Theorem 6.1. *Let* $0<s<2$; *then*

$$\max_{-1\leqslant x\leqslant 1} Q_{\mu,c}(x) \leqslant \left(\frac{\sqrt{2}+\sqrt{s}}{\sqrt{2}-\sqrt{s}}\right)^{\mu(\mathbb{C})}, \tag{6.1}$$

for every $\mu\in\mathcal{M}$ *and* $c\in\mathbb{R}$ *satisfying*

$$m\left(\left\{x\in[-1, 1]: Q_{\mu,c}(x)\leqslant 1\right\}\right)\geqslant 2-s. \tag{6.2}$$

Furthermore, if $Q_{\mu,c}$ *restricted to* $[-1, 1]$ *is continuous on* $[-1, 1]$, *then equality in* (6.1) *holds if and only if*

$$\mu=\mu(\mathbb{C})\mu^*_{[-1,1-s]} \quad or \quad \mu=\mu(\mathbb{C})\mu^*_{[-1+s,1]}$$

and

$$c=-\mu(\mathbb{C})\log\left(\tfrac{1}{4}(2-s)\right),$$

where μ^*_K *denotes the equilibrium measure* (*cf.* [61, *Section* III.2]) *of a compact set* $K\subset\mathbb{C}$.

We remark that $Q_{\mu,c}$ is upper semi-continuous on $\mathbb{C}$, so the maximum on $[-1, 1]$ is attained. We believe that the assumption that $Q_{\mu,c}$ restricted to $[-1, 1]$ is continuous on $[-1, 1]$ can be dropped from the second statement of Theorem 6.1.

Concerning pointwise upper bounds for $Q_{\mu,c}(x)$ we shall prove the following result that extends the validity of Theorem 4.3.

Theorem 6.2. *We have*

$$Q_{\mu,c}(x) \leqslant \exp\left(c_1\mu(\mathbb{C}) \min\left\{\frac{s}{\sqrt{1-x^2}}, \sqrt{s}\right\}\right), \quad 0<s\leqslant 1, \ -1\leqslant x\leqslant 1, \tag{6.3}$$

for every $\mu \in \mathscr{M}$ and $c \in \mathbb{R}$ satisfying (6.2). *Here c_1 is the same as in Theorem* 3.3.

Theorem 6.2 is proved in [24, Theorem 2.2]. The case $1<s<2$ is not studied here, see the comment right after Theorem 3.3. The analogue of the trigonometric Remez-type inequality (Theorem 4.4) is given by the following theorem.

Theorem 6.3. *We have*

$$\max_{-\pi\leqslant t\leqslant\pi} Q_{\mu,c}(e^{it}) \leqslant \exp(c_2\mu(\mathbb{C})s), \quad 0<s\leqslant \tfrac{1}{2}\pi, \tag{6.4}$$

for every $\mu \in \mathscr{M}$ and $c \in \mathbb{R}$ satisfying

$$m\left(\left\{t \in [-\pi, \pi): Q_{\mu,c}(e^{it}) \leqslant 1\right\}\right) \geqslant 2\pi - s. \tag{6.5}$$

Here c_2 is the same as in Theorem 3.4.

Theorem 6.3 is proved in [24, Theorem 2.9]. Using Theorems 6.1 and 6.3, in [24, Theorems 2.7 and 2.10] we established Remez-type inequalities in L_p, $0<p<\infty$, for exponentials of logarithmic potentials on both $[-1, 1]$ and the unit circle. These extend the results of Theorems 4.5 and 4.6.

Theorem 6.4. *Let χ be a nonnegative, nondecreasing function defined on* $[0, \infty)$ *such that $\chi(x)/x$ is nonincreasing on* $[0, \infty)$. *There is an absoute constant $c_{11}>0$ such that*

$$\begin{aligned}\int_{-1}^{1} \left(\chi(Q_{\mu,c}(x))\right)^p \,dx &\leqslant \left(1+\left(\frac{1+\sqrt{s}}{1-\sqrt{s}}\right)^{p\mu(\mathbb{C})}\right)\int_A \left(\chi(Q_{\mu,c}(x))\right)^p \,dx \\ &\leqslant \left(1+\exp\left(c_{11}p\mu(\mathbb{C})\sqrt{s}\right)\right)\int_A \left(\chi(Q_{\mu,c}(x))\right)^p \,dx,\end{aligned} \tag{6.6}$$

for every $\mu \in \mathscr{M}$, $c \in \mathbb{R}$, $0<p<\infty$, $0<s\leqslant \frac{1}{2}$ and $A \subset [-1, 1]$ with $m(A) \geqslant 2-s$. Here $c_{11}=5\sqrt{2}$ is a suitable choice.

Theorem 6.5. *Let χ be the same as in Theorem* 6.4. *There is an absolute constant $c_{12}>0$ such that*

$$\int_{-\pi}^{\pi} \left(\chi(Q_{\mu,c}(x))\right)^p \,dx \leqslant \left(1+\exp(c_{12}p\mu(\mathbb{C})s)\right)\int_A \left(\chi(Q_{\mu,c}(x))\right)^p \,dx, \tag{6.7}$$

for every $\mu \in \mathscr{M}$, $c \in \mathbb{R}$, $0 < p < \infty$, $0 < s \leqslant \frac{1}{4}\pi$ *and* $A \subset [-1, 1]$ *with* $m(A) \geqslant 2\pi - s$. *If* $0 < s \leqslant \frac{1}{4}$, *then* $c_{12} = 4$ *is a suitable choice.*

Theorems 6.4 and 6.5 can be proved from Theorems 6.1 and 6.3, respectively, similarly to the proof of Theorem 4.5. To obtain the first inequality of Theorem 6.4, the following version of Theorem 6.1 is useful.

Corollary 6.6. *We have*

$$m\left(\left\{x \in [-1, 1] : Q_{\mu,c}(s) > \frac{1-\sqrt{t}}{1+\sqrt{t}} \max_{-1 \leqslant y \leqslant 1} Q_{\mu,c}(y)\right\}\right) \geqslant 2t, \tag{6.8}$$

for every $\mu \in \mathscr{M}$, $c \in \mathbb{R}$ *and* $0 < t < 1$.

To close this section we present the proof of Theorem 6.2, following a typical method from [24]. This will illustrate that our extension is far from a simple density argument.

Proof of Theorem 6.2. For the sake of brevity let

$$E_{\mu,c} = \{x \in [-1, 1] : Q_{\mu,c}(x) \leqslant 1\}. \tag{6.9}$$

If $K \subset \mathbb{C}$ is a compact set, we denote by $D_\infty(K)$ the unbounded component of the complement $\mathbb{C} \setminus K$. This domain is referred to as the *outer domain* of K and its boundary $\partial D_\infty(K)$ is called the outer boundary of K. If K has positive logarithmic capacity [61, p.55], we denote by $g_{D_\infty(K)}(z, \infty)$ the Green function with pole at ∞ for $D_\infty(K)$. We remark that $g_{D_\infty(K)}(z, \infty)$ is the smallest positive harmonic function on $D_\infty(K) \setminus \{\infty\}$ that behaves like $\log |z| + \text{const.}$ near ∞ (cf. [51, p.333]). We may assume that $0 < s < 1$, since the case $s = 1$ follows from Theorem 6.1. Let $\mu \in \mathscr{M}$ and $c \in \mathbb{R}$ be such that (6.2) holds. For a fixed $0 < \epsilon < 1 - s$ we choose a compact set $K \subset E_{\mu,c}$ such that

$$m(E_{\mu,c}) \leqslant \epsilon. \tag{6.10}$$

This, together with assumption (6.2), gives

$$m(K) \geqslant 2 - (s + \epsilon). \tag{6.11}$$

Note that $\mu \in \mathscr{M}$, (6.9), the definition of the Green function $g_{D_\infty(K)}$ and $K \subset E_{\mu,c}$ imply that

$$g_{D_\infty(K)}(z, \infty) - P_{\mu,c}(z)$$

(see (2.15) for the definition of $P_{\mu,c}$) is superharmonic on $D_\infty(K)$, and

$$\liminf_{\substack{z \to K \\ z \in D_\infty(K)}} \left(g_{D_\infty(K)}(z, \infty) - P_{\mu,c}(z)\right) \geqslant 0; \tag{6.12}$$

therefore the minimum principle for superharmonic functions yields

$$g_{D_\infty(K)}(z, \infty) - P_{\mu,c}(z) \geqslant 0, \tag{6.13}$$

for all $z \in D_\infty(K)$. Applying Theorem 3.3 to the nth Fekete polynomials $F_{n,k} \in \mathscr{P}_n^{\mathrm{r}}$ of K [51, p.331], we obtain

$$\frac{1}{n} \log \frac{F_{n,K}(x)}{\| F_{n,K} \|_K} \leqslant c_1 \min\left\{\frac{s+\epsilon}{\sqrt{1-x^2}}, \sqrt{s+\epsilon}\right\}, \tag{6.14}$$

for every $-1 \leqslant x \leqslant 1$ and $0 < s + \epsilon \leqslant 1$ ($\|\cdot\|_K$ denotes the uniform norm on K). By a theorem of Myrberg and Leja [51, Theorem 11.1, p.333] and $m(K) \geqslant 2 - s - \epsilon > 0$, the limit of the left-hand side of (6.14), as $n \to \infty$, exists for every $x \in [-1, 1] \setminus K$, and equals $g_{D_\infty(K)}(x)$. Therefore,

$$g_{D_\infty(K)}(x, \infty) \leqslant c_1 \min\left\{ \frac{s+\epsilon}{\sqrt{1-x^2}}, \sqrt{s+\epsilon} \right\}, \tag{6.15}$$

for every $x \in [-1, 1] \setminus K$ and $0 < s + \epsilon \leqslant 1$, and together with (6.13) this yields

$$P_{\mu,c}(x) \leqslant c_1 \min\left\{ \frac{s+\epsilon}{\sqrt{1-x^2}}, \sqrt{s+\epsilon} \right\}, \tag{6.16}$$

for every $\mu \in \mathscr{M}$, $c \in \mathbb{R}$, $x \in [-1, 1] \setminus K$ and $0 < s + \epsilon \leqslant 1$. If $x \in K \subset E_{\mu,c}$, then $P_{\mu,c}(x) \leqslant 0$, thus (6.16) holds for every $x \in [-1, 1]$. Taking the limit on the right-hand side of (6.16) as $\epsilon \to 0^+$, we get the desired result. □

7. Remez-type inequalities for exponentials of logarithmic potentials on bounded domains of $\mathbb{C}$ with C^2 boundary

In this section we establish the analogues of Theorems 6.1 (Corollary 6.6) and 6.4 for the case when the interval $[-1, 1]$ is replaced by the closure of a bounded domain $\Omega \subset \mathbb{C}$ with C^2 boundary. Throughout this section $m(A)$ denotes the two-dimensional Lebesgue measure of a measurable set $A \subset \mathbb{C}$. In [24, Theorem 2.4] we proved the following theorem.

Theorem 7.1. *Let $\Omega \subset \mathbb{C}$ be a bounded domain with C^2 boundary. There is a constant $c_{13} = c_{13}(\Omega) > 0$ depending only on Ω such that*

$$m\left(\left\{ z \in \overline{\Omega} \colon Q_{\mu,c}(z) \geqslant \exp\left(-\mu(\mathbb{C})\sqrt{s}\right) \max_{w \in \overline{\Omega}} Q_{\mu,c}(w) \right\}\right) \geqslant c_{13} s, \tag{7.1}$$

for every $\mu \in \mathscr{M}$, $c \in \mathbb{R}$ and $0 < s < m(\Omega)$.

Actually, in the above theorem we need only a slightly weaker geometric assumption for the boundary of Ω, namely the following: there is an $r > 0$ depending only on Ω such that for each $z \in \partial\Omega$ there is an open disk D_z with radius r such that $D_z \subset \Omega$ and $\overline{D}_z \cap \partial\Omega = \{z\}$. It is well known that if $\partial\Omega$ is a C^2 curve, then this property holds.

To prove Theorem 7.1, first we verified the following inequality for polynomials [24, Theorem 2.5].

Theorem 7.2. *Let $D = \{z \in \mathbb{C} \colon |z| < 1\}$. There is an absolute constant $c_{14} > 0$ such that*

$$\max_{u \in \overline{D}} |p(u)| \leqslant \exp\left(c_{14} n \sqrt{s}\right), \tag{7.2}$$

for every $p \in \mathscr{P}_n^c$ and $0 < s \leqslant \frac{1}{4}$ satisfying

$$m(\{z \in D \colon |p(z)| \leqslant 1\}) \geqslant \pi - s. \tag{7.3}$$

We also proved [24, Theorem 2.6] that the result of [24, Theorem 2.5] is essentially sharp; namely we have the following theorem.

Theorem 7.3. *There is an absolute constant $c_{15} > 0$ such that*

$$\sup |p(1)| \geqslant \exp\left(c_{15} n\sqrt{s}\right), \tag{7.4}$$

for every $n \in \mathbb{N}$ and $0 < s \leqslant \frac{1}{2}$, where the supremum in (7.4) is taken for all $p \in \mathcal{P}_{3n}^{\mathrm{r}}$ satisfying (7.3).

Since $p \in \mathcal{P}_n^{\mathrm{c}}$ implies that $q_\alpha(t) := |p(\alpha \mathrm{e}^{\mathrm{i}t})|^2 \in \mathcal{T}_n^{\mathrm{r}}$ for every $\alpha > 0$, there is a close relation between our Remez-type inequality for trigonometric polynomials (Theorem 3.4) and Theorem 7.2. The proof of Theorem 7.2 given in [24] rests on this observation.

We close this section by giving an L_p, $0 < p < \infty$, analogue of Theorem 7.1 [24, Theorem 2.8], which follows from Theorem 7.1 in the same way as Theorem 4.6 follows from Theorem 4.4.

Theorem 7.4. *Let $\Omega \subset \mathbb{C}$ be a bounded domain with C^2 boundary. There are constants $c_{16} = c_{16}(\Omega) > 0$ and $c_{17} = c_{17}(\Omega) > 0$ depending only on Ω such that*

$$\int_\Omega \left(Q_{\mu,c}(z)\right)^p \mathrm{d}m(z) \leqslant \left(1 + \exp\left(c_{16} p \mu(\mathbb{C})\sqrt{s}\right)\right) \int_A \left(Q_{\mu,c}(z)\right)^p \mathrm{d}m(z), \tag{7.5}$$

for every $\mu \in \mathcal{M}$, $c \in \mathbb{R}$, $0 < p < \infty$, $0 < s \leqslant c_{17}$ and $A \subset \overline{\Omega}$ with $m(A) > m(\Omega) - s$.

The assumption that Ω has C^2 boundary in both Theorems 7.1 and 7.4 may not be the best possible. However the domain

$$\Omega = \left\{z = x + \mathrm{i}y \colon 0 < x < 1, 0 < y < 1 - \sqrt{1 - x^2}\right\} \tag{7.6}$$

shows that the conclusion of Theorems 7.1 and 7.4 is not true for arbitrary bounded domains.

8. Nikolskii-type inequalities for exponentials of logarithmic potentials on $[-1, 1]$, on the unit circle and on bounded domains of $\mathbb{C}$ with C^2 boundary

Nikolskii-type inequalities for exponentials of logarithmic potentials can be easily obtained from Theorem 6.1 on $[-1, 1]$, from Theorem 6.3 on the unit circle, and from Theorem 7.1 on bounded domains of $\mathbb{C}$ with C^2 boundary. One can easily modify the proof of Theorem 5.2 in order to prove our theorems in this section. In [24, Theorems 3.1–3.3] we established the following three Nikolskii-type inequalities.

Theorem 8.1. *There is an absolute constant $c_{18} > 0$ such that*

$$\| Q_{\mu,c} \|_{L_p(-1,1)} \leqslant \left(c_{18}\left(1 + (q\mu(\mathbb{C}))^2\right)\right)^{1/q - 1/p} \| Q_{\mu,c} \|_{L_q(-1,1)}, \tag{8.1}$$

for every $\mu \in \mathcal{M}$, $c \in \mathbb{R}$ and $0 < q < p \leqslant \infty$.

Theorem 8.2. *There is an absolute constant $c_{19} > 0$ such that*

$$\| Q_{\mu,c}(\mathrm{e}^{\mathrm{i}t}) \|_{L_p(-\pi,\pi)} \leqslant \left(c_{19}(1 + q\mu(\mathbb{C}))\right)^{1/q - 1/p} \| Q_{\mu,c}(\mathrm{e}^{\mathrm{i}t}) \|_{L_q(-\pi,\pi)}, \tag{8.2}$$

for every $\mu \in \mathcal{M}$, $c \in \mathbb{R}$ and $0 < q < p \leqslant \infty$.

Theorem 8.3. *Let $\Omega \subset \mathbb{C}$ be a bounded domain with C^2 boundary. There exists a constant $c_{20} = c_{20}(\Omega) > 0$ depending only on Ω such that*

$$\| Q_{\mu,c} \|_{L_p(\Omega)} \leqslant \left(c_{20}\left(1 + (q\mu(\mathbb{C}))^2\right)\right)^{1/q-1/p} \| Q_{\mu,c} \|_{L_q(\Omega)}, \tag{8.3}$$

for every $\mu \in \mathscr{M}$, $c \in \mathbb{R}$ and $0 < q < p \leqslant \infty$.

We remark that Theorem 8.1 is an extension of Theorem 5.1, which states the same as Theorem 8.1 when the support of μ is a finite set. Similarly, one can easily check that Theorem 8.2 extends the results of Theorem 5.2, which states the same as Theorem 8.2 when the support of μ is a finite set (see the comment right after Theorem 7.3). Note that while a direct extension of Theorems 5.1 or 5.2 to exponentials of logarithmic potentials seems to be rather difficult, it is quite obvious via the corresponding extended Remez-type inequality.

9. Bernstein- and Markov-type inequalities for generalized nonnegative algebraic and trigonometric polynomials

Bernstein's inequality [40, pp. 39–41] asserts that

$$\max_{-\pi \leqslant t \leqslant \pi} | p'(t) | \leqslant n \max_{-\pi \leqslant t \leqslant \pi} | p(t) |, \tag{9.1}$$

for every $p \in \mathscr{T}_n^{\mathrm{r}}$ (or $p \in \mathscr{T}_n^{\mathrm{c}}$). The corresponding algebraic result [40, pp. 39–41] is known as Markov's inequality and states that

$$\max_{-1 \leqslant x \leqslant 1} | p'(x) | \leqslant n^2 \max_{-1 \leqslant x \leqslant 1} | p(x) |, \tag{9.2}$$

for all $p \in \mathscr{P}_n^{\mathrm{r}}$ (or $p \in \mathscr{P}_n^{\mathrm{c}}$). The Chebyshev polynomials $\sin(nx + \alpha) \in \mathscr{T}_n^{\mathrm{r}}$ and $T_n(x) = \cos(n \arccos x) \in \mathscr{P}_n^{\mathrm{r}}$ show that inequalities (9.1) and (9.2) are sharp. By the substitution $x = \cos t$, (9.1) yields

$$| p'(y) | \leqslant \frac{n}{\sqrt{1 - y^2}} \max_{-1 \leqslant x \leqslant 1} | p(x) |, \tag{9.3}$$

for every $p \in \mathscr{P}_n^{\mathrm{r}}$ and $-1 < y < 1$. Bernstein- and Markov-type inequalities in weighted spaces and in L_p-norm play a significant role in proving inverse theorems of approximation and have their own intrinsic interest. To extend inequalities (9.1)–(9.3) to generalized nonnegative algebraic and trigonometric polynomials, we need some assumptions to insure the differentiability. Such a natural assumption is to have $r_j \geqslant 1$, $j = 1, 2, \ldots, k$, in the representation (2.11) of an $f \in |\mathrm{GCTP}|_N$ or in the representation (2.7) of an $f \in |\mathrm{GCAP}|_N$. Under this assumption the one-sided derivatives exist, they are finite, and their absolute values are equal to each other at every real number. In the rest of this section and in Sections 10 and 11, f' means either the left- or the right-hand side derivative of f with respect to the real variable, which makes

$|f'(t)|$ be well-defined for every $t \in \mathbb{R}$, $f \in |\mathrm{GCTP}|_N$ and $f \in |\mathrm{GCAP}|_N$. Bernstein's inequality is extended to $|\mathrm{GCTP}|_N$ in [18, Theorem 3.1].

Theorem 9.1. *There is an absolute constant $c_{21} > 0$ such that*

$$\max_{-\pi \leqslant t \leqslant \pi} |f'(t)| \leqslant c_{21} N \max_{-\pi \leqslant t \leqslant \pi} f(t),$$

for every $f \in |\mathrm{GCTP}|_N$ of the form (2.11) *with each $r_j \geqslant 1$, $j = 1, 2, \ldots, k$.*

Using the substitution $x = \cos t$ in Theorem 9.1, we immediately obtain [18, Theorem 3.2].

Theorem 9.2. *We have*

$$|f'(y)| \leqslant \frac{c_{21} N}{\sqrt{1-y^2}} \max_{-1 \leqslant x \leqslant 1} f(x), \quad -1 < y < 1,$$

for every $f \in |\mathrm{GCAP}|_N$ of the form (2.7) *with each $r_j \geqslant 1$, $j = 1, 2, \ldots, k$.*

Our next result [18, Theorem 3.3] extends Markov's inequality to $|\mathrm{GCAP}|_N$.

Theorem 9.3. *There is an absolute constant $c_{22} > 0$ such that*

$$\max_{-1 \leqslant x \leqslant 1} |f'(x)| \leqslant c_{22} N^2 \max_{-1 \leqslant x \leqslant 1} f(x),$$

for every $f \in |\mathrm{GCAP}|_N$ of the form (2.7) *with each $r_j \geqslant 1$, $j = 1, 2, \ldots, k$.*

Theorems 9.1–9.3 were proved in [18] based on Remez-type inequalities. How can the Remez-type inequalities of Section 4 be used to obtain the above theorems? It is far from obvious.

Sketch of the proof of Theorem 9.1. *Step* 1. It is sufficient to prove the theorem in the case when $f \in |\mathrm{GCTP}|_N$ has only real zeros. This is shown by a variational method in [18, Lemmas 4.8 and 4.9].

Step 2. If z_j, $j = 1, 2, \ldots, k$, are distinct in the representation (2.11) of an $f \in |\mathrm{GCTP}|_N$, then we say that f has a zero at z_j with multiplicity r_j. A routine application of Rolle's Theorem and a simple calculation show that if $f \in |\mathrm{GCTP}|_N$ has only real zeros and each $r_j \geqslant 1$ in its representation (2.11), then $|f'| \in |\mathrm{GCTP}|_N$, $|f'|$ has only real zeros, and at least one of any two adjacent zeros of $|f'|$ has multiplicity 1.

Step 3. We state [18, Lemma 4.7], which is the heart of the proof.

Lemma 9.4. *Assume that $g \in |\mathrm{GCTP}|_N$ has ony real zeros, and at least one of any two adjacent zeros of g has multiplicity at least* 1. *Then, there is an absolute constant $c_{23} > 0$ and an interval $I \subset \mathbb{R}$ such that $m(I) \geqslant c_{23}/N$ and*

$$g(\tau) \geqslant \mathrm{e}^{-1} \max_{-\pi \leqslant t \leqslant \pi} g(t), \quad \text{for every } \tau \in I. \tag{9.4}$$

To prove this lemma we applied the Remez-type inequality of Theorem 4.4. Observe that Theorem 4.4 guarantees an absolute constant $c_{24} > 0$ such that

$$m(E) \geqslant \frac{c_{24}}{N}, \tag{9.5}$$

where

$$E := \left\{ \tau \in [-\pi, \pi) \colon g(\tau) \geqslant e^{-1} \max_{-\pi \leqslant t \leqslant \pi} g(t) \right\}. \tag{9.6}$$

Since E is not necessarily an interval, (9.5) may seem to be far from the conclusion of Lemma 9.4. However, by exploiting the additional pieces of information on g, we showed in [18] that E contains an interval I such that $m(I) \geqslant c_{23}/N$. This needs a few additional tricks, and the Remez-type inequality of (9.5) is the central piece of the proof.

Step 4. Because of the periodicity of f and Step 1, it is sufficient to prove that

$$|f'(\pi)| \leqslant c_{21} N \max f(t), \tag{9.7}$$

for every $f \in |\mathrm{GCTP}|_N$ of the form (2.11) with each $r_j \geqslant 1$ and $z_j \in \mathbb{R}$, $j = 1, 2, \ldots, k$. By Step 2, $g := |f'| \in |\mathrm{GCTP}|_N$ satisfies the assumption of Lemma 9.4. Denote the endpoints of the interval I of Lemma 9.4 by a and b. From Lemma 9.4 we deduce

$$\begin{aligned} \max_{-\pi \leqslant t \leqslant \pi} |f'(t)| &\leqslant \frac{e}{b-a} \int_a^b |f'(t)| \,dt \leqslant \frac{eN}{c_{23}} \int_a^b |f'(t)| \,dt \\ &\leqslant c_{21} N |f(b) - f(a)| \leqslant c_{21} N \max_{-\pi \leqslant t \leqslant \pi} f(t), \end{aligned}$$

which proves the theorem. □

Another much more obvious application of the Remez-type inequalities of Section 4 is to obtain Theorem 9.3 from Theorem 9.2.

Proof of Theorem 9.3. As a corollary of Theorem 4.2 we have

$$\max_{-1 \leqslant x \leqslant 1} f(x) \leqslant e \max_{-\alpha \leqslant x \leqslant \alpha} f(x), \tag{9.8}$$

for every $f \in |\mathrm{GCAP}|_N$, where $\alpha = 1 - (5N)^{-2}$ and $N \geqslant 1$. Combining this with Theorem 9.2, we deduce that

$$\max_{-\alpha \leqslant x \leqslant \alpha} |f'(x)| \leqslant \frac{c_{21} N}{(5N)^{-1}} \max_{-1 \leqslant x \leqslant 1} f(x) \leqslant 25 e c_{21} N^2 \max_{-\alpha \leqslant x \leqslant \alpha} f(x), \tag{9.9}$$

for every $f \in |\mathrm{GCAP}|_N$ of the form (2.7) with each $r_j \geqslant 1$, $j = 1, 2, \ldots, k$, and the result follows by a linear transformation. □

The sharp L_p version of Bernstein's inequality was first established in [65, Vol. II, (3.17), p.11] for $p \geqslant 1$. It states

$$\int_{-\pi}^{\pi} |Q'(t)|^p \,dt \leqslant n^p \int_{-\pi}^{\pi} |Q(t)|^p \,dt, \tag{9.10}$$

for every $Q \in \mathscr{T}_n^{\mathrm{r}}$. For $0<p<1$, first Klein [35] and later Osval'd [50] proved (9.10) with a multiplicative constant $c(p)$. In [46] Nevai proved that $c(p)=8/p$ is a possible choice. Subsequently, Máté and Nevai [44] showed the validity of (9.10) with a multiplicative absolute constant, and then Arestov [1] proved (9.10) (with the best constant 1) for every $0<p<1$. Recently von Golitschek and Lorentz [64] found a very elegant proof of Arestov's Theorem.

Markov's inequality in L_p states

$$\int_{-1}^{1} |Q'(x)|^p \, \mathrm{d}x \leqslant c_{25}^{p+1} n^{2p} \int_{-1}^{1} |Q(x)|^p \, \mathrm{d}x, \tag{9.11}$$

for every $Q \in \mathscr{P}_n^{\mathrm{r}}$, where $c_{25}>0$ is an absolute constant. This can be deduced from the above L_p Bernstein-type inequalities, by the substitution $x=\cos t$ and by using Nikolskii-type inequalities (cf. [41,44]). To find the best constant in (9.11) is still an open problem. The following extensions of Bernstein's and Markov's inequalities in L_p, $0<p<\infty$, to generalized nonnegative polynomials were obtained in [25, Theorems 10 and 11].

Theorem 9.5. *Let χ be a nonnegative, nondecreasing and convex function defined on* $[0, \infty)$ *and let* $0<p\leqslant 1$. *There is an absolute constant $c_{26}>0$ such that*

$$\int_{-\pi}^{\pi} \chi\left(\left|\frac{f'(t)}{N}\right|^p\right) \mathrm{d}t \leqslant \int_{-\pi}^{\pi} \chi\left(c_{26}(f(t))^p\right) \mathrm{d}t,$$

for every $f \in |\mathrm{GCTP}|_N$ of the form (2.11) *with each $r_j \geqslant 1$, and for every* $0<p\leqslant 1$.

Corollary 9.6. *We have*

$$\int_{-\pi}^{\pi} |f'(t)|^q \, \mathrm{d}t \leqslant c_{26}^{q+1} N^q \int_{-\pi}^{\pi} (f(t))^q \, \mathrm{d}t,$$

for every $f \in |\mathrm{GCTP}|_N$ of the form (2.11) *with each $r_j \geqslant 1$, and for every* $0<q<\infty$.

To see this, just apply Theorem 9.5 with $p=q$ and $\chi(x)=x$ for $0<q\leqslant 1$, and with $p=1$ and $\chi(x)=x^q$ for $1<q<\infty$.

Theorem 9.7. *There is an absolute constant $c_{27}>0$ such that*

$$\int_{-1}^{1} |f'(x)|^p \, \mathrm{d}x \leqslant c_{27}^{p+1} N^{2p} \int_{-1}^{1} (f(x))^p \, \mathrm{d}x,$$

for every $f \in |\mathrm{GCAP}|_N$ of the form (2.7) *with each $r_j \geqslant 1$, $j=1, 2, \dots, k$, and for every* $0<p<\infty$.

In the proof of Theorems 9.5 and 9.7 we used the L_∞ Bernstein- and Markov-type inequalities of Theorems 9.1–9.3, the L_p Remez-type inequalities of Section 4 (Theorems 4.5 and 4.6) and the Nikolskii-type inequalities of Section 5 (Theorems 5.1 and 5.2). The nice method, developed in [44], to prove Bernstein-type inequalities for ordinary trigonometric polynomials in L_p, $0<p\leqslant\infty$, is presented in the next section, where some weighted analogues of Theorems 9.5 and 9.7 are discussed.

10. Weighted Bernstein- and Markov-type inequalities for generalized nonnegative polynomials

In this section weighted Markov- and Bernstein-type inequalities are established for generalized nonnegative polynomials and generalized nonnegative polynomial weight functions. The magnitude of

$$\frac{\max_{-1 \leqslant x \leqslant 1} |f'(x)w(x)|}{\max_{-1 \leqslant x \leqslant 1} |f(x)w(x)|}, \tag{10.1}$$

$$\frac{|f'(y)w(y)|}{\max_{-1 \leqslant x \leqslant 1} |f(x)w(x)|}, \quad -1 \leqslant y \leqslant 1, \tag{10.2}$$

and their corresponding L_p analogues, respectively, for polynomials $f \in \mathscr{P}_n^r$ and generalized Jacobi weight functions

$$w(z) = \prod_{j=1}^{k} |z - z_j|^{r_j}, \quad z_j \in \mathbb{C}, \ -1 < r_j < \infty, \ j = 1, 2, \ldots, k, \tag{10.3}$$

was examined by a number of authors [12,34,36,41,45,47,52]; however, a multiplicative constant depending on the weight function appears in these estimates. In [23] the magnitude of (10.1) and (10.2) was examined, when both f and w are generalized nonnegative polynomials. In these inequalities only the generalized degree of f, the generalized degree of w and a multiplicative *absolute* constant appear. The results are new and (in a sense) sharp, even when f is an ordinary polynomial.

Our motivation was to find tools to examine systems of orthogonal polynomials simultaneously, associated with generalized Jacobi, or at least generalized nonnegative polynomial weight functions of degree at most Γ. In Section 12 we give sharp estimates in this spirit for the Christoffel function on $[-1, 1]$ and for the distances of the consecutive zeros of orthogonal polynomials, associated with generalized nonnegative polynomial weight functions of degree at most Γ.

The following weighted Bernstein- and Markov-type inequalities are proved in [23, Theorems 1–3].

Theorem 10.1. *There is an absolute constant* $c_{28} > 0$ *such that*

$$\max_{-\pi \leqslant t \leqslant \pi} |f'(t)w(t)| c_{28}(\Gamma + 1)(N + \Gamma) \max_{-\pi \leqslant t \leqslant \pi} f(t)w(t),$$

for every $f \in |\mathrm{GCTP}|_N$ *of the form* (2.11) *with each* $r_j \geqslant 1$, $j = 1, 2, \ldots, k$, *and for every* $w \in |\mathrm{GCTP}|_\Gamma$.

By the substitution $x = \cos t$, from Theorem 10.1 we easily obtain the following theorem.

Theorem 10.2. *We have*

$$|f'(y)w(y)| \leqslant \frac{c_{28}(\Gamma + 1)(N + \Gamma)}{\sqrt{1 - y^2}} \max_{-1 \leqslant x \leqslant 1} f(x)w(x),$$

for every $f \in |\mathrm{GCAP}|_N$ *of the form* (2.7) *with each* $r_j \geqslant 1$, $j = 1, 2, \ldots, k$, *and for every* $w \in |\mathrm{GCAP}|_\Gamma$.

The weighted Markov-type inequality is given by the following theorem.

Theorem 10.3. *There is an absolute constant* $c_{29} > 0$ *such that*

$$\max_{-1 \leqslant x \leqslant 1} |f'(x)w(x)| \leqslant c_{29}(N+\Gamma)^2 \max_{-1 \leqslant x \leqslant 1} f(x)w(x),$$

for every $f \in |\mathrm{GCAP}|_N$ *of the form* (2.7) *with each* $r_j \geqslant 1$, $j = 1, 2, \ldots, k$, *and for every* $w \in |\mathrm{GCAP}|_\Gamma$.

I conjecture that in the inequalities of Theorems 10.1 and 10.2 the multiplicative factor $\Gamma + 1$ can be dropped. If this conjecture were true, we would obtain Theorem 10.3 as a simple consequence of Theorem 10.2, using the Remez-type inequality of Theorem 4.2 (see the proof of Theorem 9.3). Since this conjecture is not settled yet, Theorem 10.3 is much more than a simple consequence of the corresponding trigonometric result of Theorem 10.2.

In the rather lengthy proofs of Theorems 10.1 and 10.3 in [23], the Remez-type inequalities of Section 4 play a central role again.

Now we establish the L_p, $0 < p < \infty$, analogues of Theorems 10.1 and 10.3. The rest of this section has never been published before.

Theorem 10.4. *Let* χ *be a nonnegative, nondecreasing and convex function defined on* $[0, \infty)$. *There is an absolute constant* $c_{30} \geqslant 1$ *such that*

$$\int_{-\pi}^{\pi} \chi\left(\frac{|f'(t)|^p w(t)}{(N+\Gamma)^p(\Gamma/p+1)^p} \right) \mathrm{d}t \leqslant \int_{-\pi}^{\pi} \chi\big(c_{30}(f(t))^p w(t)\big) \mathrm{d}t,$$

for every $f \in |\mathrm{GCTP}|_N$ *of the form* (2.11) *with each* $r_j \geqslant 1$, $j = 1, 2, \ldots, k$, *for every* $w \in |\mathrm{GCTP}|_\Gamma$ *and for every* $0 < p \leqslant 1$.

Similarly to Corollary 9.6, from the above theorem we can easily deduce the following corollary.

Corollary 10.5. *Let* c_{30} *be the same as in Theorem* 10.4. *We have*

$$\int_{-\pi}^{\pi} |f'(t)|^p w(t) \,\mathrm{d}t \leqslant c_{30}^{p+1}(N+\Gamma)^p\left(\frac{\Gamma}{p}+1\right) \int_{\pi}^{\pi} (f(t))^p w(t) \,\mathrm{d}t,$$

for every $f \in |\mathrm{GCTP}|_N$ *of the form* (2.11) *with each* $r_j \geqslant 1$, $j = 1, 2, \ldots, k$, *for every* $w \in |\mathrm{GCTP}|_\Gamma$ *and for every* $0 < p < \infty$.

Corollary 10.5 plays a central role in the proof of the following theorem.

Theorem 10.6. *There is an absolute constant* $c_{31} > 0$ *such that*

$$\int_{-1}^{1} |f'(x)|^p w(x) \,\mathrm{d}x \leqslant c_{31}^{p+1}(N+\Gamma)^{2p}\left(\frac{\Gamma}{p}+1\right)^{2p} \int_{-1}^{1} (f(x))^p w(x) \,\mathrm{d}x,$$

for every $f \in |\mathrm{GCAP}|_N$ *of the form* (2.7) *with each* $r_j \geqslant 1$, $j = 1, 2, \ldots, k$, *for every* $w \in |\mathrm{GCAP}|_\Gamma$ *and for every* $0 < p < \infty$.

The proof of these quite general L_p Bernstein- and Markov-type inequalities illustrates well how to use our inequalities for generalized nonnegative polynomials, as pieces of the proof of some other inequalities. We present the proof of Theorems 10.4 and 10.6 in details.

Proof of Theorem 10.4. The inequality of Theorem 10.4 is obvious if $N = 0$; hence in the sequel we assume that $N \geqslant 1$ (if $f \in |\mathrm{GCTP}|_N$ is of the form (2.11) with each $r_j \geqslant 1$, then $N > 0$ implies $N \geqslant 1$). Let $n := [N + \Gamma]$, $0 < p \leqslant 1$, $m := 2/p$, $D_n(t) := |\sum_{j=-n}^{n} \mathrm{e}^{ijt}|$, let $g \in |\mathrm{GCTP}|_N$ be of the form (2.11) with each $r_j \geqslant 1$, $j = 1, 2, \ldots, k$, and let $v \in |\mathrm{GCTP}|_\Gamma$. Let

$$G := gD_n^m \in |\mathrm{GCTP}|_{N+2n/p} \tag{10.4}$$

and

$$F := gD_n^m v^{1/p} \in |\mathrm{GCTP}|_{N+2n/p+\Gamma/p}. \tag{10.5}$$

Applying the Nikolskii-type inequality of Theorem 5.2 to F defined by (10.5), we obtain

$$\begin{aligned} &\max_{-\pi \leqslant \tau \leqslant \pi} \left(g(\tau)(D_n(\tau))^m (v(\tau))^{1/p}\right)^p \\ &\leqslant c_9\left(1 + p\left(N + \frac{2n}{p} + \frac{\Gamma}{p}\right)\right)\int_{-\pi}^{\pi} \left(g(\theta)(D_n(\theta))^m (v(\theta))^{1/p}\right)^p \,\mathrm{d}\theta \\ &\leqslant c_{32}(N+\Gamma)\int_{-\pi}^{\pi} \left(g(\theta)(D_n(\theta))^m\right)^p v(\theta)\,\mathrm{d}\theta \\ &= c_{32}(N+\Gamma)\int_{-\pi}^{\pi} (g(\theta))^p v(\theta)(D_n(\theta))^2 \,\mathrm{d}\theta, \end{aligned} \tag{10.6}$$

where $c_{32} > 0$ is a suitable absolute constant. Applying the weighted L_∞ Bernstein-type inequality of Theorem 10.1 to G defined by (10.4) and $v^{1/p} \in |\mathrm{GCTP}|_{\Gamma/p}$, we obtain for every $t \in \mathbb{R}$ that

$$|G'(t)|(v(t))^{1/p} \leqslant c_{28}\left(N\left(1 + \frac{2}{p}\right) + \frac{\Gamma}{p}\right)\left(\frac{\Gamma}{p} + 1\right) \max_{-\pi \leqslant \tau \leqslant \pi} G(\tau)(v(\tau))^{1/p}. \tag{10.7}$$

Combining (10.6) and (10.7) we conclude

$$\begin{aligned} &|g'(t)(D_n(t))^m + mg(t)D_n'(t)(D_n(t))^{m-1}|^p v(t) \\ &\leqslant (c_{28}(N+\Gamma))^p\left(1 + \frac{2}{p}\right)^p\left(\frac{\Gamma}{p} + 1\right)^p \max_{-\pi \leqslant r \leqslant \pi} (g(\tau))^p v(\tau)(D_n(\tau))^2 \\ &\leqslant c_{33}(N+\Gamma)^p\left(\frac{\Gamma}{p} + 1\right)^p c_{32}(N+\Gamma)\int_{-\pi}^{\pi} (g(\theta))^p v(\theta)(D_n(\theta))^2 \,\mathrm{d}\theta, \end{aligned} \tag{10.8}$$

with some absolute constant $c_{33} > 0$. Putting $t = 0$ in (10.8) and noticing that

$$D_n'(0) = 0 \quad \text{and} \quad (D_n(0))^m = (2n+1)^{2/p} \geqslant (N+\Gamma)^{2/p}, \tag{10.9}$$

we can deduce that

$$|g'(0)|^p v(0) \leqslant c_{34}(N+\Gamma)^p \left(\frac{\Gamma}{p}+1\right)^p \int_{-\pi}^{\pi} (g(\theta))^p v(\theta)(2\pi)^{-1}(2n+1)^{-1}(D_n(\theta))^2 \, d\theta, \tag{10.10}$$

where $c_{34} = c_{32}c_{33} > 0$ is an absolute constant. Now let $f \in |\text{GCTP}|_N$ be of the form (2.11) with each $r_j \geqslant 1$, $j = 1, 2, \ldots, k$, and let $w \in |\text{GCTP}|_\Gamma$. Applying (10.10) to $g(\tau) = f(\tau + t)$ and $v(\tau) = w(\tau + t)$, we obtain

$$|f'(t)|^p w(t) \leqslant c_{34}(N+\Gamma)^p \left(\frac{\Gamma}{p}+1\right)^p \int_{-\pi}^{\pi} (f(\theta))^p w(\theta)(2\pi)^{-1}(2n+1)^{-1}(D_n(\theta - t))^2 \, d\theta. \tag{10.11}$$

Hence,

$$\frac{|f'(t)|^p w(t)}{(N+\Gamma)^p(\Gamma/p+1)^p} \leqslant \int_{-\pi}^{\pi} c_{34}(f(\theta))^p w(\theta)(2\pi)^{-1}(2n+1)^{-1}(D_n(\theta - t))^2 \, d\theta. \tag{10.12}$$

Observe that

$$\int_{-\pi}^{\pi} (2\pi)^{-1}(2n+1)^{-1}(D_n(\theta - t))^2 \, d\theta = 1; \tag{10.13}$$

therefore (10.12), together with the Jensen inequality, yields

$$\chi\left(\frac{|f'(t)|^p w(t)}{(N+\Gamma)^p(\Gamma/p+1)^p}\right) \leqslant \int_{-\pi}^{\pi} \chi\left(c_{34}(f(\theta))^p w(\theta)\right)(2\pi)^{-1}(2n+1)^{-1}(D_n(\theta - t))^2 \, d\theta. \tag{10.14}$$

Integrating both sides of (10.14) with respect to t, and using Fubini's Theorem and (10.13), we get the theorem. □

Proof of Theorem 10.6. We may assume that $N \geqslant 1$ as in the proof of Theorem 10.4. We distinguish two cases.

Case 1: $p \geqslant 1$. Let $f \in |\text{GCAP}|_N$ be of the form (2.7) with each $r_j \geqslant 1$, $j = 1, 2, \ldots, k$, and let $w \in |\text{GCAP}|_\Gamma$. Then $g(t) := f(\cos t) \in |\text{GCTP}|_N$ is of the form (2.11) with each $r_j \geqslant 1$, $j = 1, 2, \ldots, k$, and $v(t) = w(\cos t) \in |\text{GCTP}|_\Gamma$. Applying Corollary 10.5 to g and v, we obtain

$$\int_{-\pi}^{\pi} |g'(t)|^p v(t) \, dt \leqslant c_{30}^{p+1}(N+\Gamma)^p \left(\frac{\Gamma}{p}+1\right)^p \int_{-\pi}^{\pi} (g(t))^p v(t) \, dt. \tag{10.15}$$

Let

$$\tilde{\delta} := (N+\Gamma+8)^{-1}, \qquad \delta := \cos \tilde{\delta} \tag{10.16}$$

and

$$A := \left[\tilde{\delta}, \pi - \tilde{\delta}\right] \cup \left[\pi + \tilde{\delta}, 2\pi - \tilde{\delta}\right]. \tag{10.17}$$

Applying the L_p Remez-type inequality of Theorem 4.6 to $g^p v \in |\mathrm{GCTP}|_{pN+\Gamma}$ with $s := 4\tilde{\delta} \leqslant 4(N+\Gamma+8)^{-1} \leqslant \frac{1}{2}$, we obtain

$$\int_{-\pi}^{\pi} g(t)^p v(t)\,\mathrm{d}t \leqslant \left(1 + \exp\left(\frac{4c_7(pN+\Gamma)}{N+\Gamma+8}\right)\right)\int_A (g(t))^p v(t)\,\mathrm{d}t$$

$$\leqslant c_{35}^p \int_A (g(t))^p v(t)\,\mathrm{d}t, \tag{10.18}$$

with a suitable absolute constant $c_{35} > 0$. Combining (10.15) and (10.18), we conclude

$$\int_A |g'(t)|^p v(t)\,\mathrm{d}t \leqslant c_{36}^p (N+\Gamma)^p \left(\frac{\Gamma}{p} + 1\right)^p \int_A (g(t))^p v(t)\,\mathrm{d}t, \tag{10.19}$$

with a suitable absolute constant $c_{36} > 0$. Using the substitution $x = \cos t$ and recalling definitions (10.16) and (10.17), we deduce

$$\int_{-\delta}^{\delta} (f'(x))^p w(x)(1-x^2)^{(p-1)/2}\,\mathrm{d}x$$

$$\leqslant c_{36}^p (N+\Gamma)^p \left(\frac{\Gamma}{p} + 1\right)^p \int_{-\delta}^{\delta} (f(x))^p w(x)(1-x^2)^{-1/2}\,\mathrm{d}x. \tag{10.20}$$

Hence, by (10.16) and $p \geqslant 1$, we obtain

$$\int_{-\delta}^{\delta} |f'(x)|^p w(x)\,\mathrm{d}x$$

$$\leqslant (\sin\tilde{\delta})^{1-p} \int_{-\delta}^{\delta} |f'(x)|^p w(x)(1-x^2)^{(p-1)/2}\,\mathrm{d}x$$

$$\leqslant (\sin\tilde{\delta})^{1-p} c_{36}^p (N+\Gamma)^p \left(\frac{\Gamma}{p} + 1\right)^p \int_{-\delta}^{\delta} (f(x))^p w(x)(1-x^2)^{-1/2}\,\mathrm{d}t$$

$$\leqslant (\sin\tilde{\delta})^{1-p} (\sin\tilde{\delta})^{-1} c_{36}^p (N+\Gamma)^p \left(\frac{\Gamma}{p} + 1\right)^p \int_{-\delta}^{\delta} (f(x))^p w(x)\,\mathrm{d}x$$

$$\leqslant c_{37}^p (N+\Gamma)^{2p} \left(\frac{\Gamma}{p} + 1\right)^p \int_{-\delta}^{\delta} (f(x))^p w(x)\,\mathrm{d}x, \tag{10.21}$$

with a suitable absolute constant $c_{37} > 0$, which gives the theorem after a linear transformation.

Case 2: $0 < p \leqslant 1$. Let $f \in |\mathrm{GCAP}|_N$ be of the form (2.7) with each $r_j \geqslant 1$, $j = 1, 2, \ldots, k$, and let $w \in |\mathrm{GCAP}|_\Gamma$. Since $0 < p \leqslant 1$, we have

$$\int_A \left(|f'(\cos t)|\,|\sin t|^{1/p+1}\right)^p w(\cos t)\,\mathrm{d}t$$

$$\leqslant \int_A \left|\left(f(\cos t)|\sin t|^{1/p}\right)'\right|^p w(\cos t)\,\mathrm{d}t$$

$$+ \int_A \left(f(\cos t)p^{-1}|\sin t|^{1/p-1}|\cos t|\right)^p w(\cos t)\,\mathrm{d}t, \tag{10.22}$$

for every measurable subset of $[-\pi, \pi]$. Applying Corollary 10.5 to

$$g(t) := f(\cos t)|\sin t|^{1/p} \in |\mathrm{GCTP}|_{N+1/p} \tag{10.23}$$

(which is of the form (2.11) with each $r_j \geqslant 1$, $j = 1, 2, \ldots, k$) and

$$v(t) := w(\cos t), \tag{10.24}$$

we obtain

$$\begin{aligned}&\int_{-\pi}^{\pi} \left|\left(f(\cos t)|\sin t|^{1/p}\right)'\right|^p w(\cos t)\,\mathrm{d}t \\ &\quad\leqslant c_{30}\left(N+\Gamma+\frac{1}{p}\right)^p\left(\frac{\Gamma}{p}+1\right)^p \int_{-\pi}^{\pi}\left(f(\cos t)|\sin t|^{1/p}\right)^p w(\cos t)\,\mathrm{d}t.\end{aligned} \tag{10.25}$$

Applying the L_p Remez-type inequality of Theorem 4.6 to $g^p v \in |\mathrm{GCTP}|_{pN+\Gamma+1}$, we deduce

$$\begin{aligned}&\int_{-\pi}^{\pi}\left(f(\cos t)|\sin t|^{1/p}\right)^p w(\cos t)\,\mathrm{d}t \\ &\quad\leqslant \left(1+\exp\left(\frac{4c_7(pN+\Gamma+1)}{N+\Gamma+8}\right)\right)\int_A\left(f(\cos t)|\sin t|^{1/p}\right)^p w(\cos t)\,\mathrm{d}t \\ &\quad\leqslant c_{38}\int_A (f(\cos t))^p|\sin t|\,w(\cos t)\,\mathrm{d}t,\end{aligned} \tag{10.26}$$

where A is the same as in Case 1, and $c_{38} > 0$ is a suitable absolute constant. Combining (10.25) and (10.26), we get

$$\begin{aligned}&\int_A \left|\left(f(\cos t)|\sin t|^{1/p}\right)'\right|^p w(\cos t)\,\mathrm{d}t \\ &\quad\leqslant c_{39}\left(N+\Gamma+\frac{1}{p}\right)^p\left(\frac{\Gamma}{p}+1\right)^p \int_A\left(f(\cos t)|\sin t|^{1/p}\right)^p w(\cos t)\,\mathrm{d}t,\end{aligned} \tag{10.27}$$

where $c_{39} := c_{30}c_{38} > 0$ is a suitable absolute constant. This, together with (10.22) implies

$$\begin{aligned}&\int_A\left(f'(\cos t)|\sin t|^{1/p+1}\right)^p w(\cos t)\,\mathrm{d}t \\ &\quad\leqslant c_{39}\left(N+\Gamma+\frac{1}{p}\right)^p\left(\frac{\Gamma}{p}+1\right)^p \int_A\left(f(\cos t)|\sin t|^{1/p}\right)^p w(\cos t)\,\mathrm{d}t \\ &\qquad+\int_A\left(f(\cos t)p^{-1}|\sin t|^{1/p-1}|\cos t|\right)^p w(\cos t)\,\mathrm{d}t.\end{aligned} \tag{10.28}$$

Substituting $x = \cos t$, we obtain

$$\begin{aligned}&\int_{-\delta}^{\delta}|f'(x)|^p w(x)(1-x^2)^{p/2}\,\mathrm{d}x \\ &\quad\leqslant c_{39}\left(N+\Gamma+\frac{1}{p}\right)^p\left(\frac{\Gamma}{p}+1\right)^p\int_{-\delta}^{\delta}(f(x))^p w(x)\,\mathrm{d}x \\ &\qquad+p^{-p}\int_{-\delta}^{\delta}(f(x))^p w(x)(1-x^2)^{-p/2}\,\mathrm{d}x,\end{aligned} \tag{10.29}$$

where δ is defined by (10.16). Hence by $0<p\leqslant 1$, we deduce

$$\begin{aligned}\int_{-\delta}^{\delta}|f'(x)|^p w(x)\,\mathrm{d}x &\leqslant (\sin\tilde{\delta})^{-p}\int_{-\delta}^{\delta}|f'(x)|^p w(x)(1-x^2)^{p/2}\,\mathrm{d}x\\ &\leqslant (\sin\tilde{\delta})^{-p}c_{39}\left(N+\Gamma+\frac{1}{p}\right)^p\left(\frac{\Gamma}{p}+1\right)^p\int_{-\delta}^{\delta}(f(x))^p w(x)\,\mathrm{d}x\\ &\quad+p^{-p}(\sin\tilde{\delta})^{-p}(\sin\tilde{\delta})^{-p}\int_{-\delta}^{\delta}(f(x))^p w(x)\,\mathrm{d}x\\ &\leqslant c_{31}(N+\Gamma)^p\left(\frac{\Gamma}{p}+1\right)^p\int_{-\delta}^{\delta}(f(x))^p w(x)\,\mathrm{d}x,\end{aligned}\tag{10.30}$$

and the desired inequality follows from this by a linear transformation. □

11. Remez-, Nikolskii- and Markov-type inequalities for generalized nonnegative algebraic polynomials with restricted zeros

In this section sharp Remez-, Nikolskii- and Markov-type inequalities are established for generalized nonnegative polynomials of the form (2.7) under the assumptions

$$\sum_{j=1}^{k} r_j\leqslant N \quad\text{and}\quad \sum_{\{j:|z_j|<1\}} r_j\leqslant K,\qquad 0\leqslant K\leqslant N.\tag{11.1}$$

The Remez- and Nikolskii-type inequalities are new even for ordinary polynomials of degree at most n having at most k, $0\leqslant k\leqslant n$, zeros in the open unit disk. When $K=N$, the results of this section contain the corresponding inequalities from Sections 4, 5 and 9 for all $f\in|\mathrm{GCAP}|_N$.

Denote by $\mathscr{P}^{\mathrm{r}}_{n,k}$, $0\leqslant k\leqslant n$, the set of all $p\in\mathscr{P}_n$ which have at most k zeros (by counting multiplicities) in the open unit disk. Let $|\mathrm{GCAP}|_{N,K}$, $0\leqslant K\leqslant N$, be the set of all $f\in|\mathrm{GCAP}|_N$ of the form (2.7) for which

$$\sum_{\{j:|z_j|<1\}} r_j\leqslant K.\tag{11.2}$$

To establish a sharp Remez-type inequality for the classes $\mathscr{P}^{\mathrm{r}}_{n,k}$, we need the weighted Chebyshev polynomials $T_{n,k}$ of the form

$$T_{n,k}(x)=(x+1)^{n-k}Q(x),\quad Q\in\mathscr{P}^{\mathrm{r}}_k,\ k=0,1,\dots,n,\tag{11.3}$$

satisfying the properties

$$T_{n,k}\text{ equioscillates }k+1\text{ times on }[-1,1],\tag{11.4}$$

$$\max_{-1\leqslant x\leqslant 1}|T_{n,k}(x)|=1\tag{11.5}$$

and

$$T_{n,k}(1)=1.\tag{11.6}$$

More precisely, (11.4) and (11.5) mean that $T_{n,k}$ achieves the values

$$\pm \max_{-1 \leqslant x \leqslant 1} |T_{n,k}(x)| = \pm 1$$

$k+1$ times on $[-1, 1]$ with alternating sign. The existence and uniqueness of such weighted Chebyshev polynomials $T_{n,k}$ are well known from the theory of weighted Chebyshev approximation. Explicit formulae for $T_{n,k}$ seem to be known only when $k=0$, $k=n-1$ or $k=n$. In [6, Theorem 3.1] we proved the following sharp Remez-type inequality for $\mathscr{P}^{\mathrm{r}}_{n,k}$.

Theorem 11.1. *Given $0 \leqslant k \leqslant n$ integers and $0 < s < 2$, we have*

$$\max_{-1 \leqslant x \leqslant 1} |p(x)| \leqslant T_{n,k}\left(\frac{2+s}{2-s}\right), \tag{11.7}$$

for every $p \in \mathscr{P}^{\mathrm{r}}_{n,k}$ satisfying

$$m(\{x \in [-1, 1]: |p(x)| \leqslant 1\}) \geqslant 2-s. \tag{11.8}$$

Equality in (11.7) *holds if and only if*

$$p(x) = \pm T_{n,k}\left(\frac{\pm 2x}{2-s} + \frac{s}{2-s}\right). \tag{11.9}$$

The case $k=n$ (when there are no restrictions for the zeros of Theorem 11.1) gives the Remez inequality (Theorem 3.1). When $k=0$, Theorem 3.1 yields [17, Corollary].

By estimating $T_{n,k}((2+s)/(2-s))$ (which is by no means straightforward), we gave a sharp numerical version of Theorem 11.1 and we extended this to the classes $|\mathrm{GCAP}|_{N,K}$ [6, Theorem 3.2].

Theorem 11.2. *Given $0 < s \leqslant 1$ and $0 \leqslant K \leqslant N$, there is an absolute constant $0 < c_{40} \leqslant 9$ such that*

$$\max_{-1 \leqslant x \leqslant 1} f(x) \leqslant \exp\left(c_{40}(\sqrt{NKs} + Ns)\right), \tag{11.10}$$

for every $f \in |\mathrm{GCAP}|_{N,K}$ satisfying

$$m(\{x \in [-1, 1]: f(x) \leqslant 1\}) \geqslant 2-s. \tag{11.11}$$

We did not discuss the case $1 < s < 2$, which does not seem to be as important as the case $0 < s \leqslant 1$ in applications.

As a consequence of Theorem 11.2, we proved the following Nikolskii-type inequalities for $|\mathrm{GCAP}|_{N,K}$ [6, Theorem 3.3] similarly to the proof of Theorem 5.2.

Theorem 11.3. *Let χ be a nonnegative, nondecreasing function defined on $[0, \infty)$ such that $\chi(x)/x$ is nonincreasing. Given $0 \leqslant K \leqslant N$, $0 < q < p \leqslant \infty$, there is an absolute constant $0 < c_{41} \leqslant 81\mathrm{e}^2$ such that*

$$\|\chi(f)\|_{L_p(-1,1)} \leqslant \left(c_{41} \max\{1, q^2NK, qN\}\right)^{1/q-1/p} \|\chi(f)\|_{L_q(-1,1)}, \tag{11.12}$$

for every $f \in |\mathrm{GCAP}|_{N,K}$ satisfying (11.11).

The case $K = N$ in Theorem 11.3 gives Theorem 5.1, as a special case. If $qK \geqslant 1$, then the Nikolskii factor in the unrestricted case ($K = N$) is like $(\sqrt{c_{41}}\, qN)^{2/q - 2/p}$, while in our restricted cases it improves to $(\sqrt{c_{41}}\, q\sqrt{NK})^{2/q-2/p}$.

Theorem 11.2 plays a significant role in the proof of a sharp Markov-type inequality [6, Theorem 3.4] for $|\mathrm{GCAP}|_{N,K}$, as well. However, unlike Theorem 11.3, our next theorem is far from being a simple consequence of Theorem 11.2.

Theorem 11.4. *Given $0 \leqslant K \leqslant N$, there is an absolute constant $c_{42} > 0$ such that*

$$\max_{-1 \leqslant x \leqslant 1} |f'(x)| \leqslant c_{42} N(K+1) \max_{-1 \leqslant x \leqslant 1} f(x), \tag{11.13}$$

for every $f \in |\mathrm{GCAP}|_{N,K}$ of the form (2.7) with each $r_j \geqslant 1$, $j = 1, 2, \ldots, k$.

The condition that each $r_j \geqslant 1$, $j = 1, 2, \ldots, k$, is needed to insure that $|f'(z_j)| < \infty$ if $z_j \in \mathbb{R}$ and $f(z_j) = 0$. Theorem 11.4 is a generalization of a number of earlier results. Inequality (11.13) is proved in [2] for polynomials $f \in \mathscr{P}^{\mathrm{r}}_{n,k}$, $0 \leqslant k \leqslant n$, having only real zeros. Another proof of (11.13) is given in [19, Theorem 1] for all $f \in \mathscr{P}^{\mathrm{r}}_{n,k}$, $0 \leqslant k \leqslant n$. Less general or less sharp results can be found in [30,39,43,54,57,58]. The unrestricted generalized polynomial case ($K = N$) of Theorem 11.4 extends the results of Theorem 9.3. Up to the constant c_{41}, Theorem 11.4 is sharp even for the class $\mathscr{P}^{\mathrm{r}}_{n,k}$ [57, Example 1].

To give a picture on the methods used to prove inequalities for (generalized) polynomials with restricted zeros would be the subject to another survey paper. Here we mention only one of the most important tools, the idea of Lorentz polynomials of the form

$$p(x) = \sum_{j=0}^{n} a_j (1+x)^j (1-x)^{n-j}, \quad \text{with all } a_j \geqslant 0 \text{ or with all } a_j \leqslant 0. \tag{11.14}$$

By an observation of Lorentz, every $p \in \mathscr{P}^{\mathrm{r}}_{n,0}$ can be written in the above form. In many cases the information on the sign of the coefficients in the above representation can be exploited in a straightforward way, while the information on the position of the zeros of p given by $p \in \mathscr{P}^{\mathrm{r}}_{n,0}$ may seem difficult to handle. As an example, we present a lemma with its proof, which plays an important role in the proof of Theorem 11.1.

Lemma 11.5. *Let $0 \leqslant k \leqslant n$ be fixed integers and let $0 < s < 2$ and $0 \leqslant A$ be fixed real numbers. If $|p(1)| \leqslant A$ holds for every $p \in \mathscr{P}^{\mathrm{r}}_{n,k}$ of the form $p(x) = (1+x)^{n-k} Q(x)$ with $Q \in \mathscr{P}^{\mathrm{r}}_{k}$ satisfying*

$$m(\{x \in [-1, 1] : |p(x)| \leqslant 1\}) \geqslant 2 - s, \tag{11.15}$$

then $|p(1)| \leqslant A$ holds for every $p \in \mathscr{P}^{\mathrm{r}}_{n,k}$ satisfying (11.15).

Proof. Let $p \in \mathscr{P}^{\mathrm{r}}_{n,k}$ satisfy (11.15) and $p(1) \neq 0$. Then there are polynomials $w \in \mathscr{P}_{n-k,0}$ and $Q \in \mathscr{P}^{\mathrm{r}}_{k}$ such that $p = wQ$ and $w(x) \geqslant 0$ for every $-1 \leqslant x \leqslant 1$. Since $w \in \mathscr{P}^{\mathrm{r}}_{n-k,0}$ and it is nonnegative on $[-1, 1]$, it is of the form

$$w(x) = \sum_{j=0}^{n-k} a_j (1+x)^j (1-x)^{n-k-j}, \quad \text{with all } a_j \geqslant 0. \tag{11.16}$$

Using the nonnegativity of the coefficients a_j, we obtain

$$|p(x)| \geqslant |a_0(1+x)^{n-k}Q(x)|, \tag{11.17}$$

for every $-1 \leqslant x \leqslant 1$. Hence (11.15) implies that $q(x) := a_0(1+x)^{n-k}Q(x) \in \mathscr{P}^{\mathrm{r}}_{n,k}$ satisfies (11.15), and the assumption of the lemma yields $|q(1)| = |p(1)| \leqslant A$, thus the lemma is proved. □

A Bernstein-type analogue of Theorem 11.4 is established in [19, Theorem 3] for the classes $\mathscr{P}^{\mathrm{r}}_{n,k}$, $k = 0, 1, \ldots, n$, $n = 0, 1, \ldots$. Very recently with P. Borwein [8] we have found the "right" Bernstein-type analogue of Theorem 11.4 for the classes $\mathscr{P}^{\mathrm{r}}_{n,k}$, $k = 0, 1, \ldots, n$, $n = 0, 1, \ldots$. Markov- and Bernstein-type inequalities for other classes of constrained polynomials may be found in, e.g., [13,15,16,28]. The close relation between the location of the zeros of a polynomial p and the smallest positive integer d for which p can be written as

$$p(x) = \sum_{j=0}^{d} a_j(1-x)^j(1+x)^{d-j}, \quad \text{with all } a_j \geqslant 0 \text{ or with all } a_j \leqslant 0, \tag{11.18}$$

and the corresponding trigonometric results are discussed in [21,27,29].

12. Generalized Jacobi weight functions, Christoffel functions, and zeros of orthogonal polynomials

In this section we give various applications of the inequalities of the previous sections in the theory of orthogonal polynomials.

Let α be a nonnegative, finite Borel measure on $[-1, 1]$. Given $0 < p < \infty$, we define the Christoffel functions

$$\lambda_n(\alpha, p, z) := \min_{Q \in \mathscr{P}^{\mathrm{r}}_{n-1}} \int_{-1}^{1} \frac{|Q(t)|^p}{|Q(z)|^p} \, \mathrm{d}\alpha(t), \tag{12.1}$$

and the generalized Christoffel functions

$$\lambda_n^*(\alpha, p, z) := \inf_{f \in |\mathrm{GCAP}|_{n-1}} \int_{-1}^{1} \frac{(f(t))^p}{(f(z))^p} \, \mathrm{d}\alpha(t), \tag{12.2}$$

for $z \in \mathbb{C}$ and $n = 1, 2, \ldots$. For $M > 0$ and $-1 \leqslant x \leqslant 1$ we also introduce the functions

$$\Delta_M(x) := \max\{M^{-1}\sqrt{1-x^2},\, M^{-2}\} \tag{12.3}$$

and

$$w_M(x) := \int_{|t-x| \leqslant \Delta_M(x)} w(t) \, \mathrm{d}t, \tag{12.4}$$

where we assume that $w(t)$ is defined if $|t-x| \leqslant \Delta_M(x)$.

For $g(\geqslant 0) \in L_1(0, 2\pi)$, the Szegő function $D(g, z)$ is defined by

$$D(g, z) := \exp\left(\frac{1}{4\pi}\int_0^{2\pi} \log g(\theta) \frac{1 + z\,\mathrm{e}^{-\mathrm{i}\theta}}{1 - z\,\mathrm{e}^{-\mathrm{i}\theta}}\,\mathrm{d}\theta\right), \quad |z| < 1. \tag{12.5}$$

The boundary value $D(g, \mathrm{e}^{\mathrm{i}t})$ can be defined as the nontangential limit of $D(g, z)$; this exists for almost every real t. It is important to note that $D(g, z) \in H^2(|z| \leqslant 1)$ and $|D(g, \mathrm{e}^{\mathrm{i}t})|^2 = |g(t)|$ holds for almost every real t. Other properties of the Szegő function may be found in, e.g., [32, Chapter V].

In [26, Theorems 2.1, 2.2, 3.1 and 3.2] we give sharp lower and upper bounds for the Christoffel functions and generalized Christoffel functions on $[-1, 1]$ associated with generalized Jacobi weight functions. The lower bounds are given by the following pair of theorems.

Theorem 12.1. *Given $0 < p < \infty$, $0 \leqslant \Gamma < \infty$, and $n = 1, 2, \ldots$, let $M = 1 + p(n-1)/(\Gamma + p + 1)$. There exists an absolute constant $c_{43} > 0$ such that*

$$\lambda_n(\alpha, p, x) \leqslant c_{43}^{\Gamma+p+1} w_M(x), \quad -1 \leqslant x \leqslant 1,$$

for every measure α satisfying $\mathrm{d}\alpha = w\,\mathrm{d}t$ with $w \in |\mathrm{GCAP}|_\Gamma$.

Theorem 12.2. *Let $0 < p < \infty$, $0 \leqslant \Gamma < \infty$, and $n = 1, 2, \ldots$. Let $w = w^{(\mathrm{T})}/w^{(\mathrm{B})}$, where $w^{(\mathrm{T})}$ and $w^{(\mathrm{B})}$ belong to $|\mathrm{GCAP}|_\Gamma$, and let $\mathrm{d}\alpha = w\,\mathrm{d}t$. Let d denote the number of different zeros of $w^{(\mathrm{B})}$. There exists an absolute constant $c_{44} > 0$ such that*

$$\lambda_n(\alpha, p, x) \leqslant c_{44}^{\Gamma+pd+p+1} w_M(x), \quad -1 \leqslant x \leqslant 1,$$

with

$$M = \frac{p(n-1-d) - \Gamma}{2\Gamma + 4 + p},$$

whenever $M \geqslant 1$, and

$$\lambda_n^*(\alpha, p, x) \leqslant c_{44}^{\Gamma+1} w_M(x), \quad -1 \leqslant x \leqslant 1,$$

with

$$M = \frac{p(n-1) - \Gamma}{2\Gamma + 4},$$

whenever $M > 0$.

Since $\lambda_n^*(\alpha, p) \leqslant \lambda_n(\alpha, p)$, we give lower bounds for $\lambda_n^*(\alpha, p)$ instead of $\lambda_n(\alpha, p)$. Our first theorem deals with the case $\mathrm{d}\alpha(t) = w(t)\,\mathrm{d}t$, where $w \in |\mathrm{GCAP}|_\Gamma$, and the second one gives a lower bound in a more general case, when the weight function w satisfies the Szegő condition of logarithmic integrability.

Theorem 12.3. *Given $0 < p < \infty$, $0 \leqslant \Gamma < \infty$ and $1 \leqslant n < \infty$, let $M = 1 + p(n-1)/(\Gamma + p + 1)$. There exists an absolute constant $c_{45} > 0$ such that*

$$\lambda_n^*(\alpha, p, x) \geqslant c_{45}^{\Gamma+p+1} w_M(x), \quad -1 \leqslant x \leqslant 1,$$

for every measure α satisfying $\mathrm{d}\alpha = w\,\mathrm{d}t$ with $w \in |\mathrm{GCAP}|_\Gamma$.

Theorem 12.4. *Let w be a nonnegative integrable weight function on $[-1, 1]$ such that $\log(w(\cos\cdot)) \in L_1(-\pi, \pi)$. Let $0<p<\infty$, and let $\mathrm{d}\alpha = w\ \mathrm{d}t$. There exists an absolute constant $c_{46} > 0$ such that*

$$\lambda_n^*(\alpha, p, x) \geqslant c_{46} \Delta_{p(n-1)+1}(x) \left| D(w(\cos\cdot), re^{i\theta}) \right|^2, \quad -1 \leqslant x \leqslant 1,$$

where $r = (p(n-1)+1)/(p(n-1)+3)$, $x = \cos\theta$ and $D(g, z)$ is the Szegő function defined by (12.5).

As an application of our Nikolskii-type inequalities of Section 5 for generalized nonnegative polynomials, we give an upper bound for $\lambda_n^*(\alpha, p, x)$ for every measure α such that $\mathrm{d}\alpha = w\ \mathrm{d}t$ with $w \in |\mathrm{GCAP}|_\Gamma$ and for every $x \in [-1, 1]$. To eliminate some technical details, instead of Theorem 12.3 we prove the following theorem.

Theorem 12.3*. *Given $0<p<\infty$, $0 \leqslant \Gamma < \infty$ and $1 \leqslant N \in \mathbb{R}$, we have*

$$\lambda_n^*(\alpha, p, x) \geqslant \frac{1}{2c_9} \frac{w(x)\sqrt{1-x^2}}{p(n-1)+\Gamma+1}, \quad -1<x<1,$$

for every measure α satisfying $\mathrm{d}\alpha = w\ \mathrm{d}t$ with $w \in |\mathrm{GCAP}|_\Gamma$. Here c_9 is the same as in Theorem 5.2.

Proof. If $f \in |\mathrm{GCAP}|_{n-1}$, where $1 \leqslant N < \infty$, then $g(\theta) := |f(\cos\theta)\sin\theta| \in |\mathrm{GCTP}|_n$, and thus, applying Theorem 5.2 to the function g with $q=1$ and $p=\infty$, after the substitution $x = \cos\theta$, we obtain

$$f(x) \leqslant 2c_9 \frac{n+1}{\sqrt{1-x^2}} \int_{-1}^{1} f(t)\ \mathrm{d}t, \quad -1<x<1. \tag{12.6}$$

Replacing $f \in |\mathrm{GCAP}|_{n-1}$ by $f^p w \in |\mathrm{GCAP}|_{p(n-1)+\Gamma}$, $0<p<\infty$, we obtain

$$(f(x))^p w(x) \leqslant 2c_9 \frac{p(n-1)+\Gamma+1}{\sqrt{1-x^2}} \int_{-1}^{1} (f(t))^p w(t)\ \mathrm{d}t, \quad -1<x<1, \tag{12.7}$$

for every $f \in |\mathrm{GCAP}|_{n-1}$ and $w \in |\mathrm{GCAP}|_\Gamma$, and the theorem follows by the definition of $\lambda_n^*(\alpha, p, x)$. □

To study the zeros of orthogonal polynomials, we use the standard notation. Let α be a nonnegative finite Borel measure with supp $\alpha \subset [-1, 1]$, and let $\{p_n\}_{n=0}^\infty$ denote the corresponding orthonormal polynomials. In addition, $\{x_{j,n}\}_{j=1}^n$ denote the zeros of p_n in decreasing order, $x_{0,n} := 1$, $x_{n+1,n} := -1$, and $\theta_{j,n}$ are defined by $x_{j,n} = \cos\theta_{j,n}$ for $j = 0, 1, \ldots, n+1$. In [26, Theorem 4.1] sharp lower and upper bounds for the distance of consecutive zeros of orthogonal polynomials associated with generalized Jacobi weight functions with positive exponents (in other words generalized nonnegative polynomial weight functions) are established. The novelty of these estimates lies in the fact that our constants depend only on the degree of the weight function (and not on the weight function itself).

Theorem 12.5. *Let* $0 \leqslant \Gamma < \infty$ *and let* $d\alpha = w\, dt$, *where* $w \in |\mathrm{GCAP}|_{\Gamma}$. *There exist two absolute constants* $c_{47} > 0$ *and* $c_{48} > 0$ *such that the zeros of the corresponding orthogonal polynomials satisfy*

$$\theta_{j,n} - \theta_{j-1,n} \leqslant \frac{c_{47}^{\Gamma+1}}{n}, \quad j = 1, 2, \ldots, n+1, \tag{12.8}$$

and

$$\theta_{j,n} - \theta_{j-1,n} \geqslant \frac{c_{48}^{\Gamma+1}}{n}, \quad j = 2, 3, \ldots, n, \tag{12.9}$$

for $n = 1, 2, \ldots$.

We believe that (12.9) actually holds for $j = 1$ and $j = n + 1$ as well, but, alas, our method does not seem to work in these cases. The latter would generalize [45, Theorem 3, p.367] and [47, Theorem 9.22, p.166]. To prove the upper estimate (12.8), we used the Markov–Stieltjes inequality [32, formula (5.4), p.29] for the Christoffel numbers $\lambda_{m,n} = \lambda_n(\alpha, 2, x_{m,n})$, $m = 0, 1, \ldots, n+1$, and Theorem 12.1. To prove the lower estimate (12.9), our method was based on that of Erdős and Turán (cf. [32, pp. 111, 112], [45, p.369] and [47, pp. 164, 165]) combined with the weighted Markov- and Bernstein-type inequalities of Section 10 (Theorems 10.1 and 10.2).

The zero estimates of the following three theorems were given in [20, Theorems 6–8]. To formulate these, we use the notation introduced in Section 5, right after the proof of Theorem 5.2.

Theorem 12.6. *Let* $0 < a < 1$, $p = 2/a - 2$, $d\alpha = w\, dt$ *and* $\log^-(w(t)) \in \mathrm{WL}_p(-1, 1)$. *There is a constant* $c(a, K)$ *depending only on* a, $K = K(\log^-(w))$ *(see (5.12)) and* $\| w \|_{L_1}(-1, 1)$ *such that*

$$\theta_{j,n} - \theta_{j-1,n} \leqslant c(a, K) n^{a-1}, \quad j = 1, 2, \ldots, n+1.$$

Theorem 12.7. *Let* $0 < a < 1$, $p = 1/a - 1$, $d\alpha = w\, dt$ *and* $\log^-(w(\cos\theta)) \in \mathrm{WL}_p(-\pi, \pi)$. *There is a constant* $c(a, K)$ *depending only on* a, $K = K(\log^- w(\cos\theta))$ *(see (5.12)) and* $\| w \|_{L_1(-1,1)}$ *such that*

$$\theta_{j,n} - \theta_{j-1,n} \leqslant c(a, K) n^{a-1}, \quad j = 1, 2, \ldots, n+1.$$

If $a = \frac{1}{2}$, then $p = 1$ and Theorem 12.7 gives an upper bound for the distance of the consecutive zeros of orthogonal polynomials associated with weight functions from the Szegő class. This special case is proved in [47, pp. 157, 158] for $x_{j,n}$ instead of $\theta_{j,n}$.

The following theorem is due to Erdős and Turán [59, pp. 113,114] when $w^{-1} \in L_1(-1, 1)$. A generalization, when $w^{-\epsilon} \in L_1(-1, 1)$ for some $\epsilon > 0$, is established in [47, p.158], but with $x_{j,n}$ instead of $\theta_{j,n}$.

Theorem 12.8. *Let* $w^{-\epsilon} \in \mathrm{WL}_1(-1, 1)$ *for some* $\epsilon > 0$ *and let* $d\alpha = w\, dt$. *There is a constant* $c(\epsilon, K)$ *depending only on* ϵ, $K = K(w^{-\epsilon})$ *(see (5.12)) and* $\| w \|_{L_1(-1,1)}$ *such that*

$$\theta_{j,n} - \theta_{j-1,n} \leqslant c(\epsilon, K) \frac{\log n}{n}, \quad j = 1, 2, \ldots, n+1, \ n \geqslant 2.$$

We give the short proof of Theorems 12.6–12.8 from [20], where a method of Lengyel [59, pp. 112–115] is improved. Our improvement is based on the Nikolskii-type inequalities of Theorems 5.3–5.5.

Proof of Theorems 12.6–12.8. Let $1 \leqslant j \leqslant n+1$ be a fixed integer and let $\gamma = \frac{1}{2}(\theta_{j,n} + \theta_{j-1,n})$. We define $\rho(x) = \rho(\cos\theta)$ by

$$2\rho(x) = \left(\frac{\sin(\frac{1}{2}N(\gamma+\theta))}{N\sin(\frac{1}{2}(\gamma+\theta))}\right)^{2m} + \left(\frac{\sin(\frac{1}{2}N(\gamma-\theta))}{N\sin(\frac{1}{2}(\gamma-\theta))}\right)^{2m}, \tag{12.10}$$

where N and m are certain positive integers. Then $\rho \in \mathscr{P}^{\mathrm{r}}_{m(N-1)}$ (see [59, 6.11.3]). By [59, 6.11] we have

$$\rho(x_{k,n}) \leqslant \left(N\sin\left(\tfrac{1}{4}(\theta_{j,n} - \theta_{j-1,n})\right)\right)^{-2m}, \quad k = 1, 2, \ldots, n, \tag{12.11}$$

so by the Gaussian quadrature formula,

$$\int_{-1}^{1} \rho(x) w(x)\, \mathrm{d}x \leqslant \left(N\sin\left(\tfrac{1}{4}(\theta_{j,n} - \theta_{j-1,n})\right)\right)^{-2m} \int_{-1}^{1} w(x)\, \mathrm{d}x, \tag{12.12}$$

if $m(N-1) \leqslant 2n-1$. Further $\rho(\cos\gamma) \geqslant \frac{1}{2}$, hence

$$\max_{-1 \leqslant x \leqslant 1} |\rho(x)| \geqslant \tfrac{1}{2}. \tag{12.13}$$

Under the conditions of Theorems 12.6 or 12.7, Theorems 5.3 or 5.4, respectively, together with (12.12) and (12.13) give

$$\tfrac{1}{2} \leqslant \exp(c(a, K)n^a)\left(N\sin\left(\tfrac{1}{4}(\theta_{j,n} - \theta_{j-1,n})\right)\right)^{-2m} \int_{-1}^{1} w(x)\, \mathrm{d}x, \tag{12.14}$$

thus

$$\frac{\theta_{j,n} - \theta_{j-1,n}}{2\pi} \leqslant N^{-1} \exp\left(\frac{c(a, K)n^a}{m}\right)\left(2\int_{-1}^{1} w(x)\, \mathrm{d}x\right)^{1/(2m)} \tag{12.15}$$

Choosing $m = [n^a]$ and $N = [n^{1-a}]$ in (12.15), we obtain Theorems 12.6 and 12.7.
Under the conditions of Theorem 12.8, Theorem 5.5, (12.12) and (12.13) give

$$\tfrac{1}{2} \leqslant c(\epsilon, K) \exp(m \log n)\left(N\sin\left(\tfrac{1}{4}(\theta_{j,n} - \theta_{j-1,n})\right)\right)^{-2m} \int_{-1}^{1} w(x)\, \mathrm{d}x, \tag{12.16}$$

therefore

$$\frac{\theta_{j,n} - \theta_{j-1,n}}{2\pi} \leqslant c(\epsilon, K) N^{-1} \exp\left(\frac{M \log n}{m}\right)\left(2\int_{-1}^{1} w(x)\, \mathrm{d}x\right)^{1/(2m)} \tag{12.17}$$

Now the choices $m = [\log n]$, $N = [n/\log n]$ give the desired result. □

To see the sharpness of Theorems 12.6 and 12.7, we introduce the generalized Pollaczek weight functions by

$$w_\beta(x) = \exp\left(-(1-x^2)^{-\beta}\right), \quad 0 \leqslant \beta < \infty. \tag{12.18}$$

A result of Lubinsky and Saff [42, p.411, (16)] implies that for the above weight functions, we have

$$\theta_{n+1,n} - \theta_{n,n} = \pi - \theta_{n,n} \geqslant c(\beta) n^{-1/(2\beta+1)}, \quad 0 \leqslant \beta < \infty, \tag{12.19}$$

where $c(\beta)$ depends only on β. If $\beta = (2/a - 2)^{-1}$, $0 < a < 1$, then $\log^-(w_\beta(x))$ is in $WL_p(-1, 1)$ with $p = 2/a - 2$ and $\log^-(w_\beta(\cos\theta))$ is in $WL_p(-\pi, \pi)$ with $p = 1/a - 1$. Further $n^{-1/(2\beta+1)} = n^{a-1}$, therefore (12.19) shows the sharpness of Theorems 12.6 and 12.7.

Let $\int_{-1}^{1} w(x)\,\mathrm{d}x = 1$ and $w(x) > 0$ a.e. on $[-1, 1]$. In [37] the function

$$\phi(w, \epsilon) = \inf\left\{ \int_A w(\cos t) \sin t \,\mathrm{d}t\colon A \subset [0, \pi],\ m(A) \geqslant \epsilon \right\}$$

is introduced, for every w as above and $0 \leqslant \epsilon \leqslant \pi$, where $m(\cdot)$ denotes the Lebesgue measure. It is easy to see that $\phi(w, \epsilon)$ is a continuous, increasing function of ϵ on $[0, \pi]$, positive in $(0, \pi]$ and it satisfies $\phi(w, 0) = 0$ and $\phi(w, \pi) = 1$. This implies that for every $n \in \mathbb{N}$ the equation $\phi(w, \epsilon) = \exp(-n\epsilon)$ has a unique solution, which will be denoted by $\epsilon_n(w)$. The quantity $\epsilon_n(w)$ plays a central role in estimating the maximal distance between consecutive zeros of the monic polynomials $T_{n,p}(x) = x^n - B_{n-1,p}$ satisfying

$$\int_{-1}^{1} |x^n - B_{n-1,p}(x)|^p w(x)\,\mathrm{d}x = \min_{q \in \mathcal{P}_{n-1}^r} |x^n - q(x)|^p w(x)\,\mathrm{d}x. \tag{12.20}$$

Namely, it is shown in [37] that the maximal distance between consecutive zeros of $T_{n,p}$ is bounded by $c\epsilon_n(w)$, where c depends only on w. An important step of the proof is an application of the trigonometric Remez-type inequality of Theorem 3.4. The results of [37] extend Theorems 12.7 and 12.8.

13. Remez-type inequalities for Müntz polynomials and Müntz-type theorems on closed subsets of [0, 1] with positive measure

Let $\Lambda = \{\lambda_j\}_{j=0}^{\infty}$, $0 = \lambda_0 < \lambda_1 < \cdots$. A beautiful theorem of Müntz and Szász says that

$$H(\Lambda) := \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, x^{\lambda_2}, \ldots\} \tag{13.1}$$

(the span means the collection of finite linear combinations of the elements with real coefficients) is dense in $C[0, 1]$ in the uniform norm if and only if

$$\sum_{j=1}^{\infty} \lambda_j^{-1} = \infty. \tag{13.2}$$

Proofs of the Müntz–Szász Theorem may be found in [10,31,63] with various generalizations and extensions in [4,9,38,49,60]. This is very much a theorem about continuous functions on intervals. If $\inf\{\lambda_{j+1} - \lambda_j\colon j = 1, 2, \ldots\} > 0$, it can be proved that exactly the same theorem holds in $C(A)$, provided $A \subset [0, \infty)$ is a compact set with nonempty interior. This result is due to [11]. When $\bar{A}$ has no interior, it is by no means obvious what happens. In [7, Theorem 1] we proved the following theorem.

Theorem 13.1. *Suppose $\lambda_j > q^j$, $j = 1, 2, \ldots$, where $q > 1$ and suppose $A \subset [0, \infty)$ is any set of positive Lebesque measure. Then $H(\Lambda)$ fails to be dense in $C(A)$ in the uniform norm.*

In fact, under the assumptions of Theorem 13.1, if $y \in A$ is a point of positive Lebesgue density, then every function f from the uniform closure of $H(\Lambda)$ on A is of the form

$$f(x) := \sum_{j=0}^{\infty} a_j x^{\lambda_j}, \quad x \in [0, y). \tag{13.3}$$

If $\lambda_j \in \mathbb{N}$, $j = 1, 2, \ldots$, then this means that every function f from the uniform closure of $H(\Lambda)$ on A can be extended analytically throughout the open disk $\{z \in \mathbb{C}: |z| < y\}$, provided $y \in A$ is a point of positive Lebesgue density.

This in turn rests on the following "left-hand side" Remez-type inequality [7, Inequality 1].

Theorem 13.2. *Suppose $A \subset [\rho, 1]$ is a closed set of measure at least $\epsilon > 0$. Suppose $\lambda_j \geqslant q^j$, $j = 1, 2, \ldots$, where $q > 1$. Then*

$$\max_{0 \leqslant x \leqslant \rho} \left| \sum_{j=0}^{n} a_j x^{\lambda_j} \right| \leqslant c(\rho, \epsilon, q) \max_{x \in A} \left| \sum_{j=0}^{n} a_j x^{\lambda_j} \right|,$$

where $c(\rho, \epsilon, q)$ depends only on ρ, ϵ and q and not on Λ, n and A.

We conjecture that both Theorems 13.1 and 13.2 can be generalized to the case when $\sum_{j=1}^{\infty} \lambda_j^{-1} < \infty$; however, to prove these seems to be extremely difficult. On the other hand, using Tietze's and Müntz's Theorems, one can easily show that $\lambda_0 = 1$ and $\sum_{j=1}^{\infty} \lambda_j^{-1} = \infty$ imply that $H(\Lambda)$ is dense in $C[A]$ in the uniform norm for every compact set $A \subset [0, \infty)$. Consequently the inequality of Theorem 13.2 cannot hold if $\sum_{j=1}^{\infty} \lambda_j^{-1} = \infty$.

In a seminal paper [11], Clarkson and Erdős proved the following theorem.

Theorem 13.3. *Suppose $\inf\{\lambda_{j+1} - \lambda_j: j = 1, 2, \ldots\} > 0$ and $\sum_{j=1}^{\infty} \lambda_j^{-1} < \infty$. Then,*

$$\max_{0 \leqslant x \leqslant 1} \left| \sum_{j=0}^{n} a_j x^{\lambda_j} \right| \leqslant c(\Lambda, \delta) \max_{1-\delta \leqslant x \leqslant 1} \left| \sum_{j=0}^{n} a_j x^{\lambda_j} \right|, \quad 0 < \delta < 1,$$

where $c(\Lambda, \delta)$ depends only on $\Lambda = \{\lambda_j\}_{j=0}^{\infty}$ and δ, but not on n.

Theorem 13.3 is not stated explicitly in [11], it may be found in [56, p.54]; another distinct proof is given in [7, Inequality 1 (Interval Case)]. Theorem 13.3 implies immediately that if $\inf\{\lambda_{j+1} - \lambda_j: j = 1, 2, \ldots\} > 0$ and $\sum_{j=1}^{\infty} \lambda_j^{-1} < \infty$, then $H(\Lambda)$ fails to be dense in $C([a, b])$ in the uniform norm for every $[a, b] \subset [0, \infty)$. Moreover, it follows from [11] that in the above case every function f from the uniform closure of $H(\Lambda)$ on $[a, b]$ is of the form

$$f(x) := \sum_{j=0}^{\infty} a_j x^{\lambda_j}, \quad x \in [0, b). \tag{13.4}$$

This is a critical, but only a small piece of our proof of the "left-hand side" Remez-type inequality of Theorem 13.2. Our proof relies on an examination of generalized Chebyshev polynomials $T_n\{[\lambda_0, \lambda_1, \ldots, \lambda_n]: A\}$ on a compact set $A \subset [0, \infty)$. These are defined to be

$$T_n(x) := c \left(x^{\lambda_n} + \sum_{j=0}^{n-1} a_j x^{\lambda_j} \right),$$

where we choose $\{a_j\}_{j=0}^{n-1}$ to minimize

$$\max_{x \in A} \left| x^{\lambda_n} + \sum_{j=0}^{n-1} a_j x^{\lambda_j} \right|,$$

and c is chosen so that

$$\max_{x \in A} |T_n(x)| = 1 \quad \text{and} \quad \lim_{x \to \infty} T_n(x) = +\infty.$$

It is well known that such a T_n, $n = 1, 2, \ldots,$ exists and it is unique. In particular, we established estimates for the size of their zeros when $A = [0, 1]$. In this special case it is easy to see that the generalized Chebyshev polynomials $T_n = T_n\{[\lambda_0, \lambda_1, \ldots, \lambda_n]: [0, 1]\}$ are characterized by the following properties:

(i) $T_n \in H_n(\Lambda) := \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \ldots, x^{\lambda_n}\}$;
(ii) T_n equioscillates $n + 1$ times on $[0, 1]$;
(iii) $\max_{0 \leqslant x \leqslant 1} |T_n(x)| = 1$;
(iv) $T_n(1) = 1$.

To be precise, property (ii) means that T_n achieves the values $\pm \max_{0 \leqslant x \leqslant 1} |T_n(x)| = \pm 1$, $n + 1$ times on $[0, 1]$ with alternating sign.

Denseness and approximation questions in Markov spaces are intimately and essentially tied to the behavior of the associated Chebyshev polynomials, see, for example, [3,4,7]. We showed in [5, Theorems 2.1 and 2.2] that lacunary Müntz spaces (satisfying $\inf\{\lambda_{j+1}/\lambda_j: j = 1, 2, \ldots\} > 1$) are completely characterized by the property that their associated Chebyshev polynomials $T_n\{\lambda_0, \lambda_1, \ldots, \lambda_n]: [0, 1]\}$, $n = 1, 2, \ldots,$ have uniformly bounded coefficients. This allowed us to give an essentially sharp Bernstein-type inequality [5, Theorem 3.1] for these spaces.

Theorem 13.4. *If $\lambda_0 = 0$, $\lambda_1 \geqslant 1$ and $\lambda_{j+1}/\lambda_j \geqslant q > 1$ for every $j = 1, 2, \ldots,$ then there is a constant $c(\Lambda)$ depending only on $\Lambda = \{\lambda_j\}_{j=0}^{\infty}$ such that*

$$|p'(y)| \leqslant \frac{c(\Lambda)}{1 - y} \max_{0 \leqslant x \leqslant 1} |p(x)|, \tag{13.5}$$

for every $p \in H(\Lambda)$ and $0 \leqslant y < 1$.

In [5, Theorem 4.1], from Theorem 13.4 we rederived the conclusion of Theorem 13.1 for lacunary Müntz systems. On the other hand, it can be proved that if (13.5) holds for every $p \in H(\Lambda)$, then there is a $q > 1$ depending only on $c(\Lambda)$ such that $\lambda_j \geqslant q^j$, $j = 2, 3, \ldots$. The following question may be simple to answer, but I do not know the answer at the moment.

Problem 13.5. Is there a sequence $\Lambda = \{\lambda_j\}_{j=0}^{\infty}$ with $\lambda_0 = 0$, $\lambda_1 \geqslant 1$ and $\inf_{j \in \mathbb{N}} \lambda_{j+1}/\lambda_j = 1$ such that (13.5) holds for every $p \in H(\Lambda)$ and $0 \leqslant y < 1$?

Markov- and Bernstein-type inequalities for Müntz systems may be found in [3,48] as well.

To convince the reader that the generalization of Theorem 13.2 to the case when $\sum_{j=1}^{\infty} \lambda_j^{-1} < \infty$ would be highly nontrivial, we show that it would solve Newman's problem [49, P(10.5), p.50] concerning the density of the classes

$$H^k(\Lambda) := \left\{ p = \prod_{j=1}^{k} p_j \colon p_j \in H(\Lambda),\ j = 1, 2, \ldots, k \right\},$$

in $C[0, 1]$ in the uniform norm, when $\lambda_j = j^2$, $j = 0, 1, \dots$. Indeed, assume that the following Remez-type inequality is true.

Conjecture 13.6. *Let $\sum_{j=1}^{\infty} \lambda_j^{-1} < \infty$. For every $0 < \epsilon < 1$ there is a constant $c(\epsilon, \Lambda)$ depending only on ϵ and $\Lambda = \{\lambda_j\}_{j=0}^{\infty}$ such that $|p(0)| \leqslant c(\epsilon, \Lambda)$ for every $p \in H(\Lambda)$ with $m(\{x \in [0, 1]: |p(x)| \leqslant 1\}) \geqslant \epsilon$, where $m(\cdot)$ denotes the Lebesgue measure.*

If Conjecture 13.6 were true, then $H^k(\Lambda)$ would fail to be dense in $C[0, 1]$ in the uniform norm for every $k \in \mathbb{N}$, whenever $\sum_{j=1}^{\infty} \lambda_j^{-1} < \infty$. Indeed, Conjecture 13.6 implies that

$$m(\{x \in [0, 1]: |q(x)| \geqslant \alpha^{-1} |q(0)|\}) \geqslant 1 - (2k)^{-1}, \tag{13.6}$$

for every $q \in H(\Lambda)$ with $\alpha = c((2k)^{-1}, \Lambda) + 1$. Hence

$$m(\{x \in [0, 1]: |p(x)| \geqslant \alpha^{-k} |p(0)|\}) \geqslant \tfrac{1}{2}, \tag{13.7}$$

for every $p \in H^k(\Lambda)$ (if $p = p_1 p_2 \cdots p_k$ with $p_j \in H(\Lambda)$, $j = 1, 2, \dots, k$, then $|p(x)| \geqslant \alpha^{-k} |p(0)|$ holds for every $x \in [0, 1]$ satisfying $|p_j(x)| \geqslant \alpha^{-1} |p_j(0)|$ for each $j = 1, 2, \dots, k$). Now let $f \in C[0, 1]$ be such that $f(x) = 0$ if $\frac{1}{4} \leqslant x \leqslant 1$, and $f(0) = 1$. If there were a $p \in H^k(\Lambda)$ such that

$$\max_{0 \leqslant x \leqslant 1} |p(x) - f(x)| \leqslant \tfrac{1}{2} \alpha^{-k}, \tag{13.8}$$

then it would contradict (13.7). Similarly, Conjecture 13.6 would imply that if $\sum_{j=1}^{\infty} \lambda_j^{-1} < \infty$, and $A \subset [0, 1]$ is of positive measure, then $H^k(\Lambda)$ fails to be dense in $C(A)$ in the uniform norm for every $k \in \mathbb{N}$.

What happens in Theorems 13.1 and 13.2 (and in the corresponding conjectures under the assumption $\sum_{j=1}^{\infty} \lambda_j^{-1} < \infty$) if we allow sets of measure 0? We have the following pair of theorems.

Theorem 13.7. *Let $\Lambda = \{\lambda_j\}_{j=0}^{\infty}$ be an arbitrary sequence of distinct positive real numbers. Then there exists a nonempty perfect set $E \subset [0, 1]$ such that $H(\Lambda)$ is dense in $C(E)$ in the uniform norm. On the other hand, if $\sum_{j=1}^{\infty} \lambda_j^{-1} < \infty$, then there is a countable closed set $E \subset [0, 1]$ such that $H(\Lambda)$ fails to be dense in $C(E)$ in the uniform norm.*

The first statement is due to Totik (private communication), while the second one is proved in [7, Theorem 5]. In [7, Theorem 3] we also observe the following theorem.

Theorem 13.8. *Let $\Lambda = \{\lambda_j\}_{j=0}^{\infty}$ be an arbitrary sequence of distinct positive real numbers. Then there exist a nonempty perfect set $E \subset [0, 1]$ and Müntz polynomials $p_m \in H(\Lambda)$ such that $\max_{x \in E} |p_m(x)| \leqslant 1$ and $\lim_{m \to \infty} |p_m(0)| = \infty$.*

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