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Vertex-deleted subgraphs and regular factors from regular graph[☆]

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ABSTRACT

Let k, m, and r be three integers such that $2 \leq k \leq m \leq r$. Let G be a $2r$-regular, $2m$-edge-connected graph of odd order. We obtain some sufficient conditions for $G - v$ to contain a k-factor for all $v \in V(G)$.

1. Introduction

All graphs considered are multigraphs (with loops allowed) and finite. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The number of vertices of a graph G is the *order* of G and is denoted by n. On the other hand, the number of edges of G is the *size* of G and is denoted by e. We denote the degree of vertex v in G by $d_G(v)$. For two subsets $S, T \subseteq V(G)$, let $e_G(S, T)$ denote the number of edges of G joining S to T.

Let $c(G)$ and $c_o(G)$ denote the number of components and the number of odd components of G, respectively. Let k be a positive integer. A *k-factor* of a graph G is a spanning subgraph H of G such that $d_H(x) = k$ for every $x \in V(G)$. Let D and S be disjoint subsets of $V(G)$ and C be a component of $G - (D \cup S)$. C is a *k-odd component* of $G - (D \cup S)$ if $k|C| + e_G(S, V(C))$ is odd.

Petersen obtained the following theorem, which is chronologically the first result on k-factors in regular graphs.

Theorem 1.1 (*Petersen [3]*). *Every 3-regular, 2-connected graph has a 1-factor.*

For the existence of 1-factors in arbitrary graphs, Tutte also characterized graphs having k-factors.

Theorem 1.2 (*Tutte [4]*). *A graph G has a 1-factor if and only if $c_o(G - S) \leq |S|$ for all $S \subseteq V(G)$.*

The following theorem is the well-known Tutte's k-factor Theorem.

Theorem 1.3 (*Tutte [5]*). *Let k be a positive integer. A graph G has a k-factor if and only if, for all $D, S \subseteq V(G)$ with $D \cap S = \emptyset$,*

$$\delta_G(D, S) = k|D| + \sum_{x \in S} d_G(x) - k|S| - e_G(D, S) - q_G(D, S; k) \geq 0,$$

where $q_G(D, S; k)$ is the number of k-odd components C of $G - (D \cup S)$. Moreover, $\delta_G(D, S) \equiv k|V(G)| \pmod 2$.

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The following theorem examines the existence of a 1-factor in vertex-deleted subgraphs of a regular graph.

Theorem 1.4 (*Little et al. [2]*). *Let G be a $2r$-regular, $2r$-edge-connected graph of odd order. For any vertex u in G, then graph $G-u$ has a 1-factor.*

Katerinis presented the following result, which generalizes Theorem 1.4.

Theorem 1.5 (*Katerinis [1]*). *Let G be a $2r$-regular, $2r$-edge-connected graph of odd order. If m is an integer such that $1 \le m \le r$, then $G-u$ has an m-factor for any vertex $u \in V(G)$.*

In this paper, we set independent values to the degree and the edge-connectivity. We prove the following theorem, which generalizes Theorem 1.5. Furthermore, we also show that each hypothesis cannot be weakened.

Theorem 1.6. *Let m and r be two integers such that $2 \le m \le r$. Let G be a $2r$-regular, $2m$-edge-connected graph with odd order. If one of the following conditions holds, then $G-v$ has a k-factor for all $v \in V(G)$.*

(i) *k is even and $2 \le k \le m$;*
(ii) *k is odd, $3 \le k \le m$ and $2m > r$.*

Our results reveal an interesting behavior of a k-factor with respect to the parity of k and a difference between the case $k = 1$ and $k > 1$. These differences are not apparent in 1.5.

2. The proof of Theorem 1.6

In this section, we give the proof of Theorem 1.6 and show that each hypothesis cannot be weakened.

The Proof of Theorem 1.6. Suppose that the result does not hold. Now there exists $u \in V(G)$ such that $G-u$ contains no k-factor. Let $H = G-u$. By Theorem 1.3, there exist disjoint subsets D and S of $V(G)-u$ such that

$$q_H(D,S;k) + \sum_{x\in S}(k - d_{H-D}(x)) \ge k|D| + 2. \tag{1}$$

Define $S' = S \cup \{u\}$ and $W = (G-D) - S'$.

Claim 1. $c(W) \ge 2$.

Otherwise, suppose $c(W) \le 1$. We consider two cases.

Case 1. $c(W) = 0$.

Since $c(W) \ge q_H(D,S;k)$, (1) implies

$$\sum_{x\in S}(k - d_{H-D}(x)) \ge k|D| + 2. \tag{2}$$

So $k|S| \ge k|D| + 2$ and hence $|S| > |D|$. Since $V(H) = D \cup S$ and $|V(H)|$ is even, therefore

$$|S| \ge |D| + 2. \tag{3}$$

Now since G is $2r$-regular, by Tutte's Theorem we have

$$\sum_{x\in S'}(2r - d_{G-D}(x)) \le 2r|D|,$$

which implies

$$2r|S'| - \sum_{x\in S'} d_{G-D}(x) \le 2r|D|.$$

Therefore,

$$2r(|S|+1) - \sum_{x\in S'} d_{G-D}(x) \le 2r|D|,$$

and hence,

$$(2r-k)|S| + k|S| + 2r - \sum_{x\in S'} d_{G-D}(x) \le k|D| + (2r-k)|D|. \tag{4}$$

Also

$$\begin{aligned}\sum_{x\in S'} d_{G-D}(x) &= \sum_{x\in S} d_{G-D}(x) + d_{G-D}(u)\\ &= \sum_{x\in S} d_{H-D}(x) + e_G(u,S) + d_{G-D}(u).\end{aligned}$$

Therefore (4) becomes

$$k|D| + (2r-k)|D| \geq (2r-k)|S| + k|S| + 2r - \sum_{x\in S} d_{H-D}(x) - e_G(u,S) - d_{G-D}(u). \tag{5}$$

Now using (2) and $e_G(u,S) \leq d_{G-D}(u) \leq 2r$, (5) implies

$$(2r-k)(|S|-|D|) \leq 2r-2. \tag{6}$$

Moreover, since $|S| \geq |D|+2$ by (3), we can conclude from (6) that $k \geq r+1$. That is a contradiction, so Case 1 cannot occur.

Case 2. $c(W) = 1$.

Now $q_H(D,S;k) \leq 1$, and this implies

$$\sum_{x\in S}(k - d_{H-D}(x)) \geq k|D| + 1. \tag{7}$$

We conclude $k|S| \geq k|D| + 1$, and hence $|S| > |D|$.

Since G is a $2m$-edge-connected, $2r$-regular graph, we have

$$\begin{aligned}2r|D| &\geq e_G(D, V(G-D)) = e_G(D, V(W)) + e_G(D, S')\\ &= e_G(D\cup S', V(W)) - e_G(S', V(W)) + e_G(D, S')\\ &= e_G(D\cup S', V(W)) - \left(\sum_{x\in S'} d_{G-D}(x) - 2e_G(S',S')\right) + \left(2r|S'| - \sum_{x\in S'} d_{G-D}(x)\right)\\ &= e_G(D\cup S', V(W)) - 2\sum_{x\in S'} d_{G-D}(x) + 2e_G(S',S') + 2r|S'|\\ &\geq 2m - 2\left(\sum_{x\in S} d_{H-D}(x) + e_G(u,S) + d_{G-D}(u)\right) + 2e_G(S',S') + 2r|S'|\\ &= 2m - 2\sum_{x\in S} d_{H-D}(x) - 2d_{G-D}(u) + 2e_G(S,S) + 2r|S'|\\ &\geq 2m - 2\sum_{x\in S} d_{H-D}(x) + 2r|S| - 2r.\end{aligned}$$

Now (7) implies

$$\begin{aligned}2r|D| &\geq 2m - 2\sum_{x\in S} d_{H-D}(x) - 2r + 2r|S|\\ &\geq 2m - 2(k|S| - k|D| - 1) - 2r + 2r|S|.\end{aligned}$$

Thus

$$(2r-2k)(|D|-|S|) \geq 2m - 2r + 2 \geq 2k - 2r + 2,$$

from which it follows that $|D| \geq |S|$, a contradiction. Hence Case 2 also cannot occur.

We conclude that $c(W) \geq 2$. Denote the components of W by $C_1, \ldots, C_{c(W)}$. Suppose that $e_G(C_1, D\cup S') \leq \cdots \leq e_G(C_{c(W)}, D\cup S')$.

First assume condition (i) of the hypothesis. We have

$$\begin{aligned}2r|D| &\geq e_G(D\cup S', V(W)) - 2\sum_{x\in S'} d_{G-D}(x) + 2e_G(S',S') + 2r|S'|\\ &= e_G(D\cup S', V(W)) - 2\left(\sum_{x\in S} d_{H-D}(x) + e_G(u,S) + d_{G-D}(u)\right) + 2e_G(S',S') + 2r|S'|\\ &= e_G(D\cup S', V(W)) - 2\sum_{x\in S} d_{H-D}(x) - 2d_{G-D}(u) + 2e_G(S,S) + 2r|S'|\end{aligned}$$

$$\geq 2mc(W) - 2\sum_{x\in S} d_{H-D}(x) - 2r + 2r|S| \quad \text{(since } G \text{ is } 2m\text{-connected)}$$
$$\geq (2m-2)c(W) + (2r-2k)|S| - 2r + 2k|D| + 4 \quad \text{(by inequality (1))}$$
$$\geq 4m - 2r + (2r-2k)|S| + 2k|D|.$$

Thus we have $(2r-2k)(|D|-|S|+1) \geq 2m$, from which it follows $|D| \geq |S|$. For every odd component C of W, the integer $k|V(C)| + e_H(V(C), S)$ is odd. Since k is an even integer, $e_H(V(C), S)$ must be odd. Thus $e_H(V(C), S) \geq 1$ and $\sum_{x\in S} d_{H-D}(x) \geq q_H(D, S; k)$. Hence (1) implies $k|S| \geq k|D| + 2$, from which it follows that $|S| \geq |D| + 1$, a contradiction.

Next assume condition (ii) of the hypothesis; that is, $k \geq 3$ is odd and $2m > r$. Now

$$2r|D| \geq e_G(D \cup S', V(W)) - 2\sum_{x\in S'} d_{G-D}(x) + 2e_G(S', S') + 2r|S'|$$
$$\geq (2m-2)c(W) - 4r + (2r-2k)|S| + 2r + 2k|D| + 4$$
$$\geq 4m - 2r + (2r-2k)|S| + 2k|D|$$
$$\geq (2r-2k)|S| + 2k|D| + 2.$$

Thus we have $(2r-2k)(|D|-|S|) \geq 4m - 2r \geq 2$ and hence $|D| > |S|$. Let $q = q_H(D, S; k)$. Note that

$$2r|D| \geq 2mq + 2r|S| - 2r - 2\sum_{x\in S} d_{H-D}(x).$$

We obtain

$$|D| - |S| \geq \frac{m}{r}q - 1 - \frac{1}{r}\sum_{x\in S} d_{H-D}(x). \tag{8}$$

By (1), we have

$$|D| - |S| \leq \frac{1}{k}\left(q - \sum_{x\in S} d_{H-D}(x) - 2\right), \tag{9}$$

and $q \geq k+2$ since $|D| > |S|$. By (8) and (9), we have

$$\begin{aligned}
0 &\leq \left(\frac{1}{k} - \frac{1}{r}\right)\sum_{x\in S} d_{H-D}(x) \\
&\leq \frac{q}{k} - \frac{2}{k} - \frac{mq}{r} + 1 \\
&< \frac{q}{k} - \frac{2}{k} - \frac{q}{2} + 1 \\
&\leq q\left(\frac{1}{k} - \frac{1}{2}\right) - \frac{2}{k} + 1 \\
&\leq (k+2)\left(\frac{1}{k} - \frac{1}{2}\right) - \frac{2}{k} + 1 \\
&= 1 - k/2 < 0,
\end{aligned}$$

a contradiction. This completes the proof. □

In the following discussion, let Γ be the graph obtained from the complete graph K_{2r+1} by deleting a matching of size m.

The bounds are sharp. First, we show that the upper bound is sharp. Let G_1 be the bipartite graph with bipartition (U, W) obtained by deleting a matching of size m from $K_{2r,2r}$. Let G be the $2r$-regular graph obtained by matching the $2m$ vertices of degree $2r-1$ in Γ to $2m$ vertices of degree $2r-1$ in G_1. Clearly, G is $2m$-edge-connected. Let $m^* \geq m+1$. Now we show that $G-u$ contains no m^*-factor for all $u \in U \cup W$. Without loss of generality, suppose that $u \in U$. Let $D = U - u$, $S = W$ and $G' = G - u$. Note that $q_{G'}(D, S; m^*) = 1$ if $m^* \not\equiv m \pmod 2$ and $q_{G'}(D, S; m^*) = 0$ if $m^* \equiv m \pmod 2$. Now

$$m^*|D| - m^*|S| + \sum_{x\in S} d_{G'-D}(x) - q_{G'}(D, S; m^*) \leq -2 < 0.$$

By Theorem 1.3, $G-u$ contains no m^*-factor.

Next we show that the lower bound is sharp. Let M_{r-1} be a matching of size $r-1$. Let H be the $2r$-regular graph obtained by matching the $2r-2$ vertices of degree $2r-1$ in each of $2r-1$ disjoint copies of Γ to the vertex set S of M_{r-1}. Clearly, H is $2r$-regular $(2r-2)$-edge-connected graph. Since $c_o(H - V(M_{r-1})) = 2r-1 > |V(M_{r-1}) - v| = 2r-3$ for all $v \in V(M_{r-1})$, by Theorem 1.2, $G - v$ contains no 1-factor for all $v \in V(M_{r-1})$. So the lower bound is sharp.

Finally, we show that the condition $2m > r$ is sharp. Otherwise, suppose that $2m \le r$. Let R_1 denote the complete bipartite graph $K_{2r,2r-1}$ with bipartition (U, W), where $|U| = 2r$ and $|W| = 2r - 1$. Take two copies of Γ. Match $4m$ vertices of degree $2r - 1$ of two copies of Γ to $4m$ vertices of degree $2r - 1$ of $K_{2r,2r-1}$, and then add a matching of size $r - 2m$ to the rest vertices of degree $2r - 1$ of $K_{2r,2r-1}$. This produces a $2r$-regular, $2m$-edge-connected graph R. Let $u \in U$ and $R' = R - u$. Let $D = U - u$ and $S = W$. Since k is odd, $q_{R'}(D, S; k) = 2$ and hence we have

$$k|D| - k|S| + \sum_{x\in S} d_{R'-D}(x) - q_{G'}(D, S; k) = -2 < 0.$$

By Theorem 1.3, $G - u$ contains no k-factor.

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