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# An energy-decreasing rearrangement of level sets preserving the domain and volume constraints

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**Abstract**

Let $\Omega \subset \mathbb{R}^2$ be a bounded and regular domain, $u \in C^3(\Omega)$ and $V \subset \Omega$ a domain where the subset $K_0$ of points where the curvature of the $t$-level sets of $u$ is zero admits a regular $t$-parameterization. We exhibit a local correction of $u$ in a neighborhood of a particular point $x^* \in K_0 \subset V$ such that the volume $\int f(u)$ is preserved and the Dirichlet integral $\int |\nabla u|^2$ decreases. Consequently, a certain monotonic property is deduced for constrained minimizers in $H^1(\Omega)$. Our result can be applied to classical variational and free-boundary problems.


*MSC:* 35J20; 35J65

*Keywords:* Constrained minimization; Rearrangement of level sets

## 1. Introduction

In [3], Gidas et al. studied symmetry properties of the positive solutions to

$$-\Delta u = g(u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{1.1}$$

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when $g$ is a $C^1$ function and $\Omega$ is a domain, possibly unbounded. Their method relied in a technique of moving parallel planes to a critical position and proving the equality of $u$ and its reflection. In particular, $u$ is radial symmetric when $\Omega$ is a ball. As for the convexity of the super-level sets when $\Omega$ is convex, partial results were obtained using methods that can be somewhat related to the maximum principle (see for instance [6] and [8]). Solutions to (1.1) may occur as constrained minimizers of the energy $\int |\nabla u|^2$ on the Sobolev space $H_0^1(\Omega)$. However, most common energy-decreasing and volume-preserving rearrangements drastically modify the original function. In particular, the restriction $u \in H_0^1(\Omega)$ might fail (see [5]). We point out that, in [1], Colesanti et al. prove that the solution $u$ to an elliptic problem in a convex ring-shaped domain coincides with its quasi-concave envelope (i.e. the function whose super-level sets are the convex hulls of the corresponding super-level sets of $u$).

The present work aims to give a clearer understanding of how the energy may be affected by the shape of $u$. Assume $u \in C^3(\Omega)$ (which is the case for a classical solution to (1.1) when $g$ is $C^{1,\alpha}$). We consider a subset $K_0 \subset \Omega$ where the curvature of level sets is zero. Using elementary analysis, we prove that, in the neighborhood of a particular point $x^* \in K_0$, $u$ can be corrected in such a way that the constraint $\int f(u)$ is preserved (for continuous $f$) and $\int |\nabla u|^2$ decreases. More specifically, an adequate system of coordinates and Cauchy–Schwarz inequality allows us to relate the Dirichlet integral with isoperimetric properties of the super-level sets and to the norm of the gradient along level sets. Our results can be applied to some classical Dirichlet and Neumann boundary value problems related to (1.1) or to the obstacle problem. Finally we prove the existence of a "correction point" $x^*$ when $K_0$ is contained in a Jordan curve and satisfies some non-degenerate assumptions.

## 2. An induced coordinate system

Let $\Omega \subset \mathbb{R}^2$ be a bounded and regular domain. We denote by $\mathfrak{C}(\Omega)$ the set of functions $u : \Omega \to \mathbb{R}$ with the following properties:

$$\text{(i)} \quad u(x) = \max_{\Omega} u \quad \Rightarrow \quad x = x_M,$$

for some $x_M \in \Omega$.

$$\text{(ii)} \quad u \in C^2(\overline{\Omega}), \qquad u(x) > 0 \quad \forall x \in \Omega, \qquad u = 0 \quad \text{on } \partial\Omega.$$

(iii) All levels of $u$ are regular (except the maximum level) i.e.

$$\Bigl(u(x) = c \text{ and } c \neq \max_{\Omega} u\Bigr) \quad \Rightarrow \quad \nabla u(x) \neq 0.$$

We build a coordinate system in $\Omega$ induced by the level sets of $u$. Fix $x_1$ on $\partial\Omega$ and consider the following parameterization of $\partial\Omega$ by arc-length:

$$L(0) = x_1, \qquad \frac{dL}{d\lambda} = \tau\bigl(L(\lambda)\bigr), \tag{2.1}$$

where $\tau(x)$ is the unitary tangent at $x$ verifying

$$\left(\tau, \frac{\nabla u}{|\nabla u|}\right) \quad \text{is a positively oriented orthonormal basis of } \mathbb{R}^2.$$

Denoting by $\Lambda$ the perimeter of $\partial\Omega$, we restrict the solution $L$ to $[0, \Lambda[$. For every $\lambda \in [0, \Lambda[$ we consider $T_\lambda$ the solution of the Cauchy problem:

$$T_\lambda(0) = L(\lambda), \qquad \frac{dT_\lambda}{dt} = \frac{\nabla u(T_\lambda(t))}{|\nabla u(T_\lambda(t))|^2}. \tag{2.2}$$

We point out that the maximal domain $[0, \bar{t}_\lambda[$ of $T_\lambda$ is $[0, M[$ where $M = \max_\Omega u$. In fact, if $x = T_\lambda(t)$ then $u(x) = t$, i.e. the parameter $t$ indicates the level set of $u$ that contains $x$. This can easily be seen from

$$\begin{aligned} u\big(T_\lambda(t)\big) &= u\big(T_\lambda(0)\big) + \int_0^t \nabla u\big(T_\lambda(v)\big) \cdot \frac{dT_\lambda}{dv}\, dv \\ &= \int_0^t \nabla u\big(T_\lambda(v)\big) \cdot \frac{\nabla u(T_\lambda(v))}{|\nabla u(T_\lambda(v))|^2}\, dv = \int_0^t 1\, dv = t. \end{aligned} \tag{2.3}$$

We conclude that $\bar{t}_\lambda \leqslant M$. Moreover, if $\bar{t}_\lambda < M$, by our assumptions on $u$, $T_\lambda$ can be extended to $[0, \bar{t}_\lambda + \epsilon[$ for some $\epsilon > 0$, a contradiction.

We redefine

$$X(\lambda, t) := T_\lambda(t) \quad (\lambda, t) \in [0, \Lambda[ \times [0, M[$$

and list some properties of $X$:

**Lemma 1.**

1. *$X(\lambda, t)$ is of class $C^2$.*
2. *$X([0, \Lambda[ \times [0, M[) = \Omega \backslash \{x_M\}$.*
3. *$|\mathcal{J}_X|(\lambda, t) = |X_\lambda||\nabla u|^{-1}(X(\lambda, t))$ where $\mathcal{J}_X$ is the Jacobian of $X$ at $(\lambda, t)$.*
4. *For every $\lambda$, $|X_\lambda|(\lambda, .)$ satisfies the following differential equation*

$$\frac{d|X_\lambda|}{dt} = \frac{-K}{|\nabla u|}|X_\lambda|,$$

   *where $K$ stands for the curvature of the level set of $u$ at $X(\lambda, t)$.*
5. *$|X_\lambda|(\lambda, t) \neq 0$ for all $(\lambda, t) \in [0, \Lambda[ \times [0, M[$. In particular,*

$$X : ]0, \Lambda[ \times ]0, M[ \mapsto \Omega \backslash \overline{\{X(0, t)\colon t \in [0, M[\}}$$

   *is a diffeomorphism.*
6. *$\int_0^\Lambda |X_\lambda|(\lambda, t)\, d\lambda$ measures the perimeter of the level set $t$ of $u$.*

**Proof.** 1. The regularity of $X$ is due to the regular dependence on parameters of the solutions of ordinary differential equations.

2. It is clear from (2.1)–(2.2) that

$$X\big([0,\Lambda[\times[0,M[\big)\subset\Omega\backslash\{x_M\}.$$

Given $x\in\Omega\backslash\{x_M\}$ we have $u(x)<M$. Consider the path defined by:

$$T\big(u(x)\big)=x,\qquad \frac{dT}{dt}=-\frac{\nabla u(T(t))}{|\nabla u(T(t))|^2}.$$

Necessarily, it must intersect $\partial\Omega$ at some point $L(\lambda)$. Then, by the Existence Uniqueness Theorem for the Cauchy Problem, we have

$$x=T_\lambda\big(u(x)\big)=X\big(\lambda,u(x)\big),$$

where $T_\lambda$ was defined in (2.2).

3. As noticed in (2.3), $u(X(\lambda,t))=t$ for all $\lambda\in[0,\Lambda[$. Differentiating in $\lambda$, one deduces

$$\nabla u\big(X(\lambda,t)\big)\cdot X_\lambda(\lambda,t)=0,$$

i.e. $\nabla u\perp X_\lambda$. Therefore

$$|\mathcal{J}_X|(\lambda,t)=|X_\lambda||X_t|\big(X(\lambda,t)\big)=|X_\lambda||\nabla u|^{-1}\big(X(\lambda,t)\big).$$

4. By the previous step we may write

$$X_\lambda(\lambda,t):=|X_\lambda|\tau\big(X(\lambda,t)\big)\quad\text{and}\quad X_t(\lambda,t)=|\nabla u|^{-1}n\big(X(\lambda,t)\big),$$

where $\tau$ denotes a unitary tangent and $n$ denotes the inward normal to the level set $t$ at $X(\lambda,t)$. Then, by Schwarz rule,

$$\frac{d}{dt}\langle X_\lambda,X_\lambda\rangle=2\langle X_{\lambda t},X_\lambda\rangle=2\langle X_{t\lambda},X_\lambda\rangle=2\left\langle\frac{d}{d\lambda}\big(|\nabla u|^{-1}n\big(X(t,\lambda)\big)\big),X_\lambda\right\rangle$$
$$=2\left\langle|\nabla u|^{-1}\frac{d}{d\lambda}n\big(X(t,\lambda)\big),X_\lambda\right\rangle.$$

By Frenet's formula, denoting by $s$ the arc-length variable,

$$\frac{d}{d\lambda}n\big(X(\lambda,t)\big)=\frac{d}{ds}n\big(X(\lambda,t)\big)\frac{ds}{d\lambda}=-K\big(X(\lambda,t)\big)X_\lambda.$$

Substituting these quantities above we obtain:

$$\frac{d|X_\lambda|^2}{dt}=-2K\big(X(\lambda,t)\big)|X_\lambda|^2|\nabla u|^{-1}$$

or

$$\frac{d|X_\lambda|}{dt}=\frac{-K}{|\nabla u|}|X_\lambda|(\lambda,t).$$

5. By 4 and the Existence Uniqueness Theorem for the Cauchy Problem, recalling that $|X_\lambda|(\lambda, 0) \equiv 1$, we conclude that

$$|X_\lambda|(\lambda, t) \neq 0 \quad \forall(\lambda, t) \in [0, \Lambda[ \times [0, M[.$$

By 1 and 3 we conclude that

$$X : ]0, \Lambda[ \times ]0, M[ \mapsto \Omega \backslash \overline{\{X(0, t)\colon t \in [0, M[\}}$$

is a diffeomorphism.

6. Just note that

$$X(., t) : [0, \Lambda[ \mapsto \overline{\Omega}$$

is a regular parameterization of the level set $t$. $\square$

We refer to $X(\lambda, t)$ as a **coordinate system induced by** $u$. The system is determined up to the choice of $X(0, 0) := x_1$.

**Remark 1** *(Local induced coordinate system).* Of course one may locally define the system $X(\lambda, t)$ in an open subset of a domain $V$ such that

$$u \in C^2(V) \quad \text{and} \quad \nabla u(x) \neq 0 \quad \text{if } x \in V.$$

Consider, for some $x_0 \in V$:

$$L(0) = x_0, \qquad \frac{dL}{d\lambda} = \tau\big(L(\lambda)\big) \quad \text{if } \lambda \neq 0, \tag{2.4}$$

$$T_\lambda\big(u(x_0)\big) = L(\lambda), \qquad \frac{dT_\lambda}{dt} = \frac{\nabla u(T_\lambda(t))}{|\nabla u(T_\lambda(t))|^2} \quad \text{if } t \neq u(x_0), \tag{2.5}$$

and assume

$$(\lambda, t) \in ]-\epsilon, \epsilon[ \times \big]u(x_0) - \delta, u(x_0) + \delta\big[,$$

provided that

$$X\big(]-\epsilon, \epsilon[ \times \big]u(x_0) - \delta, u(x_0) + \delta\big[\big) \subset V.$$

Next we prove some classical results using induced coordinates.

**Lemma 2.** *Let $\Omega_1 \subset \Omega_0$ be two convex sets (which we suppose to have regular and non-intersecting boundaries). Then $l(\partial\Omega_1) < l(\partial\Omega_0)$ where $l(.)$ stands for the perimeter.*

**Proof.** The solution $u$ to the boundary value problem

$$\Delta u = 0 \quad \text{in } \Omega_0 \backslash \overline{\Omega}_1, \qquad u|_{\partial\Omega_0} = 0, \qquad u|_{\partial\Omega_1} = 1,$$

has convex super level sets $\Omega_t$ ($t \in [0, 1]$) and $\nabla u(x) \neq 0$ for all $x \in \Omega_0 \backslash \overline{\Omega}_1$ (see [1] or [2]). Let $X(\lambda, t)$ be a coordinate system induced by $u$ in $\Omega_0 \backslash \overline{\Omega}_1$. By 4 of Lemma 1 we obtain:

$$\begin{aligned} l(\partial\Omega_1) &= \int_0^\Lambda |X_\lambda|(\lambda, 1)\, d\lambda = l(\partial\Omega_0) + \int_0^1 \frac{d}{dt} \int_0^\Lambda |X_\lambda|(\lambda, t)\, d\lambda\, dt \\ &= l(\partial\Omega_0) + \int_0^1 \int_0^\Lambda \frac{d}{dt} |X_\lambda|(\lambda, t)\, d\lambda\, dt \\ &= l(\partial\Omega_0) + \int_0^1 \int_0^\Lambda -K \frac{|X_\lambda|}{|\nabla u|}\, d\lambda\, dt \leqslant l(\partial\Omega_0) - 2\pi \int_0^1 \inf_{\partial\Omega_t} \bigl\{|\nabla u|^{-1}\bigr\}\, dt. \qquad \square \end{aligned}$$

We express energy and volume in induced coordinates.

**Lemma 3.** *Let $u \in \mathfrak{C}(\Omega)$ with induced coordinate system $X$ and $f \in C(\mathbb{R}, \mathbb{R})$. Then*

$$\int_\Omega |\nabla u|^2 = \int_0^M \int_0^\Lambda |X_\lambda| |\nabla u|\, d\lambda\, dt = \int_0^M \int_{\partial\Omega_t} |\nabla u|\, ds\, dt, \tag{2.6}$$

$$\int_\Omega f(u)\, dx = \int_0^M f(t) \int_0^\Lambda \frac{|X_\lambda|}{|\nabla u|}\, d\lambda\, dt = \int_0^M f(t) \int_{\partial\Omega_t} |\nabla u|^{-1}\, ds\, dt, \tag{2.7}$$

*where $\Omega_t := \{x \in \Omega\colon\ u(x) > t\}$.*

**Proof.** The result follows from a straightforward application of the Change of Variables Theorem under the Lebesgue integral. $\square$

We introduce a function depending on the norm of the gradient along the level sets of $u$.

**Definition.** Let $t \in [0, M[$ and $u \in \mathfrak{C}(\Omega)$ with induced coordinate system $X$. We define

$$\begin{aligned} S_u(t) &:= \int_0^\Lambda |X_\lambda| |\nabla u|(\lambda, t)\, d\lambda \int_0^\Lambda |X_\lambda| |\nabla u|^{-1}(\lambda, t)\, d\lambda - \left( \int_0^\Lambda |X_\lambda|(\lambda, t)\, d\lambda \right)^2 \\ &= \int_{\partial\Omega_t} |\nabla u|\, ds \int_{\partial\Omega_t} |\nabla u|^{-1}\, ds - \left( \int_{\partial\Omega_t} 1\, ds \right)^2. \end{aligned}$$

**Remark 2.** Note that $S(t)$ is independent of the choice of $x_1$ in (2.1). Also, by Cauchy–Schwarz inequality, $S(t) \geqslant 0$ and

$$S(t) = 0 \quad \text{iff} \quad |\nabla u|(., t) \text{ is constant along the level set } \partial\Omega_t.$$

In the next lemma we provide an alternative proof to the Polya–Szëgo inequality on the subset $\mathfrak{C}(\Omega)$. It was brought to our attention that in [9, Chapter 3] this classical result was obtained using similar arguments to the ones below.

**Lemma 4.** *Let $u \in \mathfrak{C}(\Omega)$. Let $R = \sqrt{m(\Omega)/\pi}$ and, using polar coordinates, define*

$$u^*(r,\theta) : [0, R] \times [0, 2\pi[ \mapsto \mathbb{R},$$
$$u^*(r,\theta) = t \quad \textit{iff} \quad r = \sqrt{m(\Omega_t)/\pi},$$

*where $\Omega_t = \{x \in \Omega\colon\ u(x) > t\}$. Then*

$$u^* \in C^2\big(\overline{B(0,R)}\backslash\{0\}\big) \cap C(\overline{\Omega})$$

*and*

$$\int\limits_{B(0,R)} |\nabla u^*|^2\, dx \leqslant \int\limits_{\Omega} |\nabla u|^2\, dx,$$

*with equality iff $u$ is radially symmetric.*

**Proof.** By Lemma 3 we may write

$$m(\Omega_t) = \int\limits_t^M \int\limits_0^\Lambda \frac{|X_\lambda|}{|\nabla u|}\, d\lambda\, dt = \int\limits_t^M \int\limits_{\partial\Omega_t} |\nabla u|^{-1}\, ds\, dt \tag{2.8}$$

and, since $u \in \mathfrak{C}(\Omega)$, this quantity is twice differentiable with strictly negative first derivative at $t \in [0, M[$. We may conclude $u^* \in C^2(\overline{B(0,R)}\backslash\{0\}) \cap C(\overline{\Omega})$.

In view of Remark 1, consider an induced coordinate system $X^*$ for $u^*$. Denote by $\Omega_t^*$ the super level set $t$ of $u^*$. Since

$$m(\Omega_t) = m\big(\Omega_t^*\big)$$

for all $t \in [0, M[$, we conclude, differentiating a formula similar to (2.8),

$$\int\limits_{\partial\Omega_t} |\nabla u|^{-1}\, ds = \int\limits_{\partial\Omega_t^*} |\nabla u^*|^{-1}\, ds. \tag{2.9}$$

Let $l(t)$ and $l^*(t)$ be the perimeters of the level sets $t$ of $u$ and $u^*$ respectively. By the definition of $u^*$ and the isoperimetric characterization of the circle, we have

$$l^*(t) \leqslant l(t), \tag{2.10}$$

the inequality being strict in case $\Omega_t \neq \Omega_t^*$. Also, by Remark 2,

$$S_u(t) \geqslant 0 = S_{u^*}(t) \quad \forall t \in [0, M[, \tag{2.11}$$

(we recall that the equality to zero results from the invariance of the norm of the gradient along the level set $t$ of $u^*$). We conclude, by (2.9)–(2.11),

$$\begin{aligned}\int_\Omega |\nabla u|^2\, dx &= \int_0^M \int_{\partial\Omega_t} |\nabla u|\, ds\, dt = \int_0^M \frac{S_u(t) + l^2(t)}{\int_{\partial\Omega_t} |\nabla u|^{-1} ds}\, dt \\ &\geqslant \int_0^M \frac{S_{u^*}(t) + l^{*2}(t)}{\int_{\partial\Omega_t^*} |\nabla u^*|^{-1} ds}\, dt = \int_0^M \int_{\partial\Omega_t} |\nabla u^*|\, ds\, dt = \int_{B(0,R)} |\nabla u^*|^2\, dx,\end{aligned}$$

the inequality being strict if $u$ is not radially symmetric. □

## 3. A local rearrangement of level sets

In this section we consider a local rearrangement of $u \in C^3(\Omega)$. Let $V \subset \Omega$ be a domain such that

$$\nabla u(x) \neq 0 \quad \forall x \in V. \tag{3.1}$$

We define the curvature function:

$$K(x) : V \to \mathbb{R}, \qquad x \mapsto -\frac{\tau D_u^2(x)\tau}{|\nabla u|},$$

where $\tau$ is a unitary tangent to the level set $u(x)$ at $x$ and $D_u^2$ the Hessian matrix of $u$. Note that by our regularity assumption on $u$ and (3.1), we have that $K \in C^1(V)$. We consider the following subsets of $V$:

$$\begin{aligned} K_0 &:= K^{-1}(\{0\}) \cap V, \\ K^- &:= K^{-1}(]-\infty, 0[) \cap V, \\ K^+ &:= K^{-1}(]0, +\infty[) \cap V. \end{aligned}$$

Additionally, we assume the following relation to hold:

$$\nabla K(x) \cdot \tau(x) \neq 0 \quad \text{for all } x \in K_0. \tag{3.2}$$

**Lemma 5.** *Under assumptions* (3.1)–(3.2), *a connected subset of $K_0$ can be $C^1$-parameterized by the level $t$ of the function $u$, i.e.*

$$\exists x_0 \in C^1\big(]\alpha,\beta[, V\big) \quad \textit{such that} \quad x_0\big(]\alpha,\beta[\big) \subset K_0 \quad \textit{and} \quad u\big(x_0(t)\big) = t.$$

**Proof.** Define a local induced coordinate system $X$ as in Section 1, Remark 1. Then

$$K\big(X(\lambda,t)\big) = 0$$

implicitly defines $\lambda$ as a function of $t$ iff

$$\nabla K \cdot X_\lambda \neq 0$$

or, by Lemmas 1–5,

$$\nabla K \cdot \tau\big(X(t,\lambda)\big) \neq 0,$$

which is precisely assumption (3.2). $\square$

In what follows, for every $x_0 \in K_0$, we will denote by $\tau(x_0)$ the unitary tangent to the level set $u = u(x_0)$ at $x_0$ satisfying

$$\nabla K(x_0) \cdot \tau(x_0) > 0. \tag{3.2$'$}$$

In the sequel we will also refer to

$$n(x) := \frac{\nabla u(x)}{|\nabla u(x)|}.$$

For every $x_0 \in K_0$ we define an associated reference frame

$$\mathfrak{R}_{x_0} := \big(x_0; \tau(x_0), n(x_0)\big).$$

A point $x \in \Omega \subset \mathbb{R}^2$ will be represented in $\mathfrak{R}_{x_0}$ by a pair $(s_x, v_x)_{x_0}$ or simply by $(s, v)$ when there is no risk of confusion.

We introduce the fundamental notion of a correction point.

**Definition.** Let $V$ be a domain such that (3.1)–(3.2) are satisfied. We say that $x^* \in V$ is a correction point of $u$ iff there exist $t_1 < t^* < t_2$ such that

$$x^* = x_0(t^*), \qquad \tau\big(x_0(t_1)\big) = \tau\big(x_0(t_2)\big), \tag{3.3}$$

where $x_0(t)$ is defined in Lemma 5, and, with $t_1$ fixed, the function

$$c : t \mapsto n\big(x_0(t_1)\big) \cdot \tau\big(x_0(t)\big) \quad \text{has a negative minimum at } t^*. \tag{3.4}$$

For simplicity we assume this minimum to be isolated, i.e. there exists $\delta^* > 0$ such that

$$c(t) > c(t^*) \quad \text{if } t \in [t^* - \delta^*, t^* + \delta^*],\ t \neq t^*.$$

We postpone to Remark 3 the more general case. The reader may verify, by a continuity argument, that $t_1$ and $t_2$ can be chosen arbitrarily close to $t^*$. We consider $r > 0$ and $0 < \delta \leqslant \delta^*$ sufficiently small so that in the ball $B_r^*$ of center $x^*$ and radius $r$ the set $K_0 \cap B_r^*$ divides $B_r^*$ in two connected components, $K_+ \cap B_r^*$ and $K_- \cap B_r^*$, and, for all $t, t' \in [t^* - \delta, t^* + \delta]$, the level set $t$ of $u$ when intersected with $B_r^*$ is the graph of a function of a real variable defined on the reference frame $\mathfrak{R}_{x_0(t')}$. We shall denote these functions by $v_{x_0(t')}(s,t)$. The following relation is satisfied in $B_r^*$:

$$(s_x, v_x)_{x_0} = \big(s, v_{x_0}(s,t)\big)_{x_0} \quad \text{iff} \quad u(x) = t.$$

Note that by our regularity assumption on $u$ and by the Implicit Function Theorem we have that $v_{x_0}(s,t)$ is of class $C^3$ in the variables $s$ and $t$. We denote by $v'_{x_0}(s,t)$ the partial derivative $\frac{\partial v_{x_0}}{\partial s}(s,t)$.

We recall the curvature formula for the graph of a $C^2$-function $v$:

$$k(s) = \frac{v''(s)}{(1 + v'(s)^2)^{3/2}}. \tag{3.5}$$

By (3.5),

$$v''_{x_0}(s,t) > 0\ (<0) \quad \Leftrightarrow \quad \big(s, v_{x_0}(s,t)\big)_{x_0} \in K^+ \cap B_r^*\big(K^- \cap B_r^*\big), \tag{3.6}$$

$$v''_{x_0}(s,t) = 0 \quad \Leftrightarrow \quad \big(s, v_{x_0}(s,t)\big)_{x_0} \in K_0 \cap B_r^*. \tag{3.7}$$

**Lemma 6.** *There exist $0 < \delta' \leqslant \delta$, $0 < r' \leqslant r$ and $C > 0$ such that, for all $t, t' \in [t^* - \delta', t^* + \delta']$,*

$$v'''_{x_0(t')}(s,t) \geqslant C \quad \text{and} \quad \frac{\partial v_{x_0(t')}}{\partial t}(s,t) \geqslant C > 0, \tag{3.8}$$

*the derivatives being evaluated at*

$$\big\{s\colon \big(s, v_{x_0(t')}(s,t)\big)_{x_0(t')} \in B_{r'}^*\big\}.$$

**Proof.** Note that

$$\frac{\partial v_{x_0(t^*)}}{\partial t}(0,t^*) = |\nabla u|^{-1}\big(x_0(t^*)\big) \geqslant C_1 > 0.$$

By (3.2′) and (3.5), recalling that

$$v''_{x_0(t^*)}(0,t^*) = v'_{x_0(t^*)}(0,t^*) = 0,$$

we have,

$$(\nabla K \cdot \tau)\big(x_0(t^*)\big) = k'(0) = v'''_{x_0(t^*)}(0, t^*) \geqslant C_2 > 0.$$

Define $C := \frac{1}{2}\min\{C_1, C_2\}$. By the $C^3$ regularity of the level sets of $u$, in particular of the functions $v_{x_0(t^*)}(s,t)$, we may extend the previous inequalities to

$$\big\{s\colon \big(s, v_{x_0(t*)}(s,t)\big)_{x_0(t*)} \in B^*_{r'}\big\},$$

with $r' \leqslant r$. Moreover, we may suppose the above inequalities hold for all functions $v_{x_0(t')}$ with $t' \in [t^* - \delta', t^* + \delta']$ provided $\delta' \in ]0, \delta]$ is small. In fact, by the continuity of $x_0(t')$, $\tau(x_0(t'))$ and $n(x_0(t'))$, the graph of $v_{x_0(t')}(., t)$ is obtained from the one of $v_{x_0(t^*)}(., t)$ by a rotation of an angle $\theta(t')$ and a translation of some vector, that can be made arbitrarily small by choosing $t'$ sufficiently close to $t^*$. □

In the sequel, for fixed $x_0$ and $t$, we assume the domain $I$ of $v_{x_0}(., t)$ to be the maximal interval containing $s^*$, where $(s^*, v_{x_0}(s^*, t))_{x_0} \in K_0$, and such that

$$s \in I \quad \Rightarrow \quad \big(s, v_{x_0}(s,t)\big)_{x_0} \in B^*_{r'}.$$

We establish an auxiliary estimate on the class of functions with third derivative bounded below by a positive constant.

**Lemma 7.** *Let $I = [a, b]$ $(a < b)$, $C > 0$ and $w \in C^3(I)$ be such that*

$$w'''(s) > C \quad \forall s \in I. \tag{3.9}$$

*Assume that, for some $\epsilon > 0$, and $s^* \in ]a, b[$*

$$\min_I w'(s) = w'(s^*) = -\epsilon.$$

*If $s_1, s_2 \in I$, $s_1 \neq s_2$, are such that $w(s_1) = w(s_2)$, then*

$$\sqrt{(s_i - s^*)^2 + \big(w(s_i) - w(s^*)\big)^2} \leqslant \sqrt{\frac{24\epsilon}{C}\big(1 + \epsilon^2\big)} \quad (i = 1, 2).$$

**Proof.** For convenience we extend $w$ to the real line with $C^3$ regularity and lower bound $C$ for the third derivative. Assume

$$s_1 \leqslant s^* \leqslant s_2.$$

Noting that $w''(s^*) = 0$, write

$$0 = w(s_2) - w(s_1) = \int_{s_1}^{s^*} w'(s)\,ds + \int_{s^*}^{s_2} w'(s)\,ds$$

$$= \int_{s_1}^{s^*} \left( w'(s^*) + \int_{s^*}^{s} w''(y)\,dy \right) ds + \int_{s^*}^{s_2} \left( w'(s^*) + \int_{s^*}^{s} w''(y)\,dy \right) ds$$

$$\geqslant -\epsilon(s_2 - s_1) + C/24(s_2 - s_1)^3.$$

We conclude,

$$|s_i - s^*| \leqslant |s_2 - s_1| \leqslant \sqrt{\frac{24\epsilon}{C}} \quad (i = 1, 2).$$

In case $s^* < s_1 < s_2$ (or $s_2 < s_1 < s_*$), assumption (3.9) implies the existence of $s_1' < s^*$ ($s_1' > s^*$) such that $w(s_1') = w(s_2)$. In particular, $|s_1 - s^*| < |s_2 - s_1'|$. The lemma follows from

$$\left| w(s_i) - w(s^*) \right| \leqslant \epsilon |s_i - s^*|. \qquad \square$$

In the next theorem we relate the existence of a correction point to a variational property of $u$. Near the correction point we replace sections of the original level sets by parallel line segments (see Fig. 1, page 146) in such a way that the areas of the corresponding super-level sets are preserved. We prove that the rearranged function $\underline{u}$ satisfies the same volume constrains and that its Dirichlet integral is less than the one of $u$.

**Theorem 1** *(Local correction of level sets). Let $u \in C(\overline{\Omega}) \cap C^3(\Omega)$, $V \subset \Omega$ a domain where $u$ verifies* (3.1)–(3.2) *and $x^* \in V$ such that* (3.3)–(3.4) *are satisfied. Then there exists an open connected set $\omega$ containing $x^*$ whose boundary has zero measure and a function $\underline{u}$ in $C(\overline{\Omega}) \cap C^3(\Omega \backslash \overline{\omega}) \cap C^3(\omega)$ such that*

$$\underline{u} \equiv u \quad \text{in } \Omega \backslash \omega,$$

$$\int_{\Omega} |\nabla \underline{u}|^2 < \int_{\Omega} |\nabla u|^2,$$

*and for every continuous function $f$,*

$$\int_{\Omega} f(\underline{u}) = \int_{\Omega} f(u).$$

**Proof.** Take $\delta'$ and $r'$ as in Lemma 6 and choose $t_1, t_2$ in $[t^* - \delta', t^* + \delta']$ such that (3.3)–(3.4) are satisfied. We fix the reference frame $\mathfrak{R}_{x_0(t_1)}$. For simplicity, we denote $(s, v)_{x_0(t_1)}$ by $(s, v)$ and $v_{x_0(t_1)}(s, t)$ by $v(s, t)$.

Assumption (3.4) implies that

$$\min\left\{ v'(s, t) \right\} = v'(s_{x^*}, t^*) = -\epsilon(t_1),$$

where $\epsilon(t_1)$ is positive and can be made arbitrarily small as $t_1, t_2 \to t^*$. We choose $t_1, t_2$ to verify

$$d\big(x_0(t), \partial B^*_{r'}\big) > \sqrt{\frac{24\epsilon(t_1)}{C}\big(1+\epsilon^2(t_1)\big)} \quad \forall t \in [t_1, t_2], \tag{3.10}$$

where $d(.,.)$ is the distance function. In these conditions, (3.4), Lemmas 6 and 7 imply that the system

$$\int_{s_1(t)}^{s_2(t)} v(s,t)\,ds - v\big(s_1(t),t\big)\big(s_2(t)-s_1(t)\big) = 0, \tag{3.11}$$

$$v\big(s_1(t),t\big) - v\big(s_2(t),t\big) = 0 \tag{3.12}$$

has a non-trivial solution for $t = t^*$. By non-trivial we mean that there exist $s_1(t^*) < s_2(t^*)$ in the domain of $v(.,t^*)$ where the above equalities are verified. Geometrically, it implies that the section of the graphic of $v(.,t^*)$ between $s_1(t^*)$ and $s_2(t^*)$ can be replaced by an horizontal line segment in such a way that the continuity of $v$ and the area below the graphic are preserved.

*Step 1: Definition of a regular closed curve.*

In the reference frame $\mathfrak{R}_{x_0(t_1)}$ we define two regular paths, $\gamma_1(t)$ and $\gamma_2(t)$, such that

$$\gamma_i(t) := \big(s_i(t), v\big(s_i(t),t\big)\big) \in C\big([\tilde t_1, \tilde t_2], B^*_{r'}\big) \cap C^3\big(]\tilde t_1, \tilde t_2[, B^*_{r'}\big) \quad (i = 1,2),$$

with $t_1 \leqslant \tilde t_1 < \tilde t_2 \leqslant t_2$,

$$s_1(t) < s_2(t) \quad \text{for } t \in ]\tilde t_1, \tilde t_2[ \quad \text{and} \quad s_1(\tilde t_i) = s_2(\tilde t_i) \quad (i = 1,2),$$

and such that equalities (3.11)–(3.12) are verified. In fact, let

$$\phi_1(s_1,s_2,t) := \int_{s_1}^{s_2} v(s,t)\,ds - v(s_1,t)(s_2-s_1),$$
$$\phi_2(s_1,s_2,t) := v(s_2,t) - v(s_1,t).$$

By the Implicit Function Theorem, we have that

$$\big(\phi_1(s_1,s_2,t), \phi_2(s_1,s_2,t)\big) = (0,0) \tag{3.13}$$

defines locally $(s_1,s_2)$ as a function of $t$ provided the matrix

$$\begin{pmatrix} -v'(s_1,t)(s_2-s_1) & v(s_2,t)-v(s_1,t) \\ -v'(s_1,t) & -v'(s_2,t) \end{pmatrix}$$

is invertible at some $(s_1,s_2,t)$ satisfying (3.13). At those points, the determinant is given by

$$v'(s_1,t)v'(s_2,t)(s_2-s_1),$$

and assuming $s_1 < s_2$, it will suffice to prove

$$v'(s_1,t)v'(s_2,t) \neq 0.$$

We assert that this condition holds for all $s_1, s_2$ satisfying (3.13) with $s_1 < s_2$. In fact, since $v'''(t,s)$ is bounded below by a positive constant, there can be no more than two zeros for the first derivative $v'$ for every $t \in [t_1,t_2]$. Also $\phi_1 = 0$ and $s_1(t) < s_2(t)$ imply the existence of a local minimum and a local maximum necessarily different from $s_1(t)$ and $s_2(t)$. The assertion follows.

We implicitly define the regular function $(s_1(t), s_2(t))$ from (3.13) and the particular solution $(s_1(t^*), s_2(t^*))$. By Lemma 7 and (3.10), $\gamma_1(t)$ and $\gamma_2(t)$ remain in the ball $B^*_{r'}$ for all $t \in [t_1,t_2]$. Moreover, since $v(s,t_1)$ and $v(s,t_2)$ are strictly increasing functions of $s$ we conclude that the maximal domain of definition $]\tilde{t}_1, \tilde{t}_2[$ of $(s_1(t), s_2(t))$ must verify

$$]\tilde{t}_1, \tilde{t}_2[\, \subset [t_1,t_2] \quad \text{with } s_1(\tilde{t}_i) = s_2(\tilde{t}_i)\ (i=1,2).$$

Step 1 is concluded.

*Step 2: Definition of $\underline{u}$.*

Let

$$\omega := \bigl\{x \in B^*_{r'} \subset \Omega\colon\ u(x) \in\, ]\tilde{t}_1, \tilde{t}_2[,\ s_1\bigl(u(x)\bigr) < s_x < s_2\bigl(u(x)\bigr)\bigr\},$$

where $s_1(t), s_2(t)$ were defined in Step 1. In $\omega$, we replace the level sets of $u$ by parallel line segments. More specifically, let

$$\underline{u}(x) = \begin{cases} u(x) & \text{if } x \in \Omega \backslash \omega, \\ t & \text{if } x \in \omega \text{ and } v_x = v(s_1(t),t). \end{cases}$$

We show that

$$\underline{u} \in C(\overline{\Omega}) \cap C^3(\Omega \backslash \overline{\omega}) \cap C^3(\omega).$$

The continuity of $\underline{u}$ is an immediate consequence of its definition. Obviously,

$$\frac{\partial \underline{u}}{\partial s}(s,v) = 0 \quad \text{if } (s,v) \in \omega.$$

Moreover, for $t \in\, ]\tilde{t}_1, \tilde{t}_2[$ and $(s,v) \in \omega$ with $v = v(s_1(t),t)$, we have

$$\frac{\partial \underline{u}}{\partial v}(s,v) = \left[\frac{d}{dt}\bigl(v\bigl(s_1(t),t\bigr)\bigr)\right]^{-1}.$$

Differentiating (3.11) in the variable $t$ and canceling terms we conclude

$$\frac{d}{dt}\bigl(v\bigl(s_1(t),t\bigr)\bigr) = \bigl(s_2(t) - s_1(t)\bigr)^{-1} \int\limits_{s_1(t)}^{s_2(t)} \frac{\partial v}{\partial t}(s,t)\, ds.$$

Since $\frac{\partial v}{\partial t}$ is positive, regular and bounded away from zero (see Lemma 6) we conclude the regularity of $\underline{u}$ in $\omega$.

*Step 3: A coordinate system adapted to $\underline{u}$ in $\omega$.*

Let $X(\lambda, t)$ be a local coordinate system induced by $u$ at $x_0(t_1)$ (see Remark 1) such that

$$\omega \subset X\big(]-\tilde{\epsilon}, \tilde{\epsilon}[\times]t_1 - \tilde{\delta}, t_1 + \tilde{\delta}[\big) \quad (\tilde{\epsilon}, \tilde{\delta} > 0).$$

Then we may write,

$$\omega = \big\{(\lambda, t)\colon\ \lambda_1(t) < \lambda < \lambda_2(t),\ t \in ]\tilde{t}_1, \tilde{t}_2[\big\},$$

where $\lambda_1, \lambda_2$ are regular functions. Define, for $t \in ]\tilde{t}_1, \tilde{t}_2[$ and $\lambda \in ]\lambda_1(t), \lambda_2(t)[$

$$\underline{X} : (\lambda, t) \to \left(s_1(t) + \big(\lambda - \lambda_1(t)\big)\frac{s_2(t) - s_1(t)}{\lambda_2(t) - \lambda_1(t)}, v\big(s_1(t), t\big)\right). \tag{3.14}$$

Note that

$$\underline{u}\big(\underline{X}(\lambda, t)\big) = t \quad \text{for all } \lambda \in \big]\lambda_1(t), \lambda_2(t)\big[,$$

and that, similarly to $X$, $\underline{X}$ is a diffeomorphism whose Jacobian has modulus

$$|\underline{X}_\lambda||\nabla \underline{u}|^{-1}.$$

*Step 4: Estimates for $\int_\omega |\nabla \underline{u}|^2$ and $\int_\omega f(\underline{u})$.*

By the definition of $\underline{u}$, for $t \in ]\tilde{t}_1, \tilde{t}_2[$,

$$m\big(\big\{(x, y) \in \Omega\colon\ u(x, y) > t\big\} \cap \omega\big) = m\big(\big\{(x, y) \in \Omega\colon\ \underline{u}(x, y) > t\big\} \cap \omega\big),$$

or

$$\int_{\tilde{t}_1}^{t} \int_{\lambda_1(s)}^{\lambda_2(s)} \frac{|X_\lambda|}{|\nabla u|}\, d\lambda\, ds = \int_{\tilde{t}_1}^{t} \int_{\lambda_1(s)}^{\lambda_2(s)} \frac{|\underline{X}_\lambda|}{|\nabla \underline{u}|}\, d\lambda\, ds.$$

Since

$$t \mapsto \int_{\lambda_1(t)}^{\lambda_2(t)} |\underline{X}_\lambda||\nabla \underline{u}|^{-1}\, d\lambda$$

is a continuous function of $t$, we conclude

$$\int_{\lambda_1(t)}^{\lambda_2(t)} \frac{|X_\lambda|}{|\nabla u|}\,d\lambda = \int_{\lambda_1(t)}^{\lambda_2(t)} \frac{|\underline{X}_\lambda|}{|\nabla \underline{u}|}\,d\lambda. \tag{3.15}$$

Trivially,

$$\int_{\Omega\setminus\omega} |\nabla u|^2 = \int_{\Omega\setminus\omega} |\nabla \underline{u}|^2 \quad \text{and} \quad \int_{\Omega\setminus\omega} f(u) = \int_{\Omega\setminus\omega} f(\underline{u}).$$

By (3.15),

$$\int_\omega f(u) = \int_{\tilde{t}_1}^{\tilde{t}_2} f(t) \int_{\lambda_1(t)}^{\lambda_2(t)} \frac{|X_\lambda|}{|\nabla u|}\,d\lambda\,dt = \int_{\tilde{t}_1}^{\tilde{t}_2} f(t) \int_{\lambda_1(t)}^{\lambda_2(t)} \frac{|\underline{X}_\lambda|}{|\nabla \underline{u}|}\,d\lambda\,dt = \int_\omega f(\underline{u}),$$

so that the volume constraint is preserved. Moreover

$$\int_\omega |\nabla \underline{u}|^2 = \int_{\tilde{t}_1}^{\tilde{t}_2} \int_{\lambda_1(t)}^{\lambda_2(t)} |\underline{X}_\lambda||\nabla \underline{u}|\,d\lambda\,dt,$$

$$\int_\omega |\nabla u|^2 = \int_{\tilde{t}_1}^{\tilde{t}_2} \int_{\lambda_1(t)}^{\lambda_2(t)} |X_\lambda||\nabla u|\,d\lambda\,dt.$$

As in Section 2, define

$$\underline{S}(t) := \int_{\lambda_1(t)}^{\lambda_2(t)} |\underline{X}_\lambda||\nabla \underline{u}|\,d\lambda \cdot \int_{\lambda_1(t)}^{\lambda_2(t)} |\underline{X}_\lambda||\nabla \underline{u}|^{-1}\,d\lambda - \left(\int_{\lambda_1(t)}^{\lambda_2(t)} |\underline{X}_\lambda|\,d\lambda\right)^2,$$

$$S(t) := \int_{\lambda_1(t)}^{\lambda_2(t)} |X_\lambda||\nabla u|\,d\lambda \cdot \int_{\lambda_1(t)}^{\lambda_2(t)} |X_\lambda||\nabla u|^{-1}\,d\lambda - \left(\int_{\lambda_1(t)}^{\lambda_2(t)} |X_\lambda|\,d\lambda\right)^2,$$

and write

$$\int_\omega |\nabla \underline{u}|^2 = \int_{\tilde{t}_1}^{\tilde{t}_2} \frac{\underline{S}(t) + \underline{l}^2(t)}{\int_{\lambda_1(t)}^{\lambda_2(t)} |\underline{X}_\lambda||\nabla \underline{u}|^{-1}(\lambda, t)\,d\lambda}\,dt,$$

$$\int_\omega |\nabla u|^2 = \int_{\tilde{t}_1}^{\tilde{t}_2} \frac{S(t) + l^2(t)}{\int_{\lambda_1(t)}^{\lambda_2(t)} |X_\lambda||\nabla u|^{-1}(\lambda, t)\,d\lambda}\,dt,$$

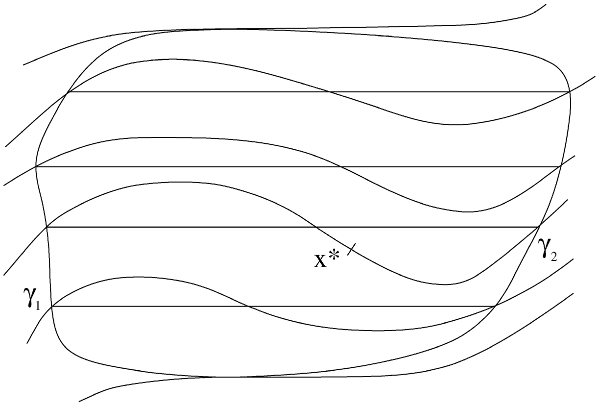


Fig. 1. Rearrangement of level sets near a correction point.

(here $l$ and $\underline{l}$ stand for the length of the level curve $t$ restricted to $\omega$ of $u$ and $\underline{u}$ respectively). Concerning the rearranged function $\underline{u}$ we have:

$$\underline{l} < l \quad \text{in } ]\tilde{t}_1, \tilde{t}_2[,$$

as line segments are geodesics in the plane. Also, by Cauchy–Schwarz inequality,

$$\underline{S}(t) = 0 \leqslant S(t)$$

(the equality follows from the invariance of the norm of the gradient along the rearranged section of the level set). Therefore

$$\int_\omega |\nabla \underline{u}|^2 < \int_\omega |\nabla u|^2.$$

This concludes the proof of Theorem 1. $\square$

Some remarks are now in order.

**Remark 3.** In case the function

$$c : t \mapsto n\big(x_0(t_1)\big) \cdot \tau\big(x_0(t)\big)$$

attains at $x_0(t^*)$ a negative non-isolated minimum the previous arguments can be adapted. If $t^* \in \overline{T_1^*} \cap \overline{T_2^*}$ where

$$T_1^* := \big\{t \colon\ t < t^* \text{ and } c(t) > c(t^*)\big\}, \tag{3.16}$$

$$T_2^* := \big\{t \colon\ t > t^* \text{ and } c(t) > c(t^*)\big\} \tag{3.17}$$

then the same approach as in Theorem 1 yields: we may choose $t_i \in T_i^*$ $(i = 1, 2)$ arbitrarily close to $t^*$ such that (3.3)–(3.4) are satisfied. If $\overline{T_1^*} \cap \overline{T_2^*} = \emptyset$ let

$$t_1^* = \sup T_1^* \quad \text{and} \quad t_2^* = \inf T_2^*.$$

Then $t_1^* < t_2^*$ and $c(t)$ is constant in $[t_1^*, t_2^*]$.

Take a finite covering of open balls of the set $x_0([t_1^*, t_2^*])$

$$V_n = \bigcup B^*_{\frac{1}{n}}$$

and $x_0(t_1), x_0(t_2) \in V_n$ satisfying (3.3)–(3.4). We may suppose $n$ large and $|t_i - t_i^*|$ $(i = 1, 2)$ small so that the level sets of $u$ intersected with $V_n$ are graphs of functions $v(., t)$ in $\mathfrak{R}_{x_0(t_1)}$ and, for some $C > 0$ (independent of $t_1$),

$$v'''_{x_0(t_1)}(x, t) \geqslant C > 0 \quad \text{and} \quad \frac{\partial v_{x_0(t_1)}}{\partial t}(x, t) \geqslant C,$$

for all $t \in [t_1, t_2]$. Since

$$v'_{x_0(t_1)}(., t) = -o(1),$$

where $o(1)$ is some positive quantity that can be made arbitrarily small as $t_i \to t_i^*$ $(i = 1, 2)$, we may define a correction set $\omega$ contained in $V_n$. The estimates in Theorem 1 prove the decreasing of the energy and the conservation of the volume.

**Remark 4.** The corrected function $\underline{u}$ used in the proof of Theorem 1 is continuous and differentiable almost everywhere with bounded energy $\int_\Omega |\nabla \underline{u}|^2$. In particular if $u$ is in $H^1(\Omega)$ or $H_0^1(\Omega)$ then $\underline{u}$ also belongs to these spaces. We may therefore apply our results to some constrained variational problems where $C^3$ regularity of the solution is expected. In the conditions of Lemma 5 and Theorem 1, we conclude that a minimizer $u$ of the Dirichlet integral under a constraint $\int_\Omega f(u) = 1$ has no correction points along the set $K_0$. Equivalently: the angle $\theta(t)$ of $\nabla u(x_0(t))$ with a fixed vector $\tau$, outward to $K^-$, has no local minima. In particular, if, for some $t_0 \in ]\alpha, \beta[$

$$\frac{d\theta}{dt}(t_0) < 0$$

then $\theta(t)$ is decreasing in $[t_0, \beta[$.

**Remark 5.** A rearrangement of isotherms by parallel line segments verifying (3.11)–(3.12) can be found in the study of Van der Waal's interpolation formula for the equation of estate. The method is applied to isotherms that are below a critical temperature and do not correspond to any real estate of matter (see [7, 84, pp. 260–262]).

Next, we list some classical boundary value problems where our result can be applied.

**Example 1** *(Dirichlet problem).* Let $\Omega$ be a regular domain, and $g \in C^{1,\alpha}(\mathbb{R})$ such that

$$\lim_{|u|\to\infty} \frac{g(u)}{|u|^{p-1}} = 0 \tag{3.18}$$

for some $p < 2^* - 1$. Let $u \in C^{3,\alpha}(\Omega)$ be a minimizer of $\int_\Omega |\nabla u|^2$ in $H_0^1(\Omega)$ under a volume constraint

$$\int_\Omega G(u) = 1$$

where $G(u) = \int_0^u g(s)\,ds$. If $V \subset \Omega$ is a domain where $u$ verifies conditions (3.1)–(3.2) then $V$ cannot contain a correction point.

**Example 2** *(Neumann problem).* Under the assumptions of the previous example, let $u \in C^{3,\alpha}(\Omega)$ be a minimizer of $\int_\Omega |\nabla u|^2$ over the class

$$\left\{u \in H^1(\Omega)\colon \int_\Omega u = 0,\ \int_\Omega G(u) = 1\right\}.$$

Then $V$ cannot contain a correction point.

**Example 3** *(Obstacle problem).* Let $\Omega$ be a regular domain and $\phi$ a positive continuous function with support strictly contained in $\Omega$. Let $u$ be a minimizer of $\int_\Omega |\nabla u|^2$ in $H_0^1(\Omega)$ over the class of functions

$$\left\{w \in H_0^1(\Omega)\colon\ w \geqslant \phi\right\}.$$

Let

$$W = \left\{x\colon\ u(x) > \phi(x)\right\},$$

and $V$ as in Example 1. Then $V \cap W$ cannot contain a correction point.

In the next proposition we observe that, if super-level sets loose convexity, although remaining connected, then optimal variational properties of a function $u$ can be affected. More precisely, we prove, under some non-degenerate assumptions, the existence of a correction point when a region where level sets have negative curvature is enclosed by some Jordan-type regular curve.

**Proposition 2.** *Let $u \in C^3(\Omega)$ and assume zero to be a regular value of the curvature function $K$. Moreover suppose $\nabla u(x) \neq 0$ for all $x \in K^{-1}(0)$ and (3.2) is verified, except at two points (necessarily where $u|_{K^{-1}(0)}$ attains its maximum and minimum). Let $x_0 \in K^{-1}(0)$ (with associated reference frame $\mathfrak{R}_{x_0}$) be such that $u(x_0)$ is minimum, and assume the following property to hold:*

$(C)$ *There exist a cone*

$$V^- := \left\{v \leqslant -\epsilon|s|\right\}_{\mathfrak{R}_{x_0}}$$

*such that for all $t \in \left]u(x_0) - \delta, u(x_0)\right[$ we have $[v'_{x_0}(.,t)]^{-1}(0) \cap V^- \neq \emptyset$.*

*Then $K^{-1}(0)$ contains a correction point.*

**Proof.** Let $x_0$ be the point where $u|_{K^{-1}(0)}$ is minimum. We fix the reference frame $\mathfrak{R}_{x_0}$ and consider the $t$-level set as the graph of a function $s \mapsto v(s,t)$ in some open ball containing the origin. The function $(s,t) \mapsto v(s,t)$ is of class $C^3$. Translating the $t$-variable, we may assume $(s, v(s,0))$ is the graph of the $u(x_0)$-level set. We have

$$v(0,0) = v'(0,0) = v''(0,0) = 0$$

and since $s \mapsto v''(s,0)$ has a minimum at 0 we also have

$$v'''(0,0) = 0.$$

We write the Taylor polynomial for $v(s,t)$ at $(0,0)$:

$$v(s,t) = v_t t + v_{st} st + \frac{1}{2} v_{tt} t^2 + \frac{1}{2} v_{sst} s^2 t + \frac{1}{2} v_{stt} st^2 + \frac{1}{6} v_{ttt} t^3 + \rho_3(s,t) \tag{3.19}$$

where $v_{s^i t^j}$ stand for the $i,j$-partial derivative at the origin and

$$\lim_{(s,t)\to(0,0)} \rho_k(s,t)/\big|(s,t)\big|^k = 0.$$

Also,

$$v'(s,t) = v_{st} t + v_{sst} st + \frac{1}{2} v_{stt} t^2 + \rho_2.$$

Since, by our assumptions,

$$\nabla K(x_0) = -\lambda \nabla u(x_0) \quad \text{with } \lambda > 0,$$

we have

$$v_{sst}(0,0) < 0. \tag{3.20}$$

We claim that $v_{st} = 0$. In fact, if we assume $v_{st} \neq 0$ we write

$$v'(s,t) = t\left(v_{st} + v_{sst} s + \frac{1}{2} v_{stt} t\right) + \rho_2, \tag{3.21}$$

and conclude that the set $v'(s,t) = 0$ is tangent to the line $t = 0$, contradicting $(C)$.

From (3.20) and (3.21) we conclude the existence of $s_a < 0 < s_b$ and $t > 0$, such that

$$v'(s_a,t) > 0 > v'(s_b,t) \quad \text{and} \quad v''(s_a,t) = v''(s_b,t) = 0,$$

or denoting by $x_a$ ($x_b$) the point of $K^{-1}(0)$ with coordinates $(s_a, v(s_a,t))$ (resp. $(s_b, v(s_b,t))$) we have

$$\tau(x_a) \cdot n(x_0) < 0 \quad \big(\tau(x_b) \cdot n(x_0) < 0\big). \tag{3.22}$$

Let $x_1$ be the point where $u|_{K^{-1}(0)}$ attains its maximum. A tangent $\tau(x_1)$ to the level set $u(x_1)$ is also tangent to $K^{-1}(0)$. Choose $\tau(x_1)$ to verify

$$\tau(x_1) \cdot n(x_0) \geqslant 0.$$

Define a parameterization $x_0(l)$ of $K^{-1}(0)$ such that

$$x_0(0) = x_0, \qquad x_0(1) = x_1, \qquad x_0'(1) \cdot \tau(x_1) < 0.$$

Then, by (3.22), the function

$$l \mapsto n(x_0) \cdot \tau\big(x_0(l)\big)$$

attains a negative minimum in ]0, 1[ at some point $l^*$. In particular $x_0(l^*)$ is a correction point of $u$. □

**Remark 6.** A similar proposition could be stated imposing non-degenerated assumptions like $(C)$ at $x_1$.

**Remark 7.** In a previous work [4], the author proved the existence of a minimizer $\underline{u}$ of the Dirichlet integral over the class of quasi-concave functions of $H_0^1(\Omega)$ verifying $\int_\Omega {u_+}^{p+1}(x)\,dx = 1$ (we recall that a function is said quasi-concave when its super-level sets are convex). The proof can be easily adapted to the more general case presented in Remark 1 or to the obstacle problem. Sufficient conditions were provided for $\underline{u}$ to be a classical ground-state solution. The present work, namely Proposition 2, was in part motivated by the following question: can a non-quasi-concave function, obtained as a slight perturbation of this "over-constrained" minimizer, be corrected in such way that its energy decreases and the volume is preserved?

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