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Independent Deuber sets in graphs on the natural numbers

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Abstract

We show that for any k, m, p, c, if G is a K_k-free graph on $\mathbb{N}$ then there is an independent set of vertices in G that contains an (m, p, c)-set. Hence if G is a K_k-free graph on $\mathbb{N}$, then one can solve any partition regular system of equations in an independent set. This is a common generalization of partition regularity theorems of Rado (who characterized systems of linear equations $A\mathbf{x} = \mathbf{0}$ a solution of which can be found monochromatic under any finite coloring of $\mathbb{N}$) and Deuber (who provided another characterization in terms of (m, p, c)-sets and a partition theorem for them), and of Ramsey's theorem itself.

Keywords: Ramsey theory; Independent sets; Partition regular equations

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1. Introduction and statement of results

In this paper we are interested in graphs whose vertices are natural numbers, and in arithmetic properties of independent sets in such graphs. We use $\mathbb{N}$ to denote the set of natural numbers—not including 0—and $\omega = \mathbb{N} \cup \{0\}$. We write $[a, b] = \{c \in \mathbb{Z} : a \leqslant c \leqslant b\}$ to denote an interval of integers.

Let A be a finite matrix with integer entries. The system of linear equations $A\mathbf{x} = \mathbf{0}$ is called *partition regular* (over $\mathbb{N}$) if for every partition of $\mathbb{N}$ into finitely many classes there exists a solution completely contained in one class.

Schur's theorem [17] says that for any positive integer r, there exists n so that for every coloring $\rho : [1, n] \to [1, r]$ there exist $x, y \in [1, n]$ with $\rho(x) = \rho(y) = \rho(x + y)$. The equation $x + y - z = 0$ describes these Schur triples, and so is partition regular. Van der Waerden's theorem [19] states that for any positive integers r, ℓ, there exists n so that for any coloring $\rho : [1, n] \to [1, r]$ there is a monochromatic ℓ-term arithmetic progression. Solutions to equations $x - 2y + z = 0$ are 3-term arithmetic progressions or are constant and so this system is also partition regular. Similarly, systems of equations describing any longer arithmetic progressions form partition regular systems. An example of a simple system which is not partition regular is $x + y = 3z$. (See, e.g., [6] or [7] for more details.)

A characterization of partition regular systems of equations was first given by Rado [15] in terms of something (which is not relevant to our use here) called the "columns property". Deuber [2] later gave another characterization of partition regular systems using structures called "(m, p, c)-sets", which we now define.

Definition 1.1. Let $p, c \in \mathbb{N}$ with $c \leqslant p$, and let $m \in \omega$. A set of integers S is an (m, p, c)-set if $S \subset \mathbb{N}$ and there exist positive integers (generators) $x_0, x_1, \ldots, x_m$ so that $S = R_0(S) \cup R_1(S) \cup \cdots \cup R_m(S)$, where

$$\begin{aligned}
R_0(S) &= \{cx_0 + \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_m x_m \colon \lambda_1, \ldots, \lambda_m \in [-p, p]\},\\
R_1(S) &= \{cx_1 + \lambda_2 x_2 + \cdots + \lambda_m x_m \colon \lambda_2, \ldots, \lambda_m \in [-p, p]\},\\
&\vdots\\
R_{m-1}(S) &= \{cx_{m-1} + \lambda_m x_m \colon \lambda_m \in [-p, p]\},\\
R_m(S) &= \{cx_m\}.
\end{aligned}$$

In this case we write $S = (x_0, x_1, \ldots, x_m)_{p,c}$ and we say that $R_k(S)$ is the $(k + 1)$th row of S.

We note that the condition $c \leqslant p$ is for convenience only; nothing would be lost without this condition because for any $p' > p$, every (m, p', c)-set trivially contains an (m, p, c)-set.

In honor of Deuber's contributions to the field, if a set S is an (m, p, c)-set for some m, p, c, then we might simply say that S is a *Deuber set* without specifying the parameters m, p, c.

Theorem 1.2 (Deuber [2]). *A linear system $A\mathbf{x} = \mathbf{0}$ is partition regular if and only if there exist positive integers m, p, c such that every (m, p, c)-set contains a solution of $A\mathbf{x} = \mathbf{0}$.*

In proving a conjecture by Rado regarding partition regular systems, Deuber used the following partition theorem.

Theorem 1.3 (Deuber [2]). *For every $m \in \omega$, every $p, c \in \mathbb{N}$ with $c \leqslant p$ and every $k \in \mathbb{N}$, there exists $n, q, d \in \mathbb{N}$ with $d \leqslant q$ so that for every (n, q, d)-set X and every coloring $\rho : X \to [1, k]$, there exists a monochromatic (m, p, c)-set contained in X.*

To state our results, we adopt standard notation. For a set S and $n \in \omega$, let $[S]^n = \{F \subseteq S : |F| = n\}$. Let $G = (V, E)$ denote a (simple) graph on vertex set $V = V(G)$ with edge set $E = E(G) \subseteq [V]^2$. A set $Y \subset V(G)$ is called *independent* in G if $[Y]^2 \cap E(G) = \emptyset$. When $E(G) = [V(G)]^2$, we say that G is complete, and the complete graph on n vertices is denoted by K_n. A graph $G = (V, E)$ is *k-partite* if V can be partitioned into k sets, $V = V_1 \cup \cdots \cup V_k$, each V_i containing no edges, and is a *complete k-partite* graph if for each $i \neq j$, whenever $x \in V_i$ and $y \in V_j$ then $\{x, y\} \in E$. A complete bipartite graph on sets of size m and n will be denoted by $K_{m,n}$.

The main result in this paper is the following.

Theorem 1.4. *Given $k, p, c \in \mathbb{N}$ with $c \leqslant p$ and $m \in \omega$, there exist $n, q, d \in \mathbb{N}$ so that any K_k-free graph on an (n, q, d)-set contains an independent (m, p, c)-set.*

Remarks. (1) Theorem 1.4 generalizes Theorem 1.3 by the following reasoning: Fix k, m, p, c, let n, q, d be guaranteed by Theorem 1.4, and fix an (n, q, d)-set X. Color X with $r = k - 1$ colors and form the complete $(k - 1)$-partite graph G whose partite sets are color classes. Since G is K_k-free, by Theorem 1.4 some (m, p, c)-set is independent in G, and hence must be contained in one partite set, i.e., a single color class. Hence there is a monochromatic (m, p, c)-set. Since (m, p, c)-sets contain sum-sets and arithmetic progressions, Theorem 1.4 also implies theorems of van der Waerden, Schur, and others.

(2) Theorem 1.4 also generalizes Ramsey's theorem for graphs, because under any red–blue coloring of the pairs of a large set, rather than guaranteeing either a red K_k or a large blue clique, we guarantee either a red K_k, or a large blue clique on an (m, p, c)-set.

Since any (n, q, d)-set sits in some initial interval of the positive integers, Theorem 1.4 immediately implies the following statement:

Corollary 1.5. *Given $k, p, c \in \mathbb{N}$, $m \in \omega$, and any K_k-free graph G with vertex set $\mathbb{N}$, there exists an (m, p, c)-set which is independent in G.*

Corollary 1.5 can be formulated in terms of partition regular systems by using the m, p, c guaranteed in Theorem 1.2:

Corollary 1.6. *For any $k \geqslant 2$ and any K_k-free graph on $\mathbb{N}$, one can solve any partition regular system in an independent set.*

We do not know if there is a hypergraph version of Theorem 2.1. For example, is there an analogous condition on a family of triples of $\mathbb{N}$ that would imply that there is an (m, p, c)-set not containing any triple? If so, it is not so simple, as the following example indicates. Let H be the 3-uniform hypergraph on $\mathbb{N}$ defined with hyperedges of the form $\{x, x+d, x+3d\}$. Then H is $K_4^{(3)}$-free, yet every arithmetic progression of length 4 contains a hyperedge.

2. Earlier work

Ramsey's theorem for graphs [16] says that for any positive integers r, and m, there exists n so that for any coloring $\rho : [1,n]^2 \to [1,r]$, there exists $M \in [1,n]^m$ so that $[M]^2$ is monochromatic. Erdős [4] asked whether the following natural generalization of both Ramsey's theorem and Schur's theorem holds: If G is a triangle-free graph on vertex set $\mathbb{N}$, does there always exist an independent Schur triple, that is, do there exist x, y, $x \neq y$ so that $\mathrm{FS}(x,y) = \{x, y, x+y\}$ is independent in G? The answer is yes, as proved in [13] where it was shown that in fact, for fixed k and d, if G is a K_k-free graph on $\mathbb{N}$, then there exist distinct integers $a_1, a_2, \ldots, a_d$, so that the finite sum set $\mathrm{FS}(\{a_1, \ldots, a_d\})$ is an independent set in G. Harborth *et al.* (see, e.g., [1,11]) have given some sharp lower bounds on n so that if G is a graph on $[1,n]$, these results hold (except the a_i need not be distinct).

Related progress was also made for an infinite version of Erdős' question. Given a set $\{x_i\}_{i \in I}$ of distinct positive integers, let $\mathrm{FS}(\{x_i\}_{i \in I}) = \left\{\sum_{j \in J} x_j : \emptyset \neq J \subseteq I, |J| < \infty\right\}$ denote the finite sums (with no repetitions) from the set. When I is infinite, we say that $\mathrm{FS}(\{x_i\}_{i \in I})$ is a *Hindman set*. In 1995, Hajnal asked the following (see [5]): If G is a triangle-free graph on $\mathbb{N}$, does there always exist a Hindman set independent in G? Hajnal's question has been answered in the negative in [3]. Variants of Hajnal's question have been shown to indeed have a positive answer; for example, if the condition "triangle-free" is replaced by "$K_{k,k}$-free" (see [3,9,13]).

A common generalization of Ramsey's theorem and van der Waerden's theorem was also found in [9]: For fixed k and ℓ, if G is a K_k-free graph on $\mathbb{N}$, then there exists an ℓ-term arithmetic progression which spans an independent set in G.

Coloring theorems for arithmetic progressions or finite sums have abstract analogues (the Hales–Jewett theorem and the Graham–Rothschild theorem, respectively; see, e.g., [7] or [14]), from which they can be deduced instantly. In contrast, Deuber's theorem for partitioning (m, p, c)-sets cannot be accomplished by any one application of such a theorem; several iterations are required. As with Deuber's theorem, one would not expect to be able to prove our main result for (m, p, c)-sets with any single application of an abstract theorem. Indeed, the first situation where a single process does not seem to work is for an arithmetic progression together with its difference—which is the simplest kind of subset of an (m, p, c)-set not necessarily contained in any one row. In [9] it was proved that for any $k, \ell \geqslant 3$, in any K_k-free graph, there exists an ℓ-term arithmetic progression together with its difference, all contained in an independent set. This proof used a form of the Gallai–Witt theorem applied iteratively; it does not seem to follow from one application of any of the major abstract theorems (like the Hales–Jewett or Graham–Rothschild theorems).

3. Preliminary results

We now briefly describe one of our main tools, the Hales–Jewett theorem.

Let A denote a finite alphabet; write $A^s = \{(x_1, \dots, x_s) : x_i \in A\}$. Let $[1, s] = F \cup M_1 \cup \cdots \cup M_t$ be a partition with $|M_j| > 0$ for $j = 1, 2, \dots, t$ and let $(g_i)_{i \in F} \in A^F$ be a fixed $|F|$-tuple. A *t-dimensional subcube* of A^s (associated with $(g_j)_{j \in F}$ and the partition $F \cup M_1 \cup \cdots \cup M_t$) is a set of the form

$$\mathrm{HJC}(F, M_1, \dots, M_t, (g_j)_{j \in F}) = \{(x_1, \dots, x_s) : x_j = g_j \text{ for } j \in F \text{ and } x_j = x_{j'} \text{ if } j, j' \in M_\alpha \text{ for some } \alpha\}.$$

We now state the central theorem regarding parameter sets.

Theorem 3.1 (Hales–Jewett). *For every $t, r \in \mathbb{N}$ and every finite alphabet A, there exists $s = HJ(t, r, |A|)$ so that for every coloring $\rho : A^s \to [1, r]$, there exists a monochromatic t-dimensional subcube of A^s.*

The original version in [10] yields a one-dimensional subcube. That version easily implies the current version (see [7, p. 40]). See [14] for a survey of results, applications, and notation for parameter words, another language to describe the Hales–Jewett theorem.

Definition 3.2. Let $p, c, q, d \in \mathbb{N}$ with $c \leqslant p$ and $q \leqslant d$, let $U = (x_0, x_1, \dots, x_m)_{p,c}$ be an (m, p, c)-set, and let $V = (y_0, y_1, \dots, y_n)_{q,d}$ be an (n, q, d)-set. We say that U is naturally contained in V, written $U \preccurlyeq V$ if and only if there is a strictly increasing function

$$\psi : \{0, 1, \dots, m\} \to \{0, 1, \dots, n\}$$

such that for each $i \in \{0, 1, \dots, m\}$, $R_i(U) \subseteq R_{\psi(i)}(V)$.

Notice that natural containment is trivially transitive.

Example 3.3. The $(1,3,1)$-set $A=(10,1)_{3,1}$ is contained in the $(2,2,1)$-set $B=(20,5,1)_{2,1}$ but A is not naturally contained in B. In fact, $R_0(A)$ is not contained in any row of B.

We now present some simple results that guarantee that all inclusions with which we shall be concerned are natural.

For any $p,m\in\mathbb{N}$, we use

$$[a_0,a_1,\dots,a_m]_p=\{\lambda_0a_0+\lambda_1a_1+\cdots+\lambda_ma_m:\lambda_0,\lambda_1,\dots,\lambda_m\in[-p,p]\},$$

to denote the span of $a_0,a_1,\dots,a_nm$.

Lemma 3.4. *Let $m,p,x_0,x_1,\dots,x_m\in\mathbb{N}$. If for each $i\in\{0,1,\dots,m-1\}$, $x_i>2p\sum_{j=i+1}^m x_j$, then expressions in $[x_0,x_1,\dots,x_m]_p$ are unique. That is, if*

$$\lambda_0,\lambda_1,\dots,\lambda_m,\mu_0,\mu_1,\dots,\mu_m\in[-p,p]\quad\text{and}\quad\sum_{i=1}^m\lambda_ix_i=\sum_{i=1}^m\mu_ix_i,$$

then for each $i\in\{0,1,\dots,m\}$, $\lambda_i=\mu_i$.

Proof. Assume that $\lambda_0,\lambda_1,\dots,\lambda_m,\mu_0,\mu_1,\dots,\mu_m\in[-p,p]$ and $\sum_{i=1}^m\lambda_ix_i=\sum_{i=1}^m\mu_ix_i$. Suppose that there is some $i\in\{0,1,\dots,m\}$ such that $\lambda_i\neq\mu_i$ and pick the first such i. Assume without loss of generality that $\lambda_i>\mu_i$. If $\lambda_mx_m=\mu_mx_m$, then $\lambda_m=\mu_m$, so we have that $i<m$. Then

$$\sum_{j=i+1}^m(\mu_j-\lambda_j)x_j=(\lambda_i-\mu_i)x_i\geqslant x_i>2p\sum_{j=i+1}^m x_j\geqslant\sum_{j=i+1}^m(\mu_j-\lambda_j)x_j,$$

a contradiction. □

With a little more work one can show that $6pc$ can be replaced by $5pc$ in the following lemma. (Here and later, if $i=m$, then we set $\sum_{j=i+1}^m x_j=0$.)

Lemma 3.5. *Let $m,p,c\in\mathbb{N}$ with $c\leqslant p$ and assume that $x_0,x_1,\dots,x_m\in\mathbb{N}$ and $(x_0,x_1,\dots,x_m)_{6pc,c}\subseteq\mathbb{N}$. Let $U=(x_0,x_1,\dots,x_m)_{p,c}$. For each $i\in\{0,1,\dots,m-1\}$, $x_i>6p\sum_{j=i+1}^m x_j$ and $\min R_i(U)>\max R_{i+1}(U)$. Also, any length 3 arithmetic progression in U is contained in some row of U.*

Proof. Let $i\in\{0,1,\dots,m-1\}$. Then $cx_i-\sum_{j=i+1}^m 6pcx_j\in\mathbb{N}$ so $x_i>6p\sum_{j=i+1}^m x_j$. Thus

$$\begin{aligned}\min R_i(U)&=cx_i-\sum_{j=i+1}^m px_j\geqslant x_i-\sum_{j=i+1}^m px_j>\sum_{j=i+1}^m px_j\geqslant cx_{i+1}\\&\quad+\sum_{j=i+2}^m px_j=\max R_{i+1}(U).\end{aligned}$$

Now assume we have $d>0$ such that $\{a, a+d, a+2d\} \subseteq U$. Pick $i, k \in \{0, 1, \ldots, m\}$ and $\lambda_{i+1}, \lambda_{i+2}, \ldots, \lambda_m, \mu_{k+1}, \mu_{k+2}, \ldots, \mu_m \in [-p, p]$ such that

$$a = cx_i + \sum_{j=i+1}^{m} \lambda_j x_j$$

and

$$a + d = cx_k + \sum_{j=k+1}^{m} \mu_j x_j.$$

Since $a + d > a$, we have that $k \leqslant i$. Suppose that $k < i$. Then

$$d = cx_k + \sum_{j=k+1}^{i-1} \mu_j x_j + (\mu_i - c)x_i + \sum_{j=i+1}^{m} (\mu_j - \lambda_j)x_j$$

and so

$$a + 2d = 2cx_k + \sum_{j=k+1}^{i-1} 2\mu_j x_j + (2\mu_i - c)x_i + \sum_{j=i+1}^{m} (2\mu_j - \lambda_j)x_j.$$

Since the absolute value of each coefficient in the expansion of $a + 2d$ is at most $3p$, we have by Lemma 3.4 that $a + 2d \notin U$.

Thus $k = i$ and

$$d = \sum_{j=i+1}^{m} (\mu_j - \lambda_j)x_j$$

and so

$$a + 2d = cx_i + \sum_{j=i+1}^{m} (2\mu_j - \lambda_j)x_j.$$

Again by Lemma 3.4 we have that $a + 2d \in R_i(U)$. $\square$

Lemma 3.6. *Let $n, q, d \in \mathbb{N}$ with $d \leqslant q$ and assume that $x_0, x_1, \ldots, x_n \in \mathbb{N}$ and $(x_0, x_1, \ldots, x_n)_{6qd,d} \subseteq \mathbb{N}$. If $m \in \omega$, $p, c \in \mathbb{N}$ with $c \leqslant p$, $y_0, y_1, \ldots, y_m \in \mathbb{N}$, and $Y = (y_0, y_1, \ldots, y_m)_{p,c} \subseteq (x_0, x_1, \ldots, x_n)_{q,d} = X$, then for each $i \in \{0, 1, \ldots, m\}$ there exists $j \in \{0, 1, \ldots, n\}$ such that $R_i(Y) \subseteq R_j(X)$.*

Proof. We proceed by induction on m. The case $m = 0$ is trivial, so assume that $m \in \mathbb{N}$ and the assertion is true for $m - 1$. Let $i \in \{0, 1, \ldots, m\}$. If $i > 0$, then $R_i(Y) = R_{i-1}((y_1, y_2, \ldots, y_m)_{p,c})$ so the conclusion holds by the induction hypothesis. So assume that $i = 0$. Let $D = R_0((y_0, y_1, \ldots, y_{m-1})_{p,c})$ and pick by the induction hypothesis some $j \in \{0, 1, \ldots, n\}$ such that $D \subseteq R_j(X)$. Now let $\lambda_1, \lambda_2, \ldots, \lambda_m \in [-p, p]$. If $\lambda_m = 0$, then $cy_0 + \sum_{l=1}^{m} \lambda_l y_l \in D \subseteq R_j(X)$, so assume $\lambda_m \neq 0$. Then $E = \{cy_0 + \sum_{l=1}^{m-1} \lambda_l y_l - \lambda_m y_m, cy_0 + \sum_{l=1}^{m-1} \lambda_l y_l, cy_0 + \sum_{l=1}^{m-1} \lambda_l y_l + \lambda_m y_m\}$ is a three term arithmetic progression in X so is contained $R_k(X)$ for some k by Lemma 3.5. Since $D \cap E \neq \emptyset$, $k = j$. $\square$

Lemma 3.7. *Let $n, q, d \in \mathbb{N}$ with $d \leqslant q$ and assume that $x_0, x_1, \ldots, x_n \in \mathbb{N}$ and $(x_0, x_1, \ldots, x_n)_{6qd,d} \subseteq \mathbb{N}$. If $m \in \omega$, $p, c \in \mathbb{N}$ with $c \leqslant p$, $y_0, y_1, \ldots, y_m \in \mathbb{N}$, and $Y = (y_0, y_1, \ldots, y_m)_{p,c} \subseteq (x_0, x_1, \ldots, x_n)_{q,d} = X$, then $Y \preccurlyeq X$.*

Proof. Let $i \in \{0, 1, \ldots, m-1\}$ and pick by Lemma 3.6 $j, k \in \{0, 1, \ldots, n\}$ such that $R_i(Y) \subseteq R_j(X)$ and $R_{i+1}(Y) \subseteq R_k(X)$. We show that $j < k$. Pick $\lambda_{j+1}, \lambda_{j+2}, \ldots, \lambda_n, \mu_{j+1}, \mu_{j+2}, \ldots, \mu_n, \gamma_{k+1}, \gamma_{k+2}, \ldots, \gamma_n \in [-q, q]$ such that

$$cy_i = dx_j + \sum_{l=j+1}^{n} \lambda_l x_l,$$

$$cy_i + cy_{i+1} = dx_j + \sum_{l=j+1}^{n} \mu_l x_l,$$

and

$$cy_{i+1} = dx_k + \sum_{l=k+1}^{n} \gamma_l x_l.$$

Then $cy_{i+1} = \sum_{l=j+1}^{n} (\mu_l - \lambda_l) x_l$, so by Lemma 3.4 $k \geqslant j + 1$. $\square$

We shall refer later to the conclusion of the following theorem by stating that "all inclusions in $(x_0, x_1, \ldots, x_n)_{q,d}$ are natural".

Theorem 3.8. *Let $n, q, d \in \mathbb{N}$ with $d \leqslant q$ and assume that $x_0, x_1, \ldots, x_n \in \mathbb{N}$ and $(x_0, x_1, \ldots, x_n)_{6qd,d} \subseteq \mathbb{N}$. If $m, M \in \omega$, $p, c, P, C \in \mathbb{N}$ with $c \leqslant p$ and $C \leqslant P$, $y_0, y_1, \ldots, y_m, z_0, z_1, \ldots, z_M \in \mathbb{N}$, $Y = (y_0, y_1, \ldots, y_m)_{p,c} \subseteq (x_0, x_1, \ldots, x_n)_{q,d} = X$, $Z = (z_0, z_1, \ldots, z_M)_{P,C} \subseteq X$, and $Y \subseteq Z$, then $Y \preccurlyeq Z$.*

Proof. By Lemma 3.7 we have that $Y \preccurlyeq X$ and $Z \preccurlyeq X$. We show first that for each $i \in \{0, 1, \ldots, m\}$ there exists $j \in \{0, 1, \ldots, M\}$ such that $R_i(Y) \subseteq R_j(Z)$. Suppose instead that one has $i \in \{0, 1, \ldots, m\}$, $a, b \in R_i(Y)$, and $j < k$ in $\{0, 1, \ldots, M\}$ such that $a \in R_j(Z)$ and $b \in R_k(Z)$. Since $Z \preccurlyeq X$, we have $u < v$ in $\{0, 1, \ldots, n\}$ such that $a \in R_u(X)$ and $b \in R_v(X)$. By Lemma 3.5, $R_u(X) \cap R_v(X) = \emptyset$ and by Lemma 3.6, $R_i(Y) \subseteq R_u(X)$, a contradiction.

Now let $i \in \{0, 1, \ldots, m-1\}$ and pick $j, k \in \{0, 1, \ldots, M\}$ such that $R_i(Y) \subseteq R_j(Z)$ and $R_{i+1}(Y) \subseteq R_k(Z)$. Pick $u < v$ in $\{0, 1, \ldots, n\}$ such that $R_i(Y) \subseteq R_u(X)$ and $R_{i+1}(Y) \subseteq R_v(X)$. Then $R_j(Z) \cap R_u(X) \neq \emptyset$ so $R_j(Z) \subseteq R_u(X)$. Likewise $R_k(Z) \subseteq R_v(X)$, and therefore $j < k$. $\square$

We shall need a slightly strengthened version of Deuber's Theorem (Theorem 1.3).

Theorem 3.9. *For every $m \in \omega$, every $p, c \in \mathbb{N}$ with $c \leqslant p$ and every $k \in \mathbb{N}$, there exists $n, q, d \in \mathbb{N}$ with $d \leqslant q$ so that for every (n, q, d)-set X and every coloring $\rho : X \to [1, k]$, there exists a monochromatic (m, p, c)-set naturally contained in X.*

Proof. Pick n', q', d' as guaranteed by Theorem 1.3. Let $n = n'$, $q = 6q'd'$, and $d = d'$. Let $X = (x_0, x_1, \ldots, x_n)_{q,d}$ be k-colored. Then $Y = (x_0, x_1, \ldots, x_{n'})_{q',d'}$ is naturally contained in X and by Theorem 1.3 Y contains a monochromatic (m, p, c)-set. By Lemma 3.7, this inclusion is natural. □

The following technical lemma completes our preliminaries.

Lemma 3.10. *Let $m, p, c, M, P, C \in \mathbb{N}$ with $c \leqslant p$ and $C \leqslant P$. Let $w_1, w_2, \ldots, w_m, v_1, v_2, \ldots, v_M \in \mathbb{N}$. If $(w_1, w_2, \ldots, w_m)_{p,c} \subseteq (cv_1, cv_2, \ldots, cv_M)_{P,C}$, then $[w_1, w_2, \ldots, w_m]_p \subseteq [v_1, v_2, \ldots, v_M]_{(2c+p)P}$.*

Proof. First consider any $\sum_{j=2}^{m} \lambda_j w_j$ with each $\lambda_j \in [-p, p]$. Then

$$cw_1 + \sum_{j=2}^{m} \lambda_j w_j \in (w_1, w_2, \ldots, w_m)_{p,c} \subseteq (cv_1, cv_2, \ldots, cv_M)_{P,C} \subseteq [cv_1, cv_2, \ldots, cv_M]_P$$

and $cw_1 \in [cv_1, cv_2, \ldots, cv_M]_P$, so $\sum_{j=2}^{m} \lambda_j w_j \in [cv_1, cv_2, \ldots, cv_M]_{2P} \subseteq [v_1, v_2, \ldots, v_M]_{2cP}$. Also $cw_1 \in [cv_1, cv_2, \ldots, cv_M]_P$ so $w_1 \in [v_1, v_2, \ldots, v_M]_P$, so for any $\lambda_1 \in [-p, p]$ one has $\lambda_1 w_1 \in [v_1, v_2, \ldots, v_M]_{pP}$ and thus $\sum_{j=1}^{m} \lambda_j w_j \in [v_1, v_2, \ldots, v_M]_{pP+2cP}$. □

4. Main proof: existence of independent (m, p, c)-sets

In the proof of Theorem 1.4, we use the following earlier result.

Theorem 4.1. *For every $k, n, q, d \in \mathbb{N}$ there exist $n', q', d' \in \mathbb{N}$ so that for any (n', q', d')-set X and any K_k-free graph G with vertex set X, there exists an (n, q, d)-set S naturally contained in X, each of whose rows is an independent set in G.*

The proof of Theorem 4.1 (see [8]) is accomplished by repeating the standard parameter sets proof of Deuber's partition theorem (see, e.g., [12]), once one knows that in a K_k-free graph on a large-dimensional Hales–Jewett cube there is always a line (or, more generally, a d-dimensional subspace) that is independent—this latter fact is proved in [9].

In view of Theorem 4.1, it is sufficient to prove Theorem 1.4 under the assumption that the graph G on an (n, q, d)-set S has all rows as independent sets. Rather than prove Theorem 1.4 with this additional assumption, we will prove a stronger statement, Theorem 4.3, below. Since a large complete k-partite graph contains many copies of K_k, Theorem 1.4 will clearly follow. This somewhat stronger theorem turns out to be easier to prove.

Definition 4.2. Let $k,p,c,t\in\mathbb{N}$ with $c\leqslant p$ and let $m\in\omega$. Then $\varphi(k,m,p,c,t)$ is the statement "there exist $n,q,d\in\mathbb{N}$ such that whenever S is an (n,q,d)-set and G is a graph on S such that the rows of S are independent, there exist either

(a) an independent (m,p,c)-set contained in S or
(b) $z_1,z_2,\dots,z_k,a_0,a_1,\dots,a_t\in\mathbb{N}$ such that $(a_0,a_1,\dots,a_t)_{p,c}\subseteq S$ and the sets $\langle cz_i+[a_0,a_1,\dots,a_t]_p\rangle_{i=1}^{k}$ form a complete k-partite subgraph of G."

Theorem 4.3. *For all $k,p,c,t\in\mathbb{N}$ with $c\leqslant p$ and all $m\in\omega$, the statement $\varphi(k,m,p,c,t)$ holds.*

Proof. The proof is by induction on m and k.

For the base cases, note that for all m',p,c,t, part (b) of $\varphi(1,m',p,c,t)$ holds, and for all k',p,c,t, part (a) of $\varphi(k',0,p,c,t)$ holds.

So assume that $k\geqslant 2$ and $m\geqslant 1$, and for the induction hypotheses, suppose that for all m',p,c,t, statement $\varphi(k-1,m',p,c,t)$ holds, and for all p,c,t, statement $\varphi(k,m-1,p,c,t)$ holds. We need to show that $\varphi(k,m,p,c,t)$ holds for all p,c,t. Pick p,c,t such that part (a) of $\varphi(k,m,p,c,t)$ fails; we show that part (b) holds.

1. Let (M,P,C) be the (n,q,d) guaranteed by $\varphi(k,m-1,p,c,t)$.
2. Let (N,Q,D) be such that whenever an $(N-1,Q,D)$-set is $(k-1)$-colored, it naturally contains a monochromatic $(M,(p+2c)P,cC)$-set. (Such (N,Q,D) exist by Theorem 3.9.) We may assume that $N>m$.
3. Set $Q'=(2cC+(2c+p)P)Q$, $D'=cCD$, and let
$$T=\mathrm{HJ}(t+1,(2Q'+1)^{Nk},(4pQ'+1)^{N+1})+N.$$
4. Put $Q''=2cp(Q')^2(N+1)$, $D''=c^2(D')^2$, and let (n',q',d') satisfy $\varphi(k-1,N,Q'',D'',T)$.
5. Pick (n,q,d) such that any (n,q,d)-set S' naturally contains an (n',q',d')-set for which all inclusions are natural. (Such (n,q,d) exists by Theorem 3.8.)

Since part (a) of $\varphi(k,m,p,c,t)$ fails, pick an (n,q,d)-set S' and a graph G' on S' for which the rows are independent but S' does not contain an independent (m,p,c)-set.

Pick an (n',q',d')-set S which is naturally contained in S' and such that all inclusions within S are natural. Let G be the subgraph of G' induced on S. Note that S does not contain an independent (m,p,c)-set.

Claim 1. *If $U=(v_0,v_1,\dots,v_{M+1})_{(p+2c)P,cC}\subseteq S$, then there is an edge of G between a point in the first row of U and a point in some later row of U.*

Proof. We have $(cv_1, cv_2, \ldots, cv_{M+1})_{P,C} \subseteq U \subseteq S$ so by the choice of (M, P, C), pick an independent $(m-1, p, c)$-set $(w_1, w_2, \ldots, w_m)_{p,c} \subseteq (cv_1, cv_2, \ldots, cv_{M+1})_{P,C}$. (If clause (b) of the definition of $\varphi(k, m-1, p, c, t)$ applied, one would have $\varphi(k, m, p, c, t)$.)

Let $V = (Cv_0, w_1, w_2, \ldots, w_m)_{p,c}$. One has immediately that

$$\bigcup_{i=1}^{m} R_i(V) = (w_1, w_2, \ldots, w_m)_{p,c} \subseteq (cv_1, cv_2, \ldots, cv_{M+1})_{P,C} \subseteq \bigcup_{i=1}^{M+1} R_i(U) \subseteq S.$$

Also, $R_0(V) = cCv_0 + [w_1, w_2, \ldots, w_m]_p \subseteq cCv_0 + [v_1, v_2, \ldots, v_m]_{(2c+p)P} = R_0(U)$, where the inclusion holds by Lemma 3.10.

Now $R_0(V)$ is contained in a row of S, so is independent, and

$$\bigcup_{i=1}^{m} R_i(V) = (w_1, w_2, \ldots, w_m)_{p,c},$$

which is independent, so, since V is not independent, there must be an edge between a point of $R_0(V)$ and a later row of V and hence between a point of $R_0(U)$ and a later row of U. $\square$

Claim 2. *Let $w_1, w_2, \ldots, w_N \in \mathbb{N}$. Recall that $Q' = (2cC + (2c+p)P)Q$ and $D' = cCD$. Assume that for each $i \in \{1, 2, \ldots, k-1\}$, $\beta_i \in \mathbb{N}$ and $(\beta_i, w_1, w_2, \ldots, w_N)_{Q',D'} \subseteq S$. Then there is some $x \in (cCw_1, cCw_2, \ldots, cCw_N)_{Q,D}$ (and therefore some $x \in (w_1, w_2, \ldots, w_N)_{Q',D'}$) such that for each $i \in \{1, 2, \ldots, k-1\}$ there is an edge from x to a point in the first row of $(\beta_i, w_1, w_2, \ldots, w_N)_{Q',D'}$.*

Proof. Suppose not and color $x \in (cCw_1, cCw_2, \ldots, cCw_N)_{Q,D}$ by the first $i \in \{1, 2, \ldots, k-1\}$ such that there is no edge from x to a point in the first row of $(\beta_i, w_1, w_2, \ldots, w_N)_{Q',D'}$. By the choice of (N, Q, D) pick $i \in \{1, 2, \ldots, k-1\}$ and $v_0, v_1, \ldots, v_m$ such that $(v_0, v_1, \ldots, v_M)_{(p+2c)P,cC} \subseteq (cCw_1, cCw_2, \ldots, cCw_N)_{Q,D}$ and for each $x \in (v_0, v_1, \ldots, v_M)_{(p+2c)P,cC}$ there is no edge from x to any point in the first row of $(\beta_i, w_1, w_2, \ldots, w_N)_{Q',D'}$.

Let $U = (D\beta_i, v_0, v_1, \ldots, v_M)_{(p+2c)P,cC}$. Now

$$(v_0, v_1, \ldots, v_M)_{(p+2c)P,cC} \subseteq (w_1, w_2, \ldots, w_N)_{cCQ,cCD}.$$

We claim that $R_0(U) \subseteq R_0((\beta_i, w_1, w_2, \ldots, w_N)_{Q',D'})$. To see this, let $y = cCD\beta_i + \sum_{l=0}^{M} \lambda_l v_l$ where each $\lambda_l \in [-(p+2c)P, (p+2c)P]$. By Lemma 3.10,

$$\sum_{l=0}^{M} \lambda_l v_l \in [w_1, w_2, \ldots, w_N]_{(2cC+(p+2c)P)Q},$$

so $y \in R_0((\beta_i, w_1, w_2, \ldots, w_N)_{Q',D'})$ as claimed.

Thus by Claim 1, there is an edge between a point $y \in R_0(U)$ and some point x in a later row of U. But then $x \in (cCw_1, cCw_2, \ldots, cCw_N)_{Q,D}$ and $y \in R_0((\beta_i, w_1, w_2, \ldots, w_N)_{Q',D'})$, a contradiction. $\square$

We now observe that there is no independent (N, Q'', D'')-set in S. Indeed, assume one has $(x_0, x_1, \ldots, x_N)_{Q'',D''} \subseteq S$. Then, since $m < N$ and $pc(D')^2 < Q''$, one has that

$$(c(D')^2x_0, c(D')^2x_1, \ldots, c(D')^2x_m)_{p,c} \subseteq (x_0, x_1, \ldots, x_N)_{Q'',D''}.$$

Since $(c(D')^2x_0, c(D')^2x_1, \ldots, c(D')^2x_m)_{p,c}$ is not independent, neither is $(x_0, x_1, \ldots, x_N)_{Q'',D''}$.

By the choice of (n', q', d'), since there is no independent (N, Q'', D'')-set in S, pick $b_0, b_1, \ldots, b_T, z_1', z_2', \ldots, z_{k-1}'$ in $\mathbb{N}$ such that $(b_0, b_1, \ldots, b_T)_{Q'',D''} \subseteq S$ and the sets

$$\langle D''z_i' + [b_0, b_1, \ldots, b_T]_{Q''} \rangle_{i=1}^{k-1}$$

form a complete $(k-1)$-partite graph.

Now let $y_0, y_1, \ldots, y_N \in [b_{N+1}, b_{N+2}, \ldots, b_T]_{2pQ'}$ and $i \in \{1, 2, \ldots, k-1\}$, and for $j \in \{0, 1, \ldots, N\}$, pick $\lambda_{j,N+1}, \lambda_{j,N+2}, \ldots, \lambda_{j,T} \in [-2pQ', 2pQ']$ such that $y_j = \sum_{l=N+1}^{T} \lambda_{j,l}b_l$. We claim that the first row of $(c^2D'z_i' + c^2D'b_0 + cy_0, c^2D'b_1 + cy_1, c^2D'b_2 + cy_2, \ldots, c^2D'b_N + cy_N)_{Q',D'}$ is contained in $D''z_i' + [b_0, b_1, \ldots, b_T]_{Q''}$. To see this, let $\mu_1, \mu_2, \ldots, \mu_N \in [-Q', Q']$, so that

$$w = c^2(D')^2z_i' + c^2(D')^2b_0 + cD'y_0 + \sum_{j=1}^{N} c^2D'\mu_jb_j + \sum_{j=1}^{N} c\mu_jy_j$$

is a typical member of the first row of $(c^2D'z_i' + c^2D'b_0 + cy_0, c^2D'b_1 + cy_1, c^2D'b_2 + cy_2, \ldots, c^2D'b_N + cy_N)_{Q',D'}$. For each $j \in \{0, 1, \ldots, N\}$, the absolute value of the coefficient of b_j in the given expansion of w is at most $c^2D'Q' < Q''$. And for $l \in \{N+1, N+2, \ldots, T\}$, the absolute value of the coefficient of b_l in the given expansion of w is

$$|cD'\lambda_{0,l} + \sum_{j=1}^{N} c\mu_j\lambda_{j,l}| \leqslant cD'2pQ' + 2pc(Q')^2N < Q''.$$

Next, we claim that $(c^2D'b_1 + cy_1, c^2D'b_2 + cy_2, \ldots, c^2D'b_N + cy_N)_{Q',D'}$ is contained in $(b_0, b_1, \ldots, b_T)_{Q'',D''}$. To see this, let $\mu_1, \mu_2, \ldots, \mu_N \in [-Q', Q']$ such that,

$$\text{if } r = \min\{j \in \{1, 2, \ldots, N\}: \mu_j \neq 0\}, \text{ then } \mu_r = D',$$

and let

$$w = \sum_{j=1}^{N} \mu_j(c^2D'b_j + cy_j) = \sum_{j=1}^{N} \mu_jc^2D'b_j + \sum_{l=N+1}^{T} \sum_{j=1}^{N} c\mu_j\lambda_{j,l}b_l.$$

Then $\mu_rc^2D' = D''$, and for $j \in \{r+1, r+2, \ldots, N\}$, if any, the absolute value of the coefficient of b_j in the given expansion of w is at most $c^2D'Q' < Q''$. Also, for

$l \in \{N+1, N+2, \dots, T\}$, the absolute value of the coefficient of b_l in the given expansion of w is at most $\sum_{j=r}^{N} c2p(Q')^2 < Q''$.

In particular, we have established that

$$(c^2D'z_i{}' + c^2D'b_0 + cy_0, c^2D'b_1 + cy_1, c^2D'b_2 + cy_2, \dots, c^2D'b_N + cy_N)_{Q',D'} \subseteq S,$$

so we may apply Claim 2.

We define $\tau : ([-2pQ', 2pQ']^{N+1})^{T-N} \to ([-Q', Q']^N)^k$ as follows: Let

$$\bar{\lambda} = ((\lambda_{0,N+1}, \lambda_{1,N+1}, \dots, \lambda_{N,N+1}), (\lambda_{0,N+2}, \lambda_{1,N+2}, \dots, \lambda_{N,N+2}), \\ \dots, (\lambda_{0,T}, \lambda_{1,T}, \dots, \lambda_{N,T})) \in ([-2pQ', 2pQ']^{N+1})^{T-N}.$$

For $j \in \{0, 1, \dots, N\}$, let $y_j = \sum_{l=N+1}^{T} \lambda_{j,l} b_l$. Then by Claim 2 applied to

$$\langle (c^2D'z_i{}' + c^2D'b_0 + cy_0, c^2D'b_1 + cy_1, c^2D'b_2 + cy_2, \dots, c^2D'b_N + cy_N)_{Q',D'} \rangle_{i=1}^{k-1}$$

there is some point in $(c^2D'b_1 + cy_1, c^2D'b_2 + cy_2, \dots, c^2D'b_N + cy_N)_{Q',D'}$ with an edge to a point in the first row of each $(c^2D'z_i{}' + c^2D'b_0 + cy_0, c^2D'b_1 + cy_1, c^2D'b_2 + cy_2, \dots, c^2D'b_N + cy_N)_{Q',D'}$. That is there is some

$$\bar{\gamma} = ((\gamma_{1,1}, \gamma_{1,2}, \dots, \gamma_{1,N}), (\gamma_{2,1}, \gamma_{2,2}, \dots, \gamma_{2,N}), \dots, (\gamma_{k,1}, \gamma_{k,2}, \dots, \gamma_{k,N})) \in ([-Q, Q]^N)^k$$

such that, if $r = \min\{j \in \{1, 2, \dots, n\} : \gamma_{k,j} \neq 0\}$, then $\gamma_{k,r} = D'$ and for each $i \in \{1, 2, \dots, k-1\}$, there is an edge between $\sum_{j=1}^{N} \gamma_{k,j}(c^2D'b_j + \sum_{l=N+1}^{T} c\lambda_{j,l}b_l)$ and

$$c^2(D')^2z_i{}' + c^2(D')^2b_0 + \sum_{l=N+1}^{T} D'c\lambda_{0,l}b_l + \sum_{j=1}^{N} \gamma_{i,j}\left(c^2D'b_j + \sum_{l=N+1}^{T} c\lambda_{j,l}b_l\right).$$

Define $\tau(\bar{\lambda}) = \bar{\gamma}$.

Now since $T = HJ(t+1, (2Q'+1)^{Nk}, (4pQ'+1)^{N+1}) + N$, Pick $F, M_0, M_1, \dots, M_t$, $\langle (v_{0,l}, v_{1,l}, \dots, v_{N,l}) \rangle_{l \in F}$,

$$\bar{\eta} = ((\eta_{1,1}, \eta_{1,2}, \dots, \eta_{1,N}), (\eta_{2,1}, \eta_{2,2}, \dots, \eta_{2,N}), \dots, (\eta_{k,1}, \eta_{k,2}, \dots, \eta_{k,N})) \in ([-Q, Q]^N)^k,$$

and $r \in \{1, 2, \dots, N\}$ such that

(1) $F, M_0, M_1, \dots, M_t$ are pairwise disjoint;
(2) $F \cup M_0 \cup M_1 \cup \dots \cup M_t = \{N+1, N+2, \dots, T\}$;
(3) each $M_s \neq \emptyset$ and $\min M_0 < \min M_1 < \cdots < \min M_t$;
(4) for each $l \in F$, $(v_{0,l}, v_{1,l}, \dots, v_{N,l}) \in [-2pQ', 2pQ']^{N+1}$;
(5) $r = \min\{j \in \{1, 2, \dots, N\} : \eta_{k,j} \neq 0\}$ and $\eta_{k,r} = D'$; and,
(6) whenever

$$\bar{\lambda} = ((\lambda_{0,N+1}, \lambda_{1,N+1}, \dots, \lambda_{N,N+1}), (\lambda_{0,N+2}, \lambda_{1,N+2}, \dots, \lambda_{N,N+2}), \\ \dots, (\lambda_{0,T}, \lambda_{1,T}, \dots, \lambda_{N,T})) \in ([-2pQ', 2pQ']^{N+1})^{T-N}$$

satisfies
(a) for each $l \in F$, $(\lambda_{0,l}, \lambda_{1,l}, \dots, \lambda_{N,l}) = (v_{0,l}, v_{1,l}, \dots, v_{N,l})$ and

(b) for each $s \in \{0, 1, \ldots, t\}$ and each $l, v \in M_s$, $(\lambda_{0,l}, \lambda_{1,l}, \ldots, \lambda_{N,l}) = (\lambda_{0,v}, \lambda_{1,v}, \ldots, \lambda_{N,v})$ one has $\tau(\bar{\lambda}) = \bar{\eta}$, and consequently, for each $i \in \{1, 2, \ldots, k-1\}$, there is an edge between $\sum_{j=1}^{N} \eta_{k,j}(c^2 D' b_j + \sum_{l=N+1}^{T} c\lambda_{j,l} b_l)$ and

$$c^2(D')^2 z_i' + c^2(D')^2 b_0 + \sum_{l=N+1}^{T} D' c\lambda_{0,l} b_l + \sum_{j=1}^{N} \eta_{i,j}(c^2 D' b_j + \sum_{l=N+1}^{T} c\lambda_{j,l} b_l).$$

Now for each $s \in \{0, 1, \ldots, t\}$, let $a_s = \sum_{l \in M_s} (D')^2 c b_l$. For $i \in \{1, 2, \ldots, k-1\}$, let

$$z_i = (D')^2 c z_i' + (D')^2 c b_0 + \sum_{l \in F} D' v_{0,l} b_l + \sum_{j=1}^{N} D' c \eta_{i,j} b_j + \sum_{l \in F} \sum_{j=1}^{N} \eta_{i,j} v_{j,l} b_l,$$

and let

$$z_k = \sum_{j=1}^{N} D' c \eta_{k,j} b_j + \sum_{l \in F} \sum_{j=1}^{N} \eta_{k,j} v_{j,l} b_l.$$

We shall show that $(a_0, a_1, \ldots, a_t)_{p,c} \subseteq S$ and that the sets $\langle cz_i + [a_0, a_1, \ldots, a_t]_p \rangle_{i=1}^{k}$ form a complete k-partite subgraph of G, completing the proof that part (b) of $\varphi(k, m, p, c, t)$ holds.

We show first that $(a_0, a_1, \ldots, a_t)_{p,c} \subseteq (b_0, b_1, \ldots, b_T)_{Q'', D''}$. So let $x \in (a_0, a_1, \ldots, a_t)_{p,c}$, and pick $\beta \in \{0, 1, \ldots, t\}$ and $\mu_{\beta+1}, \mu_{\beta+2}, \ldots, \mu_t \in [-p, p]$ such that

$$x = ca_\beta + \sum_{s=\beta+1}^{t} \mu_s a_s.$$

Then

$$x = \sum_{l \in M_\beta} c^2 (D')^2 b_l + \sum_{s=\beta+1}^{t} \sum_{l \in M_s} \mu_s c (D')^2 b_l.$$

Since $\min M_\beta < \min \bigcup_{s=\beta+1}^{t} M_s$, we have that the leading coefficient in this expansion is D'', while all other coefficients are at most $pc(D')^2 < Q''$.

Next we show that for each $i \in \{1, 2, \ldots, k-1\}$,

$$cz_i + [a_0, a_1, \ldots, a_t]_p \subseteq D'' z_i' + [b_0, b_1, \ldots, b_T]_{Q''},$$

and consequently the sets $\langle cz_i + [a_0, a_1, \ldots, a_t]_p \rangle_{i=1}^{k-1}$ form a complete $(k-1)$-partite graph. To this end, let $\mu_0, \mu_1, \ldots, \mu_t \in [-p, p]$. Then

$$cz_i + \sum_{s=0}^{t} \mu_s a_s = (D')^2 c^2 z_i' + (D')^2 c^2 b_0 + \sum_{l \in F} cDv_{0,l} b_l + \sum_{j=1}^{N} D' c^2 \eta_{i,j} b_j + \sum_{l \in F} \sum_{j=1}^{N} c\eta_{i,j} v_{j,l} b_l + \sum_{s=0}^{t} \sum_{l \in M_s} \mu_s (D')^2 c b_l.$$

The coefficient on $z_i{}'$ in this expansion is D'' while the coefficients on the b_l's have absolute value at most $2pc(Q')^2 < Q''$.

Finally we let $i \in \{1, 2, \dots, k-1\}$, let $w \in cz_i + [a_0, a_1, \dots, a_t]_p$, let $x \in cz_k + [a_0, a_1, \dots, a_t]_p$, and show that there is an edge between w and x. Pick $\alpha_0, \alpha_1, \dots, \alpha_s, \delta_0, \delta_1, \dots, \delta_s \in [-p, p]$ such that $w = cz_i + \sum_{s=0}^{t} \alpha_s a_s$ and $x = cz_k + \sum_{s=0}^{t} \delta_s a_s$. Then

$$w = (D')^2 c^2 z_i{}' + (D')^2 c^2 b_0 + \sum_{l \in F} cD' v_{0,l} b_l + \sum_{j=1}^{N} D' c^2 \eta_{i,j} b_j + \sum_{l \in F} \sum_{j=1}^{N} c\eta_{i,j} v_{j,l} b_l + \sum_{s=0}^{t} \sum_{l \in M_s} \alpha_s (D')^2 c b_l$$

and

$$x = \sum_{j=1}^{N} D' c^2 \eta_{k,j} b_j + \sum_{l \in F} \sum_{j=1}^{N} c\eta_{k,j} v_{j,l} b_l + \sum_{s=0}^{t} \sum_{l \in M_s} \delta_s (D')^2 c b_l.$$

For $l \in F$ and $j \in \{0, 1, \dots, N\}$, let $\lambda_{j,l} = v_{j,l}$. For $s \in \{0, 1, \dots, t\}$ and $l \in M_s$, let $\lambda_{r,l} = D'\delta_s$, let $\lambda_{0,l} = D'\alpha_s - \eta_{i,r}\delta_s$, and for $j \in \{1, 2, \dots, N\} \setminus \{r\}$, let $\lambda_{j,l} = 0$. Note that each $|\lambda_{j,l}| \leqslant 2pQ'$. Note also that for $s \in \{0, 1, \dots, t\}$ and $l \in M_s$,

$$\sum_{j=1}^{N} \eta_{k,j} \lambda_{j,l} = (D')^2 \delta_s$$

and

$$D' c \lambda_{0,l} + \sum_{j=1}^{N} \eta_{i,j} \lambda_{j,l} c = (D')^2 c \alpha_s.$$

Then

$$\bar{\lambda} = ((\lambda_{0,N+1}, \dots, \lambda_{N,N+1}), (\lambda_{0,N+2}, \dots, \lambda_{N,N+2}), \dots, (\lambda_{0,T}, \dots, \lambda_{N,T})) \in ([-2pQ', 2pQ']^{N+1})^{T-N}$$

satisfies

(a) for each $l \in F$, $(\lambda_{0,l}, \lambda_{1,l}, \dots, \lambda_{N,l}) = (v_{0,l}, v_{1,l}, \dots, v_{N,l})$ and
(b) for each $s \in \{0, 1, \dots, t\}$ and each $l, v \in M_s$, $(\lambda_{0,l}, \lambda_{1,l}, \dots, \lambda_{N,l}) = (\lambda_{0,v}, \lambda_{1,v}, \dots, \lambda_{N,v})$.

Consequently, there is an edge between $\sum_{j=1}^{N} \eta_{k,j}(c^2 D' b_j + \sum_{l=N+1}^{T} c\lambda_{j,l} b_l)$ and

$$c^2 (D')^2 z_i{}' + c^2 (D')^2 b_0 + \sum_{l=N+1}^{T} D' c \lambda_{0,l} b_l + \sum_{j=1}^{N} \eta_{i,j}(c^2 D' b_j + \sum_{l=N+1}^{T} c\lambda_{j,l} b_l).$$

Now

$$\begin{aligned}
&\sum_{j=1}^{N} \eta_{k,j}(c^2 D' b_j + \sum_{l=N+1}^{T} c\lambda_{j,l} b_l) \\
&= \sum_{j=1}^{N} \eta_{k,j} c^2 D' b_j + \sum_{l\in F} \sum_{j=1}^{N} \eta_{k,j} v_{j,l} c b_l + \sum_{s=0}^{t} \sum_{l\in M_s} \sum_{j=1}^{N} \eta_{k,j} \lambda_{j,l} c b_l \\
&= \sum_{j=1}^{N} \eta_{k,j} c^2 D' b_j + \sum_{l\in F} \sum_{j=1}^{N} \eta_{k,j} v_{j,l} c b_l + \sum_{s=0}^{t} \sum_{l\in M_s} c(D')^2 \delta_s b_l \\
&= x.
\end{aligned}$$

Also,

$$\begin{aligned}
&c^2(D')^2 z_i' + c^2(D')^2 b_0 + \sum_{l=N+1}^{T} D' c\lambda_{0,l} b_l + \sum_{j=1}^{N} \eta_{i,j}(c^2 D' b_j + \sum_{l=N+1}^{T} c\lambda_{j,l} b_l) \\
&= c^2(D')^2 z_i' + c^2(D')^2 b_0 + \sum_{j=1}^{N} \eta_{i,j} c^2 D' b_j + \sum_{l\in F} (D' c\lambda_{0,l} + \sum_{j=1}^{N} \eta_{i,j} \lambda_{j,l} c) b_l \\
&\quad + \sum_{s=0}^{t} \sum_{l\in M_s} (D' c\lambda_{0,l} + \sum_{j=1}^{N} \eta_{i,j} \lambda_{j,l} c) b_l \\
&= c^2(D')^2 z_i' + c^2(D')^2 b_0 + \sum_{j=1}^{N} \eta_{i,j} c^2 D' b_j + \sum_{l\in F} (D' c v_{0,l} + \sum_{j=1}^{N} \eta_{i,j} v_{j,l} c) b_l \\
&\quad + \sum_{s=0}^{t} \sum_{l\in M_s} (D')^2 c\alpha_s b_l \\
&= w. \qquad \square
\end{aligned}$$

5. Independent arithmetic progressions, revisited

The following (in a more general form) was originally proved in [9] by application of the Hales–Jewett theorem. To illustrate a different approach in a special case, we now give a different proof, this time using Szemerédi's density theorem for arithmetic progressions [18] (which says that for any m and $\varepsilon > 0$ there exists an n so that any set of εn elements from $[1, n]$ contains an m-term arithmetic progression).

Theorem 5.1 (Gunderson et al. [9]). *Fix k and ℓ. If G is a K_k-free graph on $\mathbb{N}$, then there exists an ℓ-term arithmetic progression which spans an independent set in G.*

Proof. Denote by $S(k, \ell)$ the following statement: There exists an integer $n = n(k, \ell)$ such that for every K_k-free graph G whose vertex set is an arithmetic progression of length n, there exists an arithmetic progression of length ℓ which is an independent set in G.

If for every k and ℓ, $S(k,\ell)$ holds, then Theorem 5.1 follows. For each fixed ℓ we will prove $S(k,\ell)$ by induction on k.

Observe that $S(2,\ell)$ is trivially true with $n(2,\ell)=\ell$. Suppose, therefore, that $S(k-1,\ell)$ holds and set $n^*=n(k-1,\ell)$. Let $n=n(k,\ell)$ be very large and consider a K_k-free graph G with vertex set $\{a,a+d,\dots,a+(n-1)d\}$. Assume that G contains no independent set which is an arithmetic progression of length ℓ. Also, observe that $V(G)$ contains

$$(n-(\ell-1))+(n-2(\ell-1))+\cdots+\left(n-\left\lfloor\frac{n-1}{\ell-1}\right\rfloor(\ell-1)\right)\geqslant\frac{n^2}{3\ell}$$

arithmetic progressions of length ℓ. Since each of these arithmetic progressions contains an edge and each edge is contained in at most $\binom{\ell}{2}$ arithmetic progressions of length ℓ the graph G contains at least

$$\frac{n^2}{3\ell}\binom{\ell}{2}^{-1}\geqslant\frac{2n^2}{3\ell^3}$$

edges. This means that there exists a vertex x joined to at least $\frac{4n}{3\ell^3}$ other vertices. Now if n is sufficiently large compared to n^*, we may infer by Szemerédi's theorem that the neighborhood of x contains an arithmetic progression Y of length n^*. Since $\{x,y\}\in E(G)$ for every $y\in Y$ the subgraph $G[Y]$ of G induced by Y does not contain K_{k-1}. Thus, by the induction assumption $S(k-1,\ell)$, the set Y contains an arithmetic progression of length ℓ which is an independent set in $G[Y]$ and hence also in G. $\square$

We note that since an early draft of this paper, J. Solymosi (personal communication) has independently observed a similar proof of Theorem 5.1.

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References

[1] J.P. Bode, H.D. Gronau, H. Harborth, Some Ramsey Schur numbers, Manuscript, 2002.
[2] W. Deuber, Partitionen und lineare Gleichungssysteme, Math. Z. 133 (1973) 109–123.
[3] W. Deuber, D.S. Gunderson, N. Hindman, D. Strauss, Independent finite sums for K_m-free graphs, J. Combin. Theory Ser. A 78 (1997) 171–198.
[4] P. Erdős, personal communication with DSG, CERN, Luminy, Marseilles, France, September 4, 1995.

[5] P. Erdős, A. Hajnal, J. Pach, On a metric generalization of Ramsey's theorem, Israel J. Math. 101 (1997) 283–295.
[6] P. Frankl, R.L. Graham, V. Rödl, Quantitative theorems for regular systems of equations, J. Combin. Theory Ser. A 47 (1988) 246–261.
[7] R.L. Graham, B.L. Rothschild, J. Spencer, Ramsey Theory, 2nd Edition, Wiley, New York, 1990.
[8] D.S. Gunderson, On Deuber's partition theorem for (m, p, c)-sets, Ars Combin. 63 (2002) 15–31.
[9] D.S. Gunderson, I. Leader, H.J. Prömel, V. Rödl, Independent arithmetic progressions in clique-free graphs on the natural numbers, J. Combin. Theory Ser. A 93 (2001) 1–17.
[10] A.W. Hales, R.I. Jewett, Regularity and positional games, Trans. Amer. Math. Soc. 106 (1963) 222–229.
[11] H. Harborth, Triangle-free labelled graphs and a Schur property, in: A. Sali, M. Simonovits, V.T. Sós (Eds.), Paul Erdős and his mathematics, Budapest, Janos Bolyai Mathematical Society, Budapest, July 4–11, 1999, pp. 102–103.
[12] K. Leeb, A full Ramsey theorem for the Deuber category, in: A. Hajnal, R. Rado, V. S'os (Eds.) Infinite and Finite Sets II, Colloqium Mathematical Society János Bolyai, Vol. 10, North-Holland, Amsterdam, 1975, pp. 1043–1049.
[13] T. Łuczak, V. Rödl, T. Schoen, Independent finite sums in graphs defined on the natural numbers, Discrete Math. 181 (1998) 289–294.
[14] H.J. Prömel, B. Voigt, Graham–Rothschild parameter sets, in: J. Nešetřil, V. Rödl (Eds.), Mathematics of Ramsey Theory, Springer, Berlin, 1990, pp. 113–149.
[15] R. Rado, Studien zur Kombinatorik, Math. Z. 36 (1933) 424–480.
[16] F.P. Ramsey, On a problem of formal logic, Proc. London Math. Soc. 30 (1930) 264–286.
[17] I. Schur, Über die Kongruenz $x^m + y^m \equiv z^m \pmod p$, Jber. Deutsch. Math. Verein 25 (1916) 114–116.
[18] E. Szemerédi, On sets of integers no k elements in arithmetic progressions, Acta Arith. 27 (1975) 199–245.
[19] B.L. van der Waerden, Beweis einer Baudetschen Vermutung, Nieuw. Arch. Wisk. 15 (1927) 212–216.