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The MST of symmetric disk graphs is light ☆

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ABSTRACT

Symmetric disk graphs are often used to model wireless communication networks. Given a set S of n points in $\mathbb{R}^d$ (representing n transceivers) and a transmission range assignment $r: S \to \mathbb{R}$, the *symmetric disk graph* of S (denoted $SDG(S)$) is the undirected graph over S whose set of edges is $E = \{(u, v) \mid r(u) \geqslant |uv| \text{ and } r(v) \geqslant |uv|\}$, where $|uv|$ denotes the Euclidean distance between points u and v. We prove that the weight of the MST of *any* connected symmetric disk graph over a set S of n points in the plane, is only $O(\log n)$ times the weight of the MST of the complete Euclidean graph over S. We then show that this bound is tight, even for points on a line.

Next, we prove that if the number of different ranges assigned to the points of S is only k, $k \ll n$, then the weight of the MST of $SDG(S)$ is at most $2k$ times the weight of the MST of the complete Euclidean graph. Moreover, in this case, the MST of $SDG(S)$ can be computed efficiently in time $O(kn \log n)$.

We also present two applications of our main theorem, including an alternative proof of the Gap Theorem, and a result concerning range assignment in wireless networks.

Finally, we show that in the non-symmetric model (where $E = \{(u, v) \mid r(u) \geqslant |uv|\}$), the weight of a minimum spanning subgraph might be as big as $\Omega(n)$ times the weight of the MST of the complete Euclidean graph.

1. Introduction

Symmetric disk graphs are often used to model wireless communication networks. Given a set S of n points in $\mathbb{R}^d$ (representing n transceivers) and a transmission range assignment $r: S \to \mathbb{R}$, the *symmetric disk graph* of S (denoted $SDG(S)$) is the undirected graph over S whose set of edges is $E = \{(u, v) \mid r(u) \geqslant |uv| \text{ and } r(v) \geqslant |uv|\}$, where $|uv|$ denotes the Euclidean distance between points u and v. If $r(u) \geqslant \mathrm{diam}(S)$, for each $u \in S$, then $SDG(S)$ is simply the complete Euclidean graph over S. However, usually, the transmission ranges are much shorter than $\mathrm{diam}(S)$.

The Minimum Spanning Tree (MST) of a connected Euclidean graph G is an extremely important substructure of G. In the context of wireless networks, the MST is especially important. Besides its role in various routing protocols, it is also used to obtain good approximations when the problem being considered is NP-hard; see, e.g., the power assignment problem.

It is usually impossible to use the Euclidean MST of S (denoted $MST(S)$), under the symmetric disk graph model, simply because some of the edges of $MST(S)$ are not present in $SDG(S)$. Instead, it is natural to use the MST of $SDG(S)$ (denoted $MST_{SDG(S)}$). However, it is still desirable to (tightly) bound the approximation ratio also with respect to the weight of $MST(S)$

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(and not only with respect to the weight of $MST_{SDG(S)}$). The main result of this paper makes this possible. We prove the following, somewhat surprising, theorem. For *any* set S of points in the plane and for *any* assignment of ranges to the points of S, such that $SDG(S)$ is connected, the weight (i.e., the sum of the edge lengths) of $MST_{SDG(S)}$ is $O(\log n)$ times the weight of $MST(S)$.

Disk graphs and especially unit disk graphs have received much attention, especially in the context of wireless networks. Notice that the unit disk graph of S is the symmetric disk graph of S that is obtained when $r(u) = 1$, for each point $u \in S$. The disk graph of S, on the other hand, is a directed graph, where there is an arc from u to v if $r(u) \geqslant |uv|$. Despite their importance, symmetric disk graphs have not received as much attention as (unit) disk graphs. Before describing our results concerning the MST of symmetric disk graphs, we mention two applications of our main result (stated above).

The Gap Theorem. The proof of our main result is based on a property of $MST_{SDG(S)}$ (see Lemma 2.1). This property also allows us to obtain an alternative and possibly simpler proof of the, so-called, Gap Theorem, stated and proved by Chandra et al. [4]. The Gap Theorem is used to show that the weight of the greedy spanner is $O(\log n)$ times the weight of $MST(S)$ [1,8].

Range assignment. A *range assignment* is an assignment of transmission ranges to each of the nodes of a network, so that the induced communication graph is connected and the total power consumption is minimized. The power consumed by a node v is $r(v)^\alpha$, where $r(v)$ is the range assigned to v and $\alpha \geqslant 1$ is some constant. The range assignment problem was first studied by Kirousis et al. [7], who did not impose any restriction on the potential transmission range of a node. They proved that the problem is NP-hard in three-dimensional space, assuming $\alpha = 2$. Subsequently, Clementi et al. [5] proved that the problem remains NP-hard in two-dimensional space. Kirousis et al. [7] also presented a simple 2-approximation algorithm, based on $MST(S)$.

It is more realistic to study the range assignment problem under the symmetric disk graph model. That is, the potential transmission range of a node u is bounded by some maximum range $r(u)$, and two nodes u, v can directly communicate with each other if and only if v lies within the range assigned to u and vice versa. The range assignment problem under this model was studied in [3,2]. Blough et al. [2] show that this version of the problem is also NP-hard in 2-dimensional and in 3-dimensional space. Our main theorem enables us, assuming $\alpha = 1$, to bound the weight of an optimal range assignment with limits on the ranges with respect to an optimal range assignment without such limits.

1.1. Our results

In this paper, we prove several results concerning the minimum spanning tree of symmetric disk graphs. In Section 2, we prove that the weight of the MST of *any* connected symmetric disk graph $SDG(S)$ is bounded by $O(\log n)$ times the weight of $MST(S)$. Or, in our notation, $wt(MST_{SDG(S)}) = O(\log n) \cdot wt(MST(S))$. We also show that this bound is tight, in the sense that there exists a symmetric disk graph, such that $wt(MST_{SDG(S)}) = \Omega(\log n) \cdot wt(MST(S))$. If the ratio between the maximum range and minimum range is bounded by some constant, then we show that $wt(MST_{SDG(S)}) = O(wt(MST(S)))$. In Section 3, we consider the common case where the number of different ranges is only k, for $k \ll n$. We prove that in this case $wt(MST_{SDG(S)}) \leqslant 2k \cdot wt(MST(S))$. Moreover, we present an algorithm for computing $MST_{SDG(S)}$ in this case in time $O(kn \log n)$. In Section 4, we discuss the two applications mentioned above. In particular, we provide an alternative proof of the Gap Theorem. In Section 5, we consider disk graphs. We prove that the weight of a minimum spanning subgraph of a disk graph is bounded by $O(n)$ times the weight of $MST(S)$, and give an example where this bound is tight.

2. Symmetric disk graphs

Given a set S of n points in the plane and a function $r : S \to \mathbb{R}$, the *symmetric disk graph* of S, denoted $SDG(S)$, is the undirected graph over S whose set of edges is $E = \{(u, v) \mid r(u) \geqslant |uv| \text{ and } r(v) \geqslant |uv|\}$, where $|uv|$ denotes the Euclidean distance between points u and v. The *weight*, $wt(e)$, of an edge $e = (u, v) \in E$ is $|uv|$, and the weight, $wt(E')$, of $E' \subseteq E$ is $\sum_{e \in E'} wt(e)$.

We denote by $MST_{SDG(S)}$ the minimum spanning tree of $SDG(S)$. In this section, we show that $wt(MST_{SDG(S)}) = \Theta(\log n) \cdot wt(MST(S))$, where $MST(S)$ is the Euclidean minimum spanning tree of S (i.e., the minimum spanning tree of the complete Euclidean graph over S). More precisely, we show that if $SDG(S)$ is connected, then $wt(MST_{SDG(S)}) = O(\log n) \cdot wt(MST(S))$, and that there exists a connected symmetric disk graph (over some set S of points) whose spanning tree's weight is $\Omega(\log n) \cdot wt(MST(S))$.

Lemma 2.1. *Let $SDG(S) = (S, E)$ be a symmetric disk graph over S. Let $(a, b), (c, d) \in E(MST_{SDG(S)})$ be two edges of $MST_{SDG(S)}$ that do not share an endpoint, such that $0 < |ab| \leqslant |cd|$. Then at most one edge from the set $A = \{(a, c), (b, c), (a, d), (b, d)\}$ is shorter than (a, b).*

Proof. Assume that there are two edges $e', e'' \in A$ that are shorter than (a, b). Since e' is shorter than (a, b) (and therefore also shorter than (c, d)), it belongs to $SDG(S)$. Similarly, e'' belongs to $SDG(S)$. Therefore the edges e', e'' together with (a, b), (c, d) contain a cycle in $SDG(S)$, implying that (a, b) or (c, d) is not in $E(MST_{SDG(S)})$ — a contradiction. □

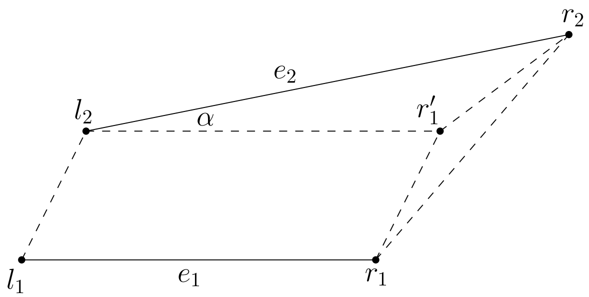

Fig. 1. Proof of Lemma 2.2.

Lemma 2.2. *Let $e_1 = (l_1, r_1)$, $e_2 = (l_2, r_2)$ be two edges of $MST_{SDG(S)}$, where r_i is to the right of l_i, $i = 1, 2$, such that* (i) $1 \leqslant \frac{|e_2|}{|e_1|} \leqslant \frac{5}{4}$, *and* (ii) *the difference α between the orientations of e_1 and e_2 is in the range* $[0, \frac{\pi}{9}]$. *Then* $|l_1 l_2| \geqslant \frac{1}{2}|e_1|$.

Proof. First, notice that it is impossible that $l_1 = l_2$. (Otherwise, the edge (r_1, r_2) is shorter than e_1 and therefore is present in $SDG(S)$, implying that e_2 is not an edge of $MST_{SDG(S)}$.) Moreover, assume that e_1 and e_2 do not share an endpoint, since, if they do (i.e., if $l_1 = r_2$, or $l_2 = r_1$, or $r_1 = r_2$), then the proof becomes much easier. Assume that $|l_1 l_2| < \frac{1}{2}|e_1|$. Let r_1' be the point to the right of l_2, such that (l_1, r_1) and (l_2, r_1') are parallel to each other and $|l_1 r_1| = |l_2 r_1'|$; see Fig. 1. By the triangle inequality,

$$|l_1 l_2| + |r_1' r_2| = |r_1 r_1'| + |r_1' r_2| \geqslant |r_1 r_2|.$$

Since $|e_1| \leqslant |e_2|$ and $|l_1 l_2| < |e_1|$, we know, by Lemma 2.1, that $|r_1 r_2| \geqslant |e_1|$. Thus, we get that

$$|r_1' r_2| \geqslant |e_1| - |l_1 l_2| > |e_1| - \frac{1}{2}|e_1| = \frac{1}{2}|e_1|.$$

By the law of cosines,

$$|r_1' r_2|^2 = |e_1|^2 + |e_2|^2 - 2|e_1||e_2|\cos(\alpha)$$

and therefore

$$|e_1|^2 + |e_2|^2 - 2|e_1||e_2|\cos(\alpha) > \frac{1}{4}|e_1|^2$$

or

$$\frac{3}{4} + \frac{|e_2|^2}{|e_1|^2} - 2\frac{|e_2|}{|e_1|}\cos(\alpha) > 0. \tag{1}$$

We now show that this is impossible. Replacing $\frac{|e_2|}{|e_1|}$ by x in (1), we get $x^2 - 2\cos(\alpha)x + 3/4 > 0$. The solutions of the equation $x^2 - 2\cos(\alpha)x + 3/4 = 0$ are $x_{1,2} = \cos(\alpha) \pm \sqrt{\cos^2(\alpha) - \frac{3}{4}}$. Notice that since $0 < \alpha \leqslant \frac{\pi}{9}$, we have $\cos(\alpha) > 37/40$, and therefore $x_1 > 5/4$ and $x_2 < 1$. Thus, for any x in the interval $[1, \frac{5}{4}]$, the left side of inequality (1) is non-positive. But this contradicts the assumption that $1 \leqslant \frac{|e_2|}{|e_1|} \leqslant 5/4$. We conclude that $|l_1 l_2| \geqslant \frac{1}{2}|e_1|$. □

We are ready to prove our main theorem.

Theorem 2.3 *(SDG Theorem).*

1. *The weight of the minimum spanning tree of a connected symmetric disk graph over a set S of n points in the plane is $O(\log n) \cdot wt(MST(S))$, where $MST(S)$ is the Euclidean minimum spanning tree of S.*
2. *There exists a set S of n points on a line, such that $wt(MST_{SDG(S)}) = \Omega(\log n) \cdot wt(MST(S))$.*

We prove the first part (i.e., the upper bound) in Section 2.1, and the second part (i.e., the lower bound) in Section 2.2.

2.1. Upper bound

Let $SDG(S) = (S, E)$ be a connected symmetric disk graph over a set S of n points in the plane. We prove that $wt(MST_{SDG(S)}) = O(\log n) \cdot wt(MST(S))$.

We partition the edge set of $MST_{SDG(S)}$ into two subsets. Let $E' = \{e \in MST_{SDG(S)} \mid |e| > wt(MST(S))/n\}$ and let $E'' = \{e \in MST_{SDG(S)} \mid |e| \leqslant wt(MST(S))/n\}$. Since $MST_{SDG(S)}$ has $n-1$ edges,

$$wt(E'') = \sum_{e \in E''} |e| \leqslant (n-1) \cdot \frac{wt(MST(S))}{n} < wt\bigl(MST(S)\bigr).$$

In order to bound $wt(E')$, we divide the edges of E' into $k \geqslant 9$ classes $\{C_1, \ldots, C_k\}$, according to their orientation (which is an angle in the range $(-\pi/2, \pi/2]$). Within each class we divide the edges into $O(\log n)$ buckets, according to their length. Specifically, for $1 \leqslant i \leqslant k$ and $1 \leqslant j \leqslant \log_p n$, where $p = 5/4$, let

$$B_{i,j} = \left\{ e \in E' \cap C_i \;\middle|\; |e| \in \left(\frac{wt(MST(S))}{n} \cdot p^{j-1}, \frac{wt(MST(S))}{n} \cdot p^j \right] \right\}.$$

(Notice that for each $e \in E'$, $|e| \leqslant \mathrm{diam}(S) \leqslant wt(MST(S))$.) Finally, let $S_{i,j} = \{s \in S \mid$ there exists a point t such that $(s,t) \in B_{i,j}\}$.

Let $s, s' \in S_{i,j}$, and let $t, t' \in S$ such that $e = (s,t)$ and $e' = (s', t')$ are edges in $B_{i,j}$. Since e, e' belong to the same class, the difference between their orientations is less than $\frac{\pi}{9}$, and since they also belong to the same bucket, we may apply Lemma 2.2 and obtain that $|ss'| \geqslant \frac{1}{2} \cdot \min\{|e|, |e'|\}$.

We now show that $wt(B_{i,j}) = O(wt(MST(S)))$. First notice that

$$wt\bigl(MST(S_{i,j})\bigr) \geqslant \bigl(|S_{i,j}| - 1\bigr) \cdot \min_{e \in MST(S_{i,j})} \bigl\{|e|\bigr\} \geqslant \frac{(|S_{i,j}| - 1)}{2} \cdot \min_{e \in B_{i,j}} \bigl\{|e|\bigr\},$$

and since for any $e_1, e_2 \in B_{i,j}$, $\min\{|e_1|, |e_2|\} \geqslant \frac{1}{p} \cdot \max\{|e_1|, |e_2|\}$ we get that

$$wt\bigl(MST(S_{i,j})\bigr) \geqslant \frac{(|S_{i,j}| - 1)}{2p} \cdot \max_{e \in B_{i,j}} \bigl\{|e|\bigr\} \geqslant \frac{1}{2p} \cdot \Bigl(wt(B_{i,j}) - \max_{e \in B_{i,j}} \bigl\{|e|\bigr\}\Bigr).$$

Rearranging,

$$\begin{aligned}
wt(B_{i,j}) &\leqslant 2p \cdot wt\bigl(MST(S_{i,j})\bigr) + \max_{e \in B_{i,j}} \bigl\{|e|\bigr\} \leqslant 2p \cdot wt\bigl(MST(S_{i,j})\bigr) + p \cdot \min_{e \in B_{i,j}} \bigl\{|e|\bigr\} \\
&\leqslant 2p \cdot wt\bigl(MST(S_{i,j})\bigr) + 2p \cdot \max_{e \in MST(S_{i,j})} \bigl\{|e|\bigr\} \leqslant 2p \cdot wt\bigl(MST(S_{i,j})\bigr) + 2p \cdot wt\bigl(MST(S_{i,j})\bigr) \\
&\leqslant 4p \cdot wt\bigl(MST(S_{i,j})\bigr).
\end{aligned}$$

Referring to $S \setminus S_{i,j}$ as Steiner points we get $wt(MST(S_{i,j})) \leqslant 2 \cdot wt(MST(S))$ (see, e.g., Lemma 1.1.4 in [8]), and therefore $wt(B_{i,j}) \leqslant 8p \cdot wt(MST(S))$.

It follows that

$$\begin{aligned}
&wt\bigl(E'\bigr) = \sum_{i=1}^{k} \sum_{j=1}^{\log_p n} wt(B_{i,j}) \leqslant 8pk \log_p n \cdot wt\bigl(MST(S)\bigr), \quad \text{and} \\
&wt(E) = wt\bigl(E'\bigr) + wt\bigl(E''\bigr) \leqslant 8pk \log_p n \cdot wt\bigl(MST(S)\bigr) + wt\bigl(MST(S)\bigr) = O(\log n) \cdot wt\bigl(MST(S)\bigr).
\end{aligned}$$

A more delicate upper bound. We showed that the weight of the minimum spanning tree of a connected symmetric disk graph is bounded by $O(\log n) \cdot wt(MST(S))$, whereas the weight of the minimum spanning tree of a connected *unit* disk graph (UDG) is equal to $wt(MST(S))$. A more delicate bound that depends also on $r_{\max}$ and $r_{\min}$, the maximum and minimum ranges, bridges between the two upper bounds.

This bound is obtained by changing the above proof in the following manner. Let $l_1 = \max\{r_{\min}, wt(MST(S))/n\}$ and $l_2 = \min\{r_{\max}, wt(MST(S))\}$. Define $E'' = \{e \in MST_{SDG(S)} \mid |e| \leqslant l_1\}$, and $E' = \{e \in MST_{SDG(S)} \mid |e| > l_1\}$. Now, if $l_1 = r_{\min}$, then we get that $E'' \subseteq E(MST(S))$ and therefore $wt(E'') \leqslant wt(MST(S))$, and if $l_1 = wt(MST(S))/n$, then we get that $wt(E'') \leqslant (n-1) \cdot wt(MST(S))/n < wt(MST(S))$. Thus, in both cases we get that $wt(E'') \leqslant wt(MST(S))$.

Concerning E', we slightly modify the division into buckets, so that $B_{i,j} = \{e \in E' \cap C_i \mid |e| \in (l_1 p^{j-1}, l_1 p^j]\}$. Since the weight of any edge in $MST_{SDG(S)}$ is at most l_2, we get that the number of buckets is $\log_p(l_2/l_1)$. The asymptotic weight of each bucket remains $O(wt(MST(S)))$ (as before).

Therefore, the new bound on the weight of $MST_{SDG(S)}$ is $O(\log(l_2/l_1) + 1) \cdot wt(MST(S))$. The following theorem summarizes our result.

Theorem 2.4.

$$wt(MST_{SDG(S)}) = O\left(\log\left(\frac{\min\{r_{\max}, wt(MST(S))\}}{\max\{r_{\min}, wt(MST(S))/n\}}\right) + 1\right) \cdot wt\bigl(MST(S)\bigr),$$

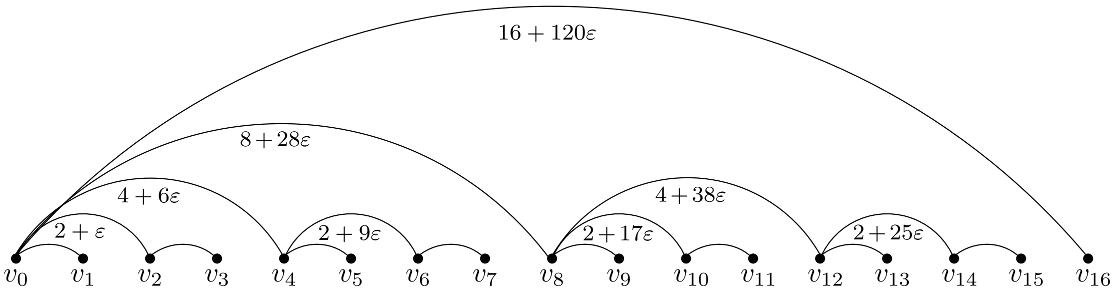

Fig. 2. Theorem 2.3 — the lower bound.

where $r_{\max}$ and $r_{\min}$ are the maximum and minimum ranges, respectively. In particular, if $(r_{\max}/r_{\min})$ is bounded by some constant, then $wt(MST_{SDG(S)}) = O(wt(MST(S)))$.

2.2. Lower bound

Consider the following set of $n+1$ points $S = (v_0, v_1, \dots, v_n)$ on a line, where $n = 2^k$ for some positive integer k. The distance between two adjacent points v_i and v_{i+1} is $1 + i\varepsilon$, where $\varepsilon = O(1/n)$, for $i = 0, \dots, n-1$. We assign a range $r(v_i)$ to each of the points $v_i \in S$; see Fig. 2.

Set

$$r(v_0) = n + \frac{(n-1)n}{2}\varepsilon.$$

That is, v_0's range is the distance between the two extreme points v_0 and v_n. For $i \neq 0$, let $m = 2^l$ be the largest power of two that divides i. Set

$$r(v_i) = m + \frac{m(2i - m - 1)}{2}\varepsilon.$$

Consider the induced symmetric disk graph, $SDG(S)$, depicted in Fig. 2. Observe that this graph is actually a tree. Since, if we build it by adding the nodes, one by one, from left to right, then each node (except for v_0) contributes exactly one new edge to $SDG(S)$. That is, $SDG(S)$ is connected and contains $n-1$ edges. Therefore, $MST_{SDG(S)}$ is simply $SDG(S)$,

$$wt(MST_{SDG(S)}) > \frac{n}{2} \cdot 1 + \frac{n}{4} \cdot 2 + \frac{n}{8} \cdot 4 + \dots + \frac{n}{n} \cdot \frac{n}{2} = \frac{n}{2} \cdot \log n = \Omega(n \log n).$$

On the other hand, $MST(S)$ is simply the path $v_0, v_1, \dots, v_n$, and therefore

$$wt\big(MST(S)\big) = n + \sum_{i=1}^{n-1} i\varepsilon = n + \frac{(n-1)n}{2}\varepsilon = O(n).$$

Therefore, in this example, $wt(MST_{SDG(S)}) = \Omega(\log n) \cdot wt(MST(S))$.

3. *k*-Range symmetric disk graphs

In this section we consider the common case where the number of different ranges assigned to the points of S is only k, for $k \ll n$. That is, the function $r : S \to \mathbb{R}$ assumes only k different values, denoted $r_1 < r_2 < \dots < r_k$. We first prove that in this case the weight of the minimum spanning tree of $SDG(S)$ is at most $2k \cdot wt(MST(S))$. Next, we present an efficient $O(kn \log n)$ algorithm for computing this minimum spanning tree. Thus, assuming k is some constant, we get that $wt(MST_{SDG(S)}) = O(wt(MST(S)))$ and $MST_{SDG(S)}$ can be constructed in time $O(n \log n)$.

3.1. The weight of the minimum spanning tree

Let $SDG(S)$ be a k-range symmetric disk graph. Let $MST_{SDG(S)}$ be the minimum spanning tree of $SDG(S)$, and let E be the set of edges of $MST_{SDG(S)}$. We divide the edges of E into k subsets according to their length. Notice that, by definition, the length of the longest edge in E is at most r_k. Put $r_0 = 0$, and let $E_i = \{e \in E \mid r_{i-1} < |e| \leqslant r_i\}$, for $i = 1, \dots, k$. Also, let $S_i = \{v \in S \mid r_i \leqslant r(v)\}$, for $i = 1, \dots, k$. Then, $S = S_1 \supseteq S_2 \supseteq \dots \supseteq S_k$.

Claim 3.1. *$E_i \subseteq E(MST(S_i))$, for $i = 1, \dots, k$.*

Proof. Let $e=(u,v)\in E_i$. We first observe that $u,v\in S_i$. Indeed, since $e\in E(SDG(S))$, we know that $r(u)\geqslant |e|$ and $r(v)\geqslant |e|$. Now, since $|e|>r_{i-1}$, it follows that $r(u)\geqslant r_i$ and $r(v)\geqslant r_i$ and therefore $u,v\in S_i$.

Let u',v' be any two vertices in S_i, such that $|u'v'|<|e|$. Then, $e'=(u',v')\in E(SDG(S))$ (since $r(u')\geqslant r_i\geqslant |e|>|e'|$ and $r(v')\geqslant r_i\geqslant |e|>|e'|$). Thus, in the construction of $MST_{SDG(S)}$ by Kruskal's algorithm (see [6]), the edge e', as well as all other edges of $SDG(S)$ with both endpoints in S_i and shorter than e, were considered before e. Nevertheless, e was selected, since there was still no path between u and v. Therefore, in the construction of $MST(S_i)$, when e is considered, there is still no path between its endpoints, and it is selected, i.e., $e\in E(MST(S_i))$. □

Theorem 3.2. $wt(MST_{SDG(S)})\leqslant 2k\cdot wt(MST(S))$.

Proof. For each $1\leqslant i\leqslant k$, $E_i\subseteq MST(S_i)$ (by the claim above), and therefore $wt(E_i)\leqslant wt(MST(S_i))$. Referring to $S\setminus S_i$ as Steiner points, we get that $wt(E_i)\leqslant 2\cdot wt(MST(S))$. Thus, $wt(MST_{SDG(S)})=\sum_{i=1}^{k} wt(E_i)\leqslant \sum_{i=1}^{k} 2\cdot wt(MST(S))=2k\cdot wt(MST(S))$. □

3.2. Constructing the minimum spanning tree

We describe below an $O(kn\log n)$ algorithm for computing the minimum spanning tree of a k-range symmetric disk graph. The algorithm applies Kruskal's minimum spanning tree algorithm (see [6]) to a subset of the edges of $SDG(S)$. We then prove that the subset E of edges that were selected by Kruskal's algorithm is $E(MST_{SDG(S)})$.

Algorithm 1 Computing the MST of a k-range symmetric disk graph

Require: S; $r_1<r_2<\cdots<r_k$; $r:S\to\{r_1,\ldots,r_k\}$
Ensure: $MST_{SDG(S)}$
1: $E\leftarrow\emptyset$
2: **for** $i=1$ to k **do**
3: $\quad S_i\leftarrow\emptyset$; $E_i\leftarrow\emptyset$
4: $\quad$ **for all** $s\in S$ such that $r(s)\geqslant r_i$ **do**
5: $\quad\quad S_i\leftarrow S_i\cup\{s\}$
6: $\quad DT(S_i)\leftarrow$ Delaunay triangulation of S_i
7: $\quad$ **for all** $e\in DT(S_i)$ such that $r_{i-1}<|e|\leqslant r_i$ **do**
8: $\quad\quad E_i\leftarrow E_i\cup\{e\}$
9: $E\leftarrow Kruskal(\bigcup_{i=1}^{k} E_i)$
10: **return** E

Lemma 3.3. $E=E(MST_{SDG(S)})$.

Proof. We prove that $E(MST_{SDG(S)})\subseteq E$, and since the algorithm assures that E does not contain any cycle (line 9), we conclude that $E(MST_{SDG(S)})=E$.

Let $e=(u,v)\in E(MST_{SDG(S)})$ and let i, $1\leqslant i\leqslant k$, such that $r_{i-1}<|e|\leqslant r_i$. (Recall that $r_0=0$.) Since e is an edge of $SDG(S)$, we have that $r(u)\geqslant |e|$ and $r(v)\geqslant |e|$, implying that $r(u),r(v)\geqslant r_i$ and therefore $u,v\in S_i$.

We show that $e\in E(MST(S_i))$. Assume that $e\notin E(MST(S_i))$, then $E(MST(S_i))\cup\{e\}$ contains a cycle C. For each $e'\in C$, $e'\neq e$, we have that $|e'|<|e|\leqslant r_i$. That is, e' is an edge of $SDG(S_i)$, and therefore also an edge of $SDG(S)$. Now, assume we apply Kruskal's minimum spanning tree algorithm to $SDG(S)$. Then, since each of the edges of $C-\{e\}$ is considered before e, e is not selected as an edge of $MST_{SDG(S)}$ — a contradiction.

Since $E(MST(S_i))\subseteq E(DT(S_i))$, e also belongs to $E(DT(S_i))$, and therefore (since $r_{i-1}<|e|\leqslant r_i$) $e\in E_i$. To complete the proof, notice that $E_i\subseteq E(SDG(S))$, for $i=1,\ldots,k$, and since, by assumption, $e\in E(MST_{SDG(S)})$, we conclude that $e\in Kruskal(\bigcup_{i=1}^{k} E_i)=E$. □

Theorem 3.4. *The minimum spanning tree of a k-range symmetric disk graph of n points can be computed in time $O(kn\log n)$.*

Proof. In each of the k iterations of the main loop, we compute the Delaunay triangulation of a subset of S. This can be done in $O(n\log n)$ time. Finally, we apply Kruskal's algorithm to a set of size $O(kn)$. Thus, the total running time is $O(kn\log n)$. □

4. Applications

4.1. An alternative proof of the Gap Theorem

Let $w\geqslant 0$ be a real number, and let E be a set of *directed* edges in $\mathbb{R}^d$. We say that E has the w-*gap property* if for any two distinct edges (p,q) and (r,s) in E, we have $|pr|>w\cdot\min\{|pq|,|rs|\}$. The gap property was introduced by Chandra et al. [4], who also proved the Gap Theorem; see below. The Gap Theorem bounds the weight of any set of edges that satisfies the gap property.

Theorem 4.1 *(Gap Theorem). Let $w > 0$, let S be a set of n points in $\mathbb{R}^d$, and let $E \subseteq S \times S$ be a set of directed edges that satisfies the w-gap property. Then $wt(E) = O(\log n) \cdot wt(MST(S))$.*

Chandra et al. use in their proof a shortest traveling salesperson tour $TSP(S)$ of S. They charge the lengths of the edges in E to portions of $TSP(S)$, and prove that $wt(E) < (1 + 2/w)\log n \cdot wt(MST(S))$. We give an alternative, simpler, proof of the Gap Theorem, which is similar to the proof of the first part of Theorem 2.3.

We now present our proof. Let $E' = \{e \in E \mid wt(e) > wt(MST(S))/n\}$ and let $E'' = \{e \in E \mid wt(e) \leqslant wt(MST(S))/n\}$. Since $w > 0$, each point of S is the source of at most one edge of E, which implies that there are at most n edges in E. Therefore,

$$wt(E'') = \sum_{e \in E''} w(e) \leqslant n \cdot \frac{wt(MST(S))}{n} = wt(MST(S)).$$

As for $wt(E')$, notice that for each $e \in E'$, $wt(e) \leqslant \mathrm{diam}(S) \leqslant wt(MST(S))$. We divide the edges of E' into $\log n$ buckets according to their size. Specifically, for $1 \leqslant i \leqslant \log n$, let $B_i = \{e \in E' \mid wt(e) \in (\frac{wt(MST(S))}{n} \cdot 2^{i-1}, \frac{wt(MST(S))}{n} \cdot 2^i]\}$ and let $S_i = \{s \in S \mid \text{there exists a point } t \text{ such that } (s,t) \in B_i\}$.

Let $s, s' \in S_i$, and let $t, t' \in S$ such that $e = (s,t)$ and $e' = (s',t')$ are edges in B_i. By the w-gap property, $wt(ss') > w \cdot \min\{wt(e), wt(e')\} \geqslant \frac{w}{2} \cdot \max\{wt(e), wt(e')\}$. We claim that $wt(B_i) \leqslant \frac{8}{w} \cdot wt(MST(S))$. We omit the details; however, this claim is very similar to the analogous claim in the proof of the first part of the SDG Theorem.

It follows that

$$wt(E') = \sum_{i=1}^{\log n} wt(B_i) \leqslant \frac{8}{w} \log n \cdot wt(MST(S)), \quad \text{and therefore}$$

$$wt(E) \leqslant \frac{8}{w} \log n \cdot wt(MST(S)) + wt(MST(S)) = O(\log n) \cdot wt(MST(S)).$$

4.2. Range assignment

Let S be a set of n points in the plane (representing transceivers). For each $v_i \in S$, let r_i be the maximum transmission range of v_i, and put $r = (r_1, \ldots, r_n)$. The following problem is known as *the Range Assignment Problem*. Assign a transmission range d_i, $d_i \leqslant r_i$, to each of the points v_i of S, such that (i) the induced symmetric disk graph (using the ranges $d_1, \ldots, d_n$) is connected, and (ii) $\sum_{i=1}^{n} d_i$ is minimized. Below, we compute a range assignment, such that the sum of ranges of the assignment is bounded by $O(\log n)$ times the sum of ranges of an optimal assignment, computed under the assumption that $r_1 = \cdots = r_n = \mathrm{diam}(S)$.

Let $SDG(S)$ be the symmetric disk graph of S. We first compute $MST_{SDG(S)}$. Next, for each $v_i \in S$, let d_i be the weight of the heaviest edge incident to v_i in $MST_{SDG(S)}$. Notice that the induced symmetric disk graph (using the ranges $d_1, \ldots, d_n$) is connected, since it contains $E(MST_{SDG(S)})$. It remains to bound the sum of ranges of the assignment with respect to $wt(MST(S))$, where $MST(S)$ is the Euclidean minimum spanning tree of S. Let $OPT(S)$ denote an optimal range assignment with respect to the complete Euclidean graph of S. It is easy to see that $wt(MST(S)) < wt(OPT(S)) < 2 \cdot wt(MST(S))$ (see Kirousis et al. [7]). Thus, $\sum_{i=1}^{n} d_i < 2 \cdot wt(MST_{SDG(S)}) = O(\log n) \cdot wt(MST(S)) = O(\log n) \cdot wt(OPT(S))$. (Of course, we also know that $\sum_{i=1}^{n} d_i < 2 \cdot wt(OPT_r(S))$, where $OPT_r(S)$ is an optimal range assignment with respect to $SDG(S)$.)

5. Disk graphs

Given a set S of n points in $\mathbb{R}^d$ and a function $r : S \to \mathbb{R}$, the *Disk Graph* of S, denoted $DG(S)$, is a directed graph over S whose set of edges is $E = \{(u,v) \mid r(u) \geqslant |uv|\}$. In this section we show that, unlike symmetric disk graphs, the weight of a minimum spanning subgraph of a disk graph might be much bigger than that of $MST(S)$.

Notice that if the corresponding symmetric disk graph $SDG(S)$ is connected, then the weight of a minimum spanning subgraph of $DG(S)$ (denoted $MST_{DG(S)}$) is bounded by $wt(MST_{DG(S)}) \leqslant 2 \cdot wt(MST_{SDG(S)}) = O(\log n) \cdot wt(MST(S))$.

We now state the main theorem of this section.

Theorem 5.1. *Let $DG(S)$ be a strongly connected disk graph over a set S of n points in $\mathbb{R}^d$. Then,* (i) *$wt(MST_{DG(S)}) = O(n) \cdot wt(MST(S))$, and* (ii) *there exists a set of n points in the plane, such that $wt(MST_{DG(S)}) = \Omega(n) \cdot wt(MST(S))$.*

Proof. *Upper bound*. Since $\max_{e \in E(DG(S))}\{wt(e)\} \leqslant wt(MST(S))$ and since the number of edges in $MST_{DG(S)}$ is less than $2n$,

$$wt(MST_{DG(S)}) < 2n \cdot \max_{e \in E(DG(S))}\{wt(e)\} \leqslant 2n \cdot wt(MST(S)) = O(n) \cdot wt(MST(S)).$$

Lower bound. Consider the following set S of $n+1$ points in the plane, where $n = 3k$ for some positive integer k. We place $\frac{2}{3}n + 1$ points on the line $y = 0$, such that the distance between any two adjacent points is $1 - \varepsilon$, where $0 < \varepsilon < 1/2$, and we place $\frac{1}{3}n$ points on the line $y = 1$, such that the distance between any two adjacent points is $2 - 2\varepsilon$; see Fig. 3. For

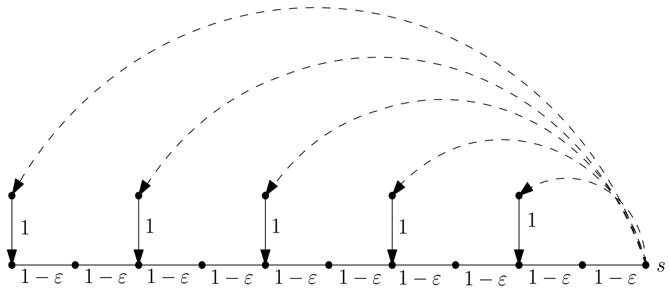

Fig. 3. Theorem 5.1 — the lower bound.

each point u on the top line, set $r(u)=1$, and for each point v on the bottom line, except for the rightmost point s, set $r(v)=1-\varepsilon$. Set $r(s)$ so that it reaches all points on the top line.

We show that $wt(MST_{DG(S)})=\Omega(n)\cdot wt(MST(S))$. First notice that the Euclidean minimum spanning tree of S has the shape of a comb, and therefore

$$wt\big(MST(S)\big)=\frac{2}{3}n\cdot(1-\varepsilon)+\frac{1}{3}n\cdot 1<n=O(n).$$

Next notice that for each point u on the top line, the minimum spanning subgraph of $DG(S)$ must include the edge (s,u), since this is the only edge that enters u. The total weight of these $n/3$ edges is at least $2+4+\cdots+2n/3=\Omega(n^2)$. Therefore, $wt(MST_{DG(S)})=\Omega(n^2)=\Omega(n)\cdot wt(MST(S))$. $\square$

6. Concluding remarks

The main result of this paper is a proof that the weight of the MST of *any* connected symmetric disk graph over a set S of n points in the plane, is only $O(\log n)$ times the weight of the MST of the complete Euclidean graph over S. Moreover, this bound is tight, even for points on a line.

Recently and independently, Michiel Smid [9] introduced the *weak gap property*: A set E of *directed* edges has the *weak w-gap property* for some constant $w>0$, if for any two distinct edges (p,q) and (r,s) in E, $|pq|\geqslant w\cdot\min\{|pq|,|rs|\}$, or $|qs|\geqslant w\cdot\min\{|pq|,|rs|\}$. Smid proved that if E has the weak w-gap property, then $wt(E)=O(\log n)\cdot wt(MST(S))$ and $|E|=O(n)$, where S is the set of endpoints of edges in E and $|S|=n$. (His proof applies to any metric space of constant doubling dimension.) Moreover, the $O(\log n)$ upper bound is tight.

Observe that if one assigns an arbitrary direction to each of the edges in $E(MST_{SDG(S)})$, then the resulting set of directed edges has the weak gap property for $w=1$; this follows immediately from Lemma 2.1. Thus, one could apply Smid's result to obtain our main result. This is not surprising, as both Smid's proof and our proof proceed essentially along the same lines. In particular, we obtain an alternative proof for the Gap Theorem.

We end this section with an open problem. In Section 3.2, we presented an algorithm for computing, in time $O(kn\log n)$, the MST of a k-range symmetric disk graph of n points. Can one always compute the MST of a symmetric disk graph efficiently?

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