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Deformation of Dijkgraaf–Vafa relation via spontaneously broken $\mathcal{N}=2$ supersymmetry

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Abstract

It is known that the fermionic shift symmetry of the $\mathcal{N}=1$, $U(N)$ gauge model with a superpotential of an adjoint chiral superfield is replaced by the second (spontaneously broken) supersymmetry in the $\mathcal{N}=2$, $U(N)$ gauge model with a prepotential and Fayet–Iliopoulos parameters. Based on a diagrammatic analysis, we demonstrate how the well-known form of the effective superpotential in the former model is modified in the latter. A set of two equations on the one-point functions stating the Konishi anomaly is modified accordingly.

1. Introduction

For more than two decades, effective superpotential has been a central object in the nonperturbative study of $\mathcal{N}=1$ supersymmetric theories. This object is protected from perturbative corrections in the conventional sense [1], and yet receives important nonperturbative corrections (see for example [2,3]). In recent years, analyses from superstring theory have revealed an interesting perturbative window into nonperturbative physics with the use of the gluino condensate superfield variable [4–7]. In [8], field theoretic discussion based on the model with $U(N)$ gauge group and rigid $\mathcal{N}=1$ supersymmetry (see Eq. (2.2) for its action $S_{\mathcal{N}=1}$) is given and this is in accord with the string theory based developments.

Superstring theory, on the other hand, insists upon maximally extended supersymmetry with no adjustable parameter. A scenario that one may draw is that this extended supersymmetry becomes spontaneously broken to $\mathcal{N}=1$. Along this vein, a field theory model with $U(N)$ gauge group and rigid $\mathcal{N}=2$ supersymmetry spontaneously broken to $\mathcal{N}=1$ has been introduced in [9–11] (see Eq. (2.1) for its action $S_{\mathcal{N}=2}$), generalizing the Abelian counterpart of [12]. (See also [13] for $\mathcal{N}=2$ supergravity and [14] for related discussions.) Several properties of this model have been derived.

In this Letter, we make a first analysis on the interplay between the effective superpotential and partially as well as spontaneously broken $\mathcal{N}=2$ supersymmetry, shedding a light upon the comparison of the two models mentioned above. A key aspect of this comparison is that the fermionic shift symmetry of $S_{\mathcal{N}=1}$ gets replaced by the second (spontaneously broken) supersymmetry of $S_{\mathcal{N}=2}$. In fact, this is one of the original motivations/results of [9].

The fermionic shift symmetry of $S_{\mathcal{N}=1}$ supplies the well-known formula [7,8] constraining the form of the effective superpotential, which is originally proposed from flux compactification of string theory [15,16]. Based on a diagrammatic analysis [17] (for a review see [18]), we are able to state how this form undergoes modifications in the model $S_{\mathcal{N}=2}$. After giving a few accounts of the model in the next section, we present a diagrammatic analysis of W_{eff} in Section 3. Our final understanding is summarized in Eq. (3.10). This is followed by a computation of the two-loop contribution to W_{eff} in Section 4. In the final section, we derive a set

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of two equations on the two generating functions $R(z)$ and $T(z)$ of the one-point functions, generalizing the argument based on the chiral ring and the Konishi anomaly in [8]. We observe a modification from that given in [8] here as well.

2. The $U(N)$ gauged model with spontaneously broken $\mathcal{N}=2$ supersymmetry

Let us briefly recall a few ingredients of the model, which are needed in what follows. The action [9] given in the Wess–Zumino gauge can be written as

$$S_{\mathcal{N}=2} = \int d^4x\, d^4\theta \left[-\frac{i}{2} \operatorname{Tr}\left(\bar{\Phi} e^{adV} \frac{\partial \mathcal{F}(\Phi)}{\partial \Phi} - \text{h.c.} \right) + \xi V^0 \right]$$
$$+ \left[\int d^4x\, d^2\theta \left(-\frac{i}{4} \frac{\partial^2 \mathcal{F}(\Phi)}{\partial \Phi^a \partial \Phi^b} \mathcal{W}^a \mathcal{W}^b + e\Phi^0 + m \frac{\partial \mathcal{F}(\Phi)}{\partial \Phi^0} \right) + \text{h.c.} \right], \tag{2.1}$$

where $V = V^a t_a$ and $\mathcal{W}^\alpha$ are the vector superfield and the gauge superfield strength respectively and $\Phi = \Phi^a t_a$ ($a = 0, 1, \ldots, N^2 - 1$) is the chiral superfield.[1] There are three Fayet–Iliopoulos parameters (e, m, ξ) which are all real. For simplicity, we choose the prepotential as a single trace function of degree $n+2$: $\mathcal{F}(\Phi) = \sum_{k=1}^{n+1} g_k \operatorname{Tr} \Phi^{k+1}/(k+1)!$. While this action is shown to be invariant under the $\mathcal{N}=2$ supersymmetry transformations [9,10], the vacuum breaks half of the $\mathcal{N}=2$ supersymmetries. Extremizing the scalar potential, we obtain the condition $\langle \frac{\partial^2 \mathcal{F}}{\partial \Phi^0 \partial \Phi^0} \rangle = -(e \pm i\xi)/m$, which is a polynomial of order n and this determines the expectation value of the scalar field.

The action $S_{\mathcal{N}=2}$ in (2.1) is to be compared with that of the $\mathcal{N}=1$, $U(N)$ gauge model with a single trace tree level superpotential $W(\Phi)$:

$$S_{\mathcal{N}=1} = \int d^4x\, d^4\theta \operatorname{Tr} \bar{\Phi} e^{adV} \Phi + \left[\int d^4x\, d^2\theta \operatorname{Tr}\big(i\tau \mathcal{W}\mathcal{W} + W(\Phi) \big) + \text{h.c.} \right], \tag{2.2}$$

where τ is a complex gauge coupling $\tau = \theta/2\pi + 4\pi i/g^2$.

In [9], it is checked that the second supersymmetry reduces to the fermionic shift symmetry in the limit $e, m, \xi \to \infty$. The action $S_{\mathcal{N}=2}$ in fact reduces to $S_{\mathcal{N}=1}$ in the limit $e, m, \xi \to \infty$ with mg_k ($k \geqslant 2$) fixed [19]. We show that our result reduces to that of [7,17] in this limit.

3. Diagrammatic analysis of the effective superpotential

In this Letter, we consider the matter-induced part of the effective superpotential by integrating out the massive degrees of freedom Φ:

$$e^{i\int d^4x\, (d^2\theta\, W_{\text{eff}} + \text{h.c.} + d^4\theta (\text{nonchiral terms}))} = \int \mathcal{D}\Phi \mathcal{D}\bar{\Phi} e^{iS_{\mathcal{N}=2}}. \tag{3.1}$$

Let us take $\mathcal{W}^\alpha$ (or V) as the background field.[2] We consider the case of unbroken $U(N)$ gauge group. For simplicity, we choose $\langle \Phi \rangle = 0$ by setting $g_1 = -(e \pm i\xi)/m$.

We are interested in the holomorphic superpotential which does not contain the anti-holomorphic couplings $\bar{g}_k$. We can take $\bar{g}_k = 0$ for $k \geqslant 3$ without loss of generality. Collecting the $\bar{\Phi}$ dependent terms, we obtain

$$S_{\bar{\Phi}} = \int d^4x\, d^4\theta \frac{-i}{2} \operatorname{Tr}\left[\bar{\Phi} e^{adV} \frac{\partial \mathcal{F}(\Phi)}{\partial \Phi} - \left(\bar{g}_1 \bar{\Phi} + \frac{\bar{g}_2}{2} \bar{\Phi}^2 \right) e^{adV} \Phi \right] + \int d^4x\, d^2\bar{\theta} \frac{m\bar{g}_2}{2} \operatorname{Tr} \bar{\Phi}^2$$
$$= \int d^4x\, d^4\theta \operatorname{Tr}\left[\tilde{\Phi} \bar{g}_2 \left(-\frac{2m}{\nabla^2} + \frac{i}{4}\Phi \right) \tilde{\Phi} + \frac{i}{2} \left(\bar{g}_1 \Phi - \frac{\partial \mathcal{F}}{\partial \Phi} \right) \tilde{\Phi} \right]. \tag{3.2}$$

In the last expression, we have introduced a covariantly anti-chiral superfield $\tilde{\Phi} = \bar{\Phi} e^{adV}$, which satisfies $\nabla_\alpha \tilde{\Phi} = 0$ ($\nabla_\alpha = e^{-adV} D_\alpha e^{adV}$). Eq. (3.2) is quadratic in $\tilde{\Phi}$ and can be integrated straightforwardly. As a result, we obtain the following terms,

$$\frac{1}{16\bar{g}_2} \left(\bar{g}_1 \Phi - \frac{\partial \mathcal{F}}{\partial \Phi} \right) \left(-\frac{2m}{\nabla^2} + \frac{i}{4}\Phi \right)^{-1} \left(\bar{g}_1 \Phi - \frac{\partial \mathcal{F}}{\partial \Phi} \right) = \frac{(\operatorname{Im} g_1)^2}{8m\bar{g}_2} \Phi \nabla^2 \Phi + \cdots, \tag{3.3}$$

where $\cdots$ denotes the higher order interaction terms, which we will not consider here. Indeed, these interaction vertices are higher order in m^{-1} compared to the vertices which we consider below. These contribute to our main result (3.10) as higher order corrections in m^{-1} and do not spoil our conclusion that the effective superpotential is modified from the case of $S_{\mathcal{N}=1}$ (2.2).

1 $a = 0$ corresponds to the overall $U(1)$ part.

2 The simplest background is that consisting of a vanishing gauge field A_μ and a constant gaugino λ^α, which satisfies $\{\lambda^\alpha, \lambda^\beta\} = 0$ [18]. This configuration implies that traces of more than two $\mathcal{W}$ vanish.

Replacing $d^2\bar\theta$ integration by $-\bar\nabla^2/4$ and collecting the terms which are not in $S_{\bar\Phi}$, we obtain an action after the $\bar\Phi$ integration:

$$\int d^4x\, d^2\theta\, \mathrm{Tr}\Bigg[-\frac{(\mathrm{Im}\, g_1)^2}{32m\bar g_2}\Phi\bar\nabla^2\nabla^2\Phi + m\sum_{k=2}^{n+1}\frac{g_k}{k!}\Phi^k - \frac{i}{4}\sum_{k=3}^{n+1}\sum_{s=0}^{k-1}\frac{g_k}{k!}\big(\mathcal{W}\Phi^s\mathcal{W}\Phi^{k-1-s}\big)\Bigg]. \tag{3.4}$$

The first two terms are already present in the integrations with regard to the action $S_{\mathcal{N}=1}$ (2.2). The last term is new and originates from the gauge kinetic term in Eq. (2.1). As we will see below, this last term does contribute to the effective superpotential and becomes responsible for the violation of the well-known relation [7,8] between the effective superpotential of the gauge theory and the planar free energy of the matrix model having the tree level (bare) superpotential as its potential.

After rescaling $\Phi \to a\Phi$ with $a^2 = m\bar g_2/(\mathrm{Im}\, g_1)^2$, the quadratic part of the action (3.4) reduces to

$$\frac{1}{2}\Phi\left(-\Box + m' + \frac{1}{2}ad\mathcal{W}^\alpha D_\alpha\right)\Phi - \frac{ig_3'}{2}\big(2\mathcal{W}\mathcal{W}\Phi^2 + \mathcal{W}\Phi\mathcal{W}\Phi\big),$$

where we have used the relation $\bar\nabla^2\nabla^2\Phi = 16(\Box\Phi - ad\mathcal{W}^\alpha D_\alpha\Phi/2)$ and introduced $m' = a^2 m g_2$ and $g_3' = a^2 g_3/12$. The propagator in the momentum space is

$$\Delta(p,\pi) = \int_0^\infty ds\, e^{-s(p^2 + m' + \frac{1}{2}ad\mathcal{W}^\alpha\pi_\alpha - ig_3' M)}.$$

The Grassmann momentum π^α is Fourier transformation of superspace coordinate θ^α and the matrix M is

$$M_{abcd} = (\mathcal{W}\mathcal{W})_{da}\delta_{bc} + (\mathcal{W}\mathcal{W})_{bc}\delta_{da} + \mathcal{W}_{da}\mathcal{W}_{bc}, \tag{3.5}$$

where we have exhibited the gauge index dependence explicitly. This matrix is not present in the propagator of [17]. Using Eq. (3.5), we are able to insert $\mathcal{W}$ without involving the momentum π^α.

The interaction terms in Eq. (3.4) are divided into the following two types:

$$\text{type I:}\quad m\frac{g_k a^k}{k!}\mathrm{Tr}\,\Phi^k, \quad k = 3,\dots,n+1;$$

$$\text{type II:}\quad -\frac{i}{4}\sum_{s=0}^{k-1}\frac{g_k a^{k-1}}{k!}\mathrm{Tr}\big(\mathcal{W}\Phi^s\mathcal{W}\Phi^{k-1-s}\big), \quad k = 4,\dots,n+1.$$

Type I vertices are already present in [17]. Type II vertices are not present in [17]. They insert two $\mathcal{W}$ in specific ways.

Before going on to consider loop diagrams, let us first demonstrate that we have only to consider planar diagrams in our case as well [17,18]. For a given diagram, we denote by V the number of vertices, by P the number of propagators and by h the number of holes (or index loops). There are V sets of chiral superspace integrations from V vertices. One of them becomes the chiral superspace integration over the effective superpotential, and the number of remaining π^α momentum integrations is $P - V + 1$. These Grassmann integrations must be saturated by $\frac{1}{2}ad\mathcal{W}^\alpha\pi_\alpha$ terms in the propagators. Furthermore, we can freely insert $\mathcal{W}$ both from the M terms in the propagators and from the type II vertices. If we denote the number of these additional insertions by 2α, the total number of $\mathcal{W}$ insertions is $2(P - V + 1 + \alpha)$. On the other hand, one index loop can accommodate at most two $\mathcal{W}$. Thus we have $h \geqslant P - V + 1 + \alpha$. This implies that only the planar diagrams contribute to the effective superpotential as the Euler number of the diagram is $\chi = V - P + h$.

A planar diagram with h index loops has $(h-1)$ loop momenta. Let us consider the $(h-1)$-loop planar diagrams (contributing to the $(h-1)$-loop vacuum amplitude) in which all vertices are type I. Let us, for a moment, ignore the M term of (3.5). The calculation is then the same as that of [17] which we briefly describe. Each diagram is a product of the bosonic part obtained by integrating over the momentum p and the fermionic one coming from the π^α integrations. As we have seen in the last paragraph, we have exactly $2(h-1)\mathcal{W}$ insertions in the fermion part. There are two possibilities for these $\mathcal{W}$ insertions. The one is to keep one of the index loops empty, filling the remaining index loops with two $\mathcal{W}$. This yields NS^{h-1} term, where $S = -\frac{1}{64\pi^2}\mathrm{Tr}_{U(N)}\mathcal{W}^\alpha\mathcal{W}_\alpha$. The other is to fill each of two index loops chosen with single $\mathcal{W}$, which yields $S^{h-2}w^\alpha w_\alpha$ terms where $w^\alpha = \frac{1}{8\pi}\mathrm{Tr}\,\mathcal{W}^\alpha$. After calculating the both parts, we perform the Schwinger parameter integrals. Clearly this procedure is universal to every $(h-1)$-loop planar diagram up to the multiplications by the symmetric factor and by the coupling constants. Therefore every such diagram is a product of these factors with the following expression

$$\left(\prod_{i=1}^h\int ds_i\right)e^{-(\sum s_i)m'}\frac{1}{4^{h-1}}\big\{NhS^{h-1} + {}_hC_2 2S^{h-2}w^\alpha w_\alpha\big\} \equiv \left(\prod_{i=1}^h\int ds_i\right)e^{-(\sum s_i)m'}\mathcal{A}_0^{(h-1)}, \tag{3.6}$$

where we have introduced $\mathcal{A}_0^{(h-1)}$. The factor h of the first term comes from the choice of the empty index loop, and ${}_hC_2$ of the second term is the combination of inserting two $\mathcal{W}$ into different index loops. The most important fact is that the dependence on

Schwinger parameters of the bosonic part is canceled by that of the fermionic part. This explains that the calculation of the effective superpotential of the gauge theory reduces to that of the matrix model [17].

There are two types of corrections to $\mathcal{A}_0^{(h-1)}$. The one is due to the presence of the M terms in the propagators, which we denote by $\mathcal{A}_1^{(h-1)}$. The other is due to the type II vertices, which is obtained by replacing one of the type I vertices in $\mathcal{A}_0^{(h-1)}$ by the corresponding type II vertex and by summing over all possibilities. We denote this by $\mathcal{A}_2^{(h-1)}$. We consider them in order.

Let us see the effects of the M term, namely, Eq. (3.5). It plays a role of inserting two $\mathcal{W}$ further. Thus we will obtain terms which are proportional to S^h. Note that we cannot insert more than two $\mathcal{W}$ because, in such case, at least one of the index loops has more than two insertions of $\mathcal{W}$. For the parts contributing to NS^{h-1}, which have an empty index loop, we can further insert $\mathcal{W}^\alpha\mathcal{W}_\alpha$ from the first two terms in (3.5). In the case in which they are inserted in the ath index loop, we obtain $(\frac{S}{4})^{h-1} ig_3'(\sum_{i_a} s_{i_a}) \operatorname{Tr}\mathcal{W}\mathcal{W}$, where i_a labels the propagators which form the ath index loop. The absence of factor N is explained by the absence of an empty index loop. The factor h is not present as we have so far restricted ourselves to the ath index loop. Summing over all index loops, we obtain the first contribution to $\mathcal{A}_1^{(h-1)}$:

$$\sum_a \left(\frac{S}{4}\right)^{h-1} ig_3'\left(\sum_{i_a} s_{i_a}\right) \operatorname{Tr}\mathcal{W}\mathcal{W} = 2ig_3'\left(\sum_i s_i\right)\left(\frac{S}{4}\right)^{h-1} \operatorname{Tr}\mathcal{W}\mathcal{W},$$

where we have used that when all index loops are summed, they pass through each double line propagator exactly twice.

Let us note that the parts contributing to the second term of Eq. (3.6) can receive further insertions of $\mathcal{W}$ as well. They have two index loops with a single $\mathcal{W}$ insertion, for which we can exploit the last term of M. An insertion of this term requires that two index loops share a propagator. Let us define the index $A = 1, \ldots, {}_hC_2$ as labeling the combinations of such two index loops and the index $\tilde{A}$ labeling the cases which have a common propagator in the two index loops chosen. Let us further introduce the index $i_{\tilde{A}}$ labeling the common propagator in case $\tilde{A}$. With these notations, we obtain the second contribution to $\mathcal{A}_1^{(h-1)}$:

$$\frac{2S^{h-2}}{4^{h-1}} ig_3'\left(\sum_{i_{\tilde{A}}} s_{i_{\tilde{A}}}\right)\frac{1}{64\pi^2}\mathcal{W}_{ab}^\alpha\mathcal{W}_{\alpha cd}\mathcal{W}_{ba}^\beta\mathcal{W}_{\beta dc} = ig_3'\left(\sum_i s_i\right)\left(\frac{S}{4}\right)^{h-1} \operatorname{Tr}\mathcal{W}^\alpha\mathcal{W}_\alpha.$$

Putting all these together, we obtain the contributions from the vertices of type I,

$$\left(\prod_{i=1}^h \int ds_i\right) e^{-(\sum s_i)m'}\left(\mathcal{A}_0^{(h-1)} + \mathcal{A}_1^{(h-1)}(s_i)\right) = \frac{h}{m'}\left(\frac{S}{4m'}\right)^{h-1}\left(N - \frac{16\pi^2 ig_3 S}{mg_2}\right) + \frac{{}_hC_2}{2m'^2}\left(\frac{S}{4m'}\right)^{h-2} w^\alpha w_\alpha. \tag{3.7}$$

It is important that the above new term has Schwinger parameter dependence aside from the exponential factor. In [17], it was pointed out that the cancellation of this dependence represents the reduction of the system to the matrix model. The appearance of this new term with Schwinger parameter dependence may spoil this reduction. Note also that this new term does not have an overall factor N, indicating the violation of the well-known relation due to Dijkgraaf–Vafa [7].

We now turn to the vertices of type II which contain two $\mathcal{W}$ insertions. The ℓth order vertex in Φ is

$$\operatorname{Tr}\left(2\mathcal{W}\mathcal{W}\Phi^\ell + \mathcal{W}\Phi\mathcal{W}\Phi^{\ell-1} + \cdots + \mathcal{W}\Phi^{\ell-1}\mathcal{W}\Phi\right), \tag{3.8}$$

where we have omitted the overall factors. The first term inserts two $\mathcal{W}$ into an index loop while the remainder insert them into two different index loops. Having done $2(h-1)\pi^\alpha$ integrations, we obtain $2(h-1)\mathcal{W}$ insertions. We can therefore use vertex (3.8) only once in a diagram. When this is done, insertion of the M term from the propagator is disallowed.

Let us consider $\mathcal{A}_2^{(h-1)}$ and suppose that one of the type I vertices, $\operatorname{Tr}\Phi^\ell$, is replaced by the above vertex (3.8). The first term connects ℓ index loops and we can insert $\mathcal{W}^2$ into ℓ different ways. Thus we obtain $(\frac{S}{4})^{h-1} 2\ell \operatorname{Tr}\mathcal{W}\mathcal{W}$ as a contribution to $\mathcal{A}_2^{(h-1)}$. For the other terms of Eq. (3.8), there are in total $\ell(\ell-1)$ ways of inserting two $\mathcal{W}$ into different index loops. These give

$$\frac{2S^{h-2}}{4^{h-1}}\ell(\ell-1)\frac{1}{64\pi^2}\mathcal{W}_{ab}^\alpha\mathcal{W}_{\alpha cd}\mathcal{W}_{ba}^\beta\mathcal{W}_{\beta dc} = \left(\frac{S}{4}\right)^{h-1}\ell(\ell-1)\operatorname{Tr}\mathcal{W}\mathcal{W}.$$

Summing the above two contributions, we obtain $(\frac{S}{4})^{h-1}\ell(\ell+1)\operatorname{Tr}\mathcal{W}\mathcal{W}$. Thus, in any $(h-1)$-loop diagram, changing a vertex from type I to type II is equivalent to considering only NS^{h-1} terms in Eq. (3.6) and changing the coupling constant by

$$mg_\ell \to \frac{16\pi^2 ig_{\ell+1}S}{Nh}, \quad \text{for } \ell \geqslant 3. \tag{3.9}$$

Considering all planar diagrams, we obtain a formula for the $(h-1)$-loop contribution to W_{eff} in (2.1),

$$W_{\text{eff}}^{(h-1)} = N\frac{\partial F^{(h-1)}}{\partial S} + \frac{\partial^2 F^{(h-1)}}{\partial S^2} w^\alpha w_\alpha - \frac{16\pi^2 img_3}{mg_2}\left(\frac{\partial F^{(h-1)}}{\partial S}\right)\frac{S}{m} + W_2^{(h-1)}, \tag{3.10}$$

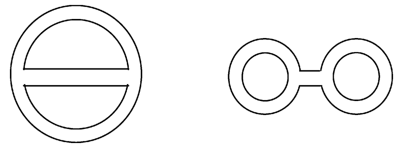

Fig. 1. Two-loop planar diagrams.

where $W_2^{(h-1)}$ is defined by replacing, in the first term, one coupling constant according to Eq. (3.9) and summing over all possibilities. We have denoted by $F^{(h-1)}$ the $(h-1)$-loop contribution to the planar free energy of the matrix model.

4. Example

As a sample computation, let us take the two-loop contribution to the effective superpotential. There are two two-loop planar diagrams depicted in Fig. 1. Collecting all possible insertions of $\mathcal{W}$, we obtain

$$W_{\mathrm{eff}}^{(2)} = -\frac{(mg_3)^2}{32(mg_2)^3} N S^2 - \frac{(mg_3)^2}{16(mg_2)^3} S w^\alpha w_\alpha + \frac{\pi^2 i (mg_3)^3}{2(mg_2)^4}\frac{S^3}{m} - \frac{\pi^2 i (mg_3)(mg_4)}{2(mg_2)^3}\frac{S^3}{m}. \tag{4.1}$$

The first two terms are the ones which are present in the computation based on [7,8] with $S_{\mathcal{N}=1}$. The third one comes from the M term in the propagator and the last one from the type II vertices. Note that, in the limit $m \to \infty$ with mg_k $(k \geqslant 2)$ fixed, we reproduce the result of [17]. In an arbitrary loop amplitude, the situation is the same: new terms are of order m^{-1} in this limit.

The overall $U(1)$ part does not decouple from the $SU(N)$ part. This can be easily seen by translating S into the glueball superfield $\hat{S} = -\frac{1}{64\pi^2} \mathrm{Tr}_{SU(N)} \mathcal{W}^\alpha \mathcal{W}_\alpha$ and extracting the factor in front of $w^\alpha w_\alpha$. By the existence of the last two terms in Eq. (3.10), it is nonvanishing. For example, in the two-loop example, this part in (4.1) reads

$$\frac{3\pi i (mg_3)[(mg_2)(mg_4) - (mg_3)^2]}{2(mg_2)^4}\frac{\hat{S}^2}{m} w^\alpha w_\alpha \neq 0.$$

5. The chiral ring and the generalized Konishi anomaly

An alternative approach to the effective superpotential is to exploit and extend the properties of the $\mathcal{N}=1$ chiral ring and the generalized Konishi anomaly equations based on reference [8,20]. The anomalous Ward identity of our model for the general transformation $\delta\Phi = f(\Phi, \mathcal{W})$ is

$$-\left\langle \frac{1}{64\pi^2}\left[\mathcal{W}^\alpha, \left[\mathcal{W}_\alpha, \frac{\partial f}{\partial \Phi_{ij}}\right]\right]_{ij}\right\rangle_\Phi = \left\langle \mathrm{Tr}\, f W'(\Phi)\right\rangle_\Phi - \left\langle \frac{i}{4}\mathrm{Tr}\big(f \mathcal{F}'''(\Phi)\mathcal{W}^\alpha \mathcal{W}_\alpha\big)\right\rangle_\Phi, \tag{5.1}$$

where $W''(\Phi) = m\mathcal{F}'''(\Phi)$. In terms of the two generating functions of chiral one-point functions

$$R(z) = -\frac{1}{64\pi^2}\left\langle \mathrm{Tr}\, \mathcal{W}^\alpha \mathcal{W}_\alpha \frac{1}{z-\Phi}\right\rangle_\Phi,$$

$$T(z) = \left\langle \mathrm{Tr}\, \frac{1}{z-\Phi}\right\rangle_\Phi,$$

the anomalous Ward identities (5.1) are

$$R(z)^2 = W'(z)R(z) + \frac{1}{4}f(z),$$

$$2R(z)T(z) = W'(z)T(z) + \frac{1}{4}c(z) + 16\pi^2 i \mathcal{F}'''(z)R(z) + \frac{1}{4}\tilde{c}(z),$$

where $f(z)$ and $c(z)$ are polynomials of degree $n-1$ in z and $\tilde{c}(z)$ is a polynomial of degree $n-2$:

$$f(z) = -\frac{1}{16\pi^2}\mathrm{Tr}\left\langle \frac{(W'(\Phi) - W'(z))\mathcal{W}^\alpha \mathcal{W}_\alpha}{z-\Phi}\right\rangle,$$

$$c(z) = 4\left\langle \frac{W'(\Phi) - W'(z)}{z-\Phi}\right\rangle,$$

$$\tilde{c}(z) = -i\left\langle \frac{(\mathcal{F}'''(\Phi) - \mathcal{F}'''(z))\mathcal{W}^\alpha \mathcal{W}_\alpha}{z-\Phi}\right\rangle.$$

The last term of Eq. (5.1) does not contribute to the equation for $R(z)$ because of the chiral ring relation $\operatorname{Tr}\mathcal{W}^\alpha\mathcal{W}_\alpha\mathcal{W}^\beta\mathcal{W}_\beta = 0$. The equation for $R(z)$ is the same as that of [8], which is the loop equation of the matrix model. On the other hand, the equation for $T(z)$ alters from that of [8].

The final step of this approach is to express the effective superpotential in terms of $R(z)$ and $T(z)$. Taking a variational derivative of (3.1) with respect to the coupling g_k, we obtain

$$\frac{\partial W_{\text{eff}}}{\partial g_k} = \frac{m}{k!}\int dz\, z^k T(z) + \frac{16\pi^2 i}{(k-1)!}\int dz\, z^{k-1} R(z).$$

Hence we can determine the effective superpotential up to g_k independent terms.

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References

[1] M.T. Grisaru, W. Siegel, M. Rocek, Nucl. Phys. B 159 (1979) 429.
[2] G. Veneziano, S. Yankielowicz, Phys. Lett. B 113 (1982) 231.
[3] I. Affleck, M. Dine, N. Seiberg, Phys. Rev. Lett. 51 (1983) 1026;
I. Affleck, M. Dine, N. Seiberg, Nucl. Phys. B 241 (1984) 493.
[4] C. Vafa, J. Math. Phys. 42 (2001) 2798, hep-th/0008142.
[5] F. Cachazo, K.A. Intriligator, C. Vafa, Nucl. Phys. B 603 (2001) 3, hep-th/0103067.
[6] F. Cachazo, C. Vafa, hep-th/0206017.
[7] R. Dijkgraaf, C. Vafa, Nucl. Phys. B 644 (2002) 3, hep-th/0206255;
R. Dijkgraaf, C. Vafa, Nucl. Phys. B 644 (2002) 21, hep-th/0207106;
R. Dijkgraaf, C. Vafa, hep-th/0208048.
[8] F. Cachazo, M.R. Douglas, N. Seiberg, E. Witten, JHEP 0212 (2002) 071, hep-th/0211170.
[9] K. Fujiwara, H. Itoyama, M. Sakaguchi, Prog. Theor. Phys. 113 (2005) 429, hep-th/0409060;
K. Fujiwara, H. Itoyama, M. Sakaguchi, hep-th/0410132.
[10] K. Fujiwara, H. Itoyama, M. Sakaguchi, Nucl. Phys. B 723 (2005) 33, hep-th/0503113;
K. Fujiwara, H. Itoyama, M. Sakaguchi, Prog. Theor. Phys. Suppl. 164 (2007) 125, hep-th/0602267.
[11] K. Fujiwara, H. Itoyama, M. Sakaguchi, Nucl. Phys. B 740 (2006) 58, hep-th/0510255;
K. Fujiwara, H. Itoyama, M. Sakaguchi, hep-th/0611284.
[12] I. Antoniadis, H. Partouche, T.R. Taylor, Phys. Lett. B 372 (1996) 83, hep-th/9512006;
I. Antoniadis, T.R. Taylor, Fortschr. Phys. 44 (1996) 487, hep-th/9604062;
H. Partouche, B. Pioline, Nucl. Phys. B (Proc. Suppl.) 56 (1997) 322, hep-th/9702115.
[13] S. Ferrara, L. Girardello, M. Porrati, Phys. Lett. B 366 (1996) 155, hep-th/9510074;
P. Fre, L. Girardello, I. Pesando, M. Trigiante, Nucl. Phys. B 493 (1997) 231, hep-th/9607032;
M. Porrati, Nucl. Phys. B (Proc. Suppl.) 55 (1997) 240, hep-th/9609073;
J. Louis, hep-th/0203138;
H. Itoyama, K. Maruyoshi, Int. J. Mod. Phys. A 21 (2006) 6191, hep-th/0603180;
K. Maruyoshi, hep-th/0607047.
[14] P. Kaste, H. Partouche, JHEP 0411 (2004) 033, hep-th/0409303;
P. Merlatti, Nucl. Phys. B 744 (2006) 207, hep-th/0511280;
L. Girardello, A. Mariotti, G. Tartaglino-Mazzucchelli, JHEP 0603 (2006) 104, hep-th/0601078.
[15] S. Gukov, C. Vafa, E. Witten, Nucl. Phys. B 584 (2000) 69, hep-th/9906070;
S. Gukov, C. Vafa, E. Witten, Nucl. Phys. B 608 (2001) 477, Erratum.
[16] T.R. Taylor, C. Vafa, Phys. Lett. B 474 (2000) 130, hep-th/9912152.
[17] R. Dijkgraaf, M.T. Grisaru, C.S. Lam, C. Vafa, D. Zanon, Phys. Lett. B 573 (2003) 138, hep-th/0211017.
[18] R. Argurio, G. Ferretti, R. Heise, Int. J. Mod. Phys. A 19 (2004) 2015, hep-th/0311066.
[19] K. Fujiwara, Nucl. Phys. B 770 (2007) 145, hep-th/0609039.
[20] K. Konishi, Phys. Lett. B 135 (1984) 439.