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Graphs without large triangle free subgraphs

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Abstract

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The main aim of the paper is to show that for $2 \leqslant r < s$ and large enough n, there are graphs of order n and clique number less than s in which every set of vertices, which is not too small, spans a clique of order r. Our results extend those of Erdős and Rogers.

Consider the set of graphs of order n not containing a K^s, a complete graph of order s, as a vertex induced subgraph. What is the maximum number of vertices, $f_{r,s}(n)$, such that any graph in our set contains a vertex induced subgraph of order $f_{r,s}(n)$ not containing a K^r as a vertex induced subgraph?

This problem, which is essentially a problem of Ramsey Theory, was first considered by Erdős and Rogers [5] in 1961, when they showed that there exist graphs of order n, not containing a K^s, such that every vertex induced subgraph of order more than $n^{1-\epsilon_s}$, contains a K^{s-1}. The value of ϵ_s obtained was $\epsilon_s \sim 1/(512 s^4 \log s)$ for large values of s. The main aim of this paper is to improve this result.

The notation used will be standard (see [1]) and as is customary, the symbols c, c_i, c_i', . . . will be used to denote constants. Most of our proofs will make use of the theory of random graphs; for an introduction to the subject see [2].

For a given graph G, define

$$h_r(G) = \max\{|W|: W \subset V(G), \operatorname{cl}(G[W]) \leqslant r-1\}.$$

That is to say, $h_r(G)$ is the order of the largest subset of $V(G)$ for which the corresponding vertex-induced subgraph of G does not contain a K^r. For $2 \leqslant r < s$

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define

$$f_{r,s}(n) = \min\{h_r(G): \mathrm{cl}(G) \leqslant s-1,\ |G| = n\}.$$

Note that $f_{2,s}(n)$ is intimately connected with the Ramsey number $R(s, t)$ (see [1, p. 103] or for a comprehensive introduction to Ramsey Theory, see [6]). To be precise

$$f_{2,s}(N) = \max\{t: R(s, t) \leqslant N\}$$

and

$$R(s, t) = \min\{N:\ f_{2,s}(N) \geqslant t\}.$$

The function $f_{r,s}(n)$ can thus be viewed as a generalized Ramsey function. The method of proof we shall use in order to give an upper bound for $f_{r,s}(n)$ is similar to that used by Erdős [3, 4] in his attack on the Ramsey number $R(s, t)$.

Initially we shall be concerned with the function $f_{3,4}(n)$. Our lower bound for $f_{3,4}(n)$ is essentially trivial.

Theorem 1. *If $n > 4$ then $f_{3,4}(n) \geqslant (2n)^{1/2}$.*

Proof. Let G be a graph of order n with $\mathrm{cl}(G) \leqslant 3$ and let $x \in V(G)$ be a vertex of maximal degree, $d_G(x) = \Delta(G)$. Define $W = \Gamma_G(x)$. It is clear that $\mathrm{cl}(G[W]) \leqslant 2$. Therefore in proving the theorem we may assume $\Delta(G) < (2n)^{1/2}$.

Since $n-1 > (2n)^{1/2}$ for $n > 4$, the graph G is not a complete graph. Furthermore, since $(2n)^{1/2} > 3$ for $n > 4$, the graph G is not an odd cycle with maximal degree at least $(2n)^{1/2} - 1$. Thus Brooks' Theorem guarantees that the graph is k-vertex-colourable for some $k < (2n)^{1/2}$. Let W_1 and W_2 be colour classes in a k-vertex-colouring of G such that $|W_1 \cup W_2|$ is maximal. Then

$$|W_1 \cup W_2| \geqslant 2\left(\frac{n}{k}\right) > (2n)^{1/2}$$

and $G[W_1 \cup W_2]$ contains no K^3. $\square$

Before establishing an upper bound a few definitions are required. Let $\boldsymbol{H}^{(3)}(n, p)$ be the probability space of 3-uniform hypergraphs with vertex set $V = [n] = \{1, 2, \ldots, n\}$ in which a 3-set of vertices is chosen to be a hyperedge with probability p, and independently of the choice for any other 3-set. Let H_p be a random element of $\boldsymbol{H}^{(3)}(n, p)$. To each such $H = H_p \in \boldsymbol{H}^{(3)}(n, p)$ we associate a graph $G = G_H$ on vertex set V in which a vertex i and a vertex j are joined by an edge if some hyperedge of H contains $\{i, j\}$. Note that two distinct hypergraphs H and H' in $\boldsymbol{H}^{(3)}(n, p)$ may have the same associated graph.

Let $\boldsymbol{G}^{(3)}(n, p)$ be the probability space of graphs obtained in this way, and write $G_p^{(3)}$ for a random element of this space. Thus for every graph G_0 on V we have

$$P(G_p^{(3)} = G_0) = P(H \in \boldsymbol{H}^{(3)}(n, p)\colon G_H = G_0).$$

We shall later show that for any $\epsilon > 0$ and n sufficiently large, $f_{3,4}(n) \leq n^{7/10+\epsilon}$, but first we give a flavour of the proofs by proving a weaker result.

Theorem 2. *If n is sufficiently large, then*

$$f_{3,4}(n) \leq (n \log n)^{3/4}.$$

Proof. For a given n, select k to be the greatest positive integer for which

$$n \geq k + 2\left\lfloor \frac{k}{(\log k)^{2/3}} \right\rfloor,$$

and select ϵ and p such that

$$\epsilon = \frac{\log \log n}{3 \log n}$$

and

$$p = n^{-(3/2)-\epsilon} = \frac{n^{-3/2}}{(\log n)^{1/3}}.$$

Note that if n is sufficiently large, then

$$k \leq n - \lfloor n^{1-2\epsilon} \rfloor \quad \text{and} \quad \tfrac{1}{2}(n \log n)^{3/4} \leq (k \log k)^{3/4}.$$

We shall assume that these inequalities hold.

(i) Let Y be the random variable on $\boldsymbol{G}^{(3)}(n, p)$ defined by putting $Y(G) = k_4(G)$, i.e., $Y(G)$ is the number of K^4's contained in G. We now estimate $E(Y)$, the expected value of Y. What is the probability that $G = G_p^{(3)}$, a random element of $\boldsymbol{G}^{(3)}(n, p)$, contains a given K^4, K_0 say, with vertex set $V(K_0) = W = \{x_1, x_2, x_3, x_4\}$? Let $H = H_p \in \boldsymbol{H}^{(3)}(n, p)$ be such that $G = G_H$. Then K_0 is a subgraph of G if one of the following four cases occurs.

(a) W contains three distinct hyperedges of H. This occurs with probability $4p^3$.

(b) W contains two distinct hyperedges of H, say σ and σ' and the vertex pair $\{x_i, x_j\} = \sigma \triangle \sigma'$ is contained in a hyperedge σ'' such that $\sigma'' \cap W = \{x_i, x_j\}$. This occurs with probability $6p^2(1 - (1-p)^{n-4}) = O(p^2(pn))$.

(c) W contains one hyperedge of H, $\sigma = \{x_{i_1}, x_{i_2}, x_{i_3}\}$ say. Furthermore, letting $\{x_{i_4}\} = W \setminus \sigma$, the vertex pairs $\{x_{i_1}, x_{i_4}\}$, $\{x_{i_2}, x_{i_4}\}$ and $\{x_{i_3}, x_{i_4}\}$ are each contained in a distinct hyperedge of H meeting W in exactly that pair of vertices. This occurs with probability $4p(1 - (1-p)^{n-4})^3 = 4p(pn + O(p^2n^2))^3$.

(d) Finally, the six distinct vertex pairs in W are each contained in a distinct hyperedge of H meeting W in exactly that pair of vertices. This occurs with probability $(1 - (1-p)^{n-4})^6 = O((pn)^6)$.

Recall that $p = n^{-(3/2)-\epsilon}$. Thus the probability that $G = G_p^{(3)}$ contains K_0 as a subgraph is $4n^{-3-4\epsilon} + O(n^{-3-6\epsilon})$, and

$$E(Y) = \binom{n}{4} P(K_0 \subset G_p^{(3)}) = \tfrac{1}{6} n^{1-4\epsilon} + O(n^{1-6\epsilon}).$$

Then, as a consequence of Markov's inequality, a.e. $G_p^{(3)}$ is such that $Y(G_p^{(3)}) \leq n^{1-2\epsilon}$.

(ii) Let Z be the random variable on $\boldsymbol{H}^{(3)}(n, p)$ whose value for $H \in \boldsymbol{H}^{(3)}(n, p)$ is the number of m-sets of vertices containing no hyperedge of H, where $m = \lfloor \frac{1}{2}(n \log n)^{3/4} \rfloor = \lfloor n^{(3/4)+\eta} \rfloor$.

Then

$$E(Z) = \binom{n}{m}(1-p)^{\binom{m}{3}}$$

$$\leq \exp\left\{ m \left[\log(2en^{(1/4)-\eta}) - \frac{p}{6} n^{(3/2)+2\eta} + pn^{(3/4)+\eta} \right] \right\} \leq e^{-m} = o(1)$$

if n is large enough, since

$$(2\eta - \epsilon)\log n > \log \log n + 1$$

for large n, and so

$$\tfrac{1}{4} \log n + 2 < \tfrac{1}{6} n^{-\epsilon+2\eta}.$$

This shows (again using Markov's inequality) that almost every $H_p \in \boldsymbol{H}^{(3)}(n, p)$ satisfies the condition that every m-set of vertices contains at least one hyperedge of H_p.

(iii) Finally by (i) and (ii), there is an $H \in \boldsymbol{H}^{(3)}(n, p)$ such that $k_4(G_H) \leq n^{1-2\epsilon}$ and every m-set of vertices contains at least one hyperedge of H.

Now choose $U \subset V$ to be a set of $n - k \geq \lfloor n^{1-2\epsilon} \rfloor$ vertices meeting every K^4 in G_H in at least one vertex. Set $G = G_H \setminus U$. Then $|G| = k$; the graph G does not contain a K^4 and $h_3(G) \leq m \leq \frac{1}{2}(n \log n)^{3/4} \leq (k \log k)^{3/4}$, completing the proof. □

To obtain a better upper bound for $f_{3,4}(n)$, we need a little more care.

Consider $\boldsymbol{H}^{(3)}(n, p)$, the probability space of 3-uniform hypergraphs with vertex set $V = [n]$, as described above. For each $H \in \boldsymbol{H}^{(3)}(n, p)$, let

$$D(H) = \{\tau \subset V: |\tau| = 2 \text{ and } \tau \subset \sigma \text{ for some } \sigma \in E(H)\}$$

and

$$F(H) = \{\mu \subset V:\ |\mu| = 4 \text{ and } \mu^{(2)} \subset D(H)\}.$$

Thus $D(H)$ is the edge set of the graph G_H and $F(H)$ is the family of 4-sets of vertices which induce a K^4 in G_H. For each $H \in \boldsymbol{H}^{(3)}(n, p)$ define a function

$$g_H : D(H) \to \mathbb{N}$$

by putting

$$g_H(\tau) = |\{\mu \in F(H): \tau \subset \mu\}|.$$

Finally, let Z_k be the random variable on $\boldsymbol{H}^{(3)}(n, p)$ defined by

$$Z_k(H) = |\{\tau \in D(H): g_H(\tau) \geqslant k\}|$$

i.e., $Z_k(H)$ is the number of edges of G_H, each of which is an edge of at least k distinct K^4's.

Lemma 3. *Let* $\delta > 0$, *let* $p = n^{-(7/5)-\delta}$ *and let* $k \geqslant \max\{\lceil 8/25\delta \rceil, 3\}$. *Then* $E(Z_k) = o(1)$.

Proof. If $\delta \geqslant \frac{1}{10}$ and $k = 3$, the result follows simply since the expected size of the set $F(H_p)$ is small. Therefore assume that $0 < \delta < \frac{1}{10}$. Let $p = n^{-(7/5)-\delta}$ and consider the probability space $\boldsymbol{H}^{(3)}(n, p)$. Suppose $k \geqslant \lceil 8/25\delta \rceil$. For a vertex pair $\tau = \{i, i'\} \in V^{(2)}$, let A_τ be the event that $\tau \in D(H)$ and that there exist k sets $\mu_1, \mu_k, \ldots, \mu_k \in F(H)$ such that $\tau \subset \mu_j$ for each $j \in \{1, 2, \ldots, k\}$. Then

$$E(Z_k) = \binom{n}{2} P(A_\tau).$$

Let $\{i_1, i_2, \ldots, i_l\} \subseteq V$. Let $B_\tau(i_1, \ldots, i_l)$ be the event that $\tau \in D(H)$ and that there exist k sets $\mu_1, \mu_2, \ldots, \mu_k \in F(H)$ such that $\tau \subset \mu_j$ for each $j \in \{1, 2, \ldots, k\}$ and $\bigcup_{j=1}^{k} \mu_j = \{i_1, \ldots, i_l\} \cup \{i, i'\}$. Without loss of generality, we may assume that $l \leqslant 2k$, since $P(B_\tau(i_1, \ldots, i_l)) = 0$ for $l > 2k$. So

$$P(A_\tau) \leqslant \sum_{l \leqslant 2k} \binom{n}{l} P(B_\tau(i_1, i_2, \ldots, i_l)).$$

If $H \in B_\tau(i_1, \ldots, i_l)$ then for each $j \in \{1, 2, \ldots, l\}$, both the vertex pair $\{i, i_j\}$ and the vertex pair $\{i', i_j\}$ must be in $D(H)$. Furthermore, at least k vertex pairs of the form $\{i_j, i_{j'}\}$ where $j, j' \in \{1, 2, \ldots, l\}$ must be in $D(H)$.

Let $C_\tau(i_1, \ldots, i_l; m)$ be the event that there exist exactly m distinct hyperedges $\sigma_1, \ldots, \sigma_m$ in $E(H)$ of the form $\sigma = \{i, i_j, i_{j'}\}$ or $\sigma = \{i', i_j, i_{j'}\}$ where $j, j' \in \{1, 2, \ldots, l\}$. Then noting that $m \leqslant 2\binom{l}{2}$, it follows that

$$\begin{aligned} &P(B_\tau(i_1, i_2, \ldots, i_l)) \\ &\quad = \sum_{m=0}^{2\binom{l}{2}} P(B_\tau(i_1, \ldots, i_l) \mid C_\tau(i_1, \ldots, i_l; m)) P(C_\tau(i_1, \ldots, i_l; m)) \\ &\quad \leqslant \sum_{m=0}^{2\binom{l}{2}} \{(pn)^{2l+k-3m+1} + o((pn)^{2l+k-3m+1})\} \left\{ \binom{2\binom{l}{2}}{m} p^m \right\}. \end{aligned}$$

So

$$P(A_\tau) \leqslant \sum_{l \leqslant 2k} \binom{n}{l} \sum_{m=0}^{2\binom{l}{2}} \{(pn)^{2l+k-3m+1} + o((pn)^{2l+k-3m+1})\} \left\{ \binom{2\binom{l}{2}}{m} p^m \right\}.$$

Noting that

$$(pn)^{2l+k-3m+1}p^m = n^{(-(2/5)-\delta)(2l+k+1)+(-(1/5)+2\delta)m},$$

and $\delta < \frac{1}{10}$, this expression has a maximum when $m = 0$. Therefore

$$P(A_\tau) \leqslant \sum_{l \leqslant 2k} \binom{n}{l} \{n^{(-(2/5)-\delta)(2l+k+1)} + o(n^{(-(2/5)-\delta)(2l+k+1)})\}$$
$$= O(n^{2k+(-(2/5)-\delta)(5k+1)}) = O(n^{-(2/5)-(5k+1)\delta}).$$

Thus

$$E(Z_k) \leqslant O(n^{-(2/5)-(5k+1)\delta}n^2) = O(n^{(8/5)-(5k+1)\delta}).$$

But $k \geqslant \lceil 8/25\delta \rceil$, and so $E(Z_k) = o(1)$. □

Let $p = n^{-(7/5)-\delta}$. We make the remark that if X_j is the random variable on $\boldsymbol{H}^{(3)}(n, p)$ defined by

$$X_j(H) = |\{\tau \in V^{(2)}: |\{\sigma \in E(H): \tau \subset \sigma\}| = j\}|,$$

(i.e., $X_j(H)$ is the number of vertex pairs in $V^{(2)}$, each of which is common to exactly j hyperedges of H) then $E(X_j) = o(1)$ for $j \geqslant 5$. Further, if $X_j^* = \sum_{i \geqslant j} X_i$, then $E(X_j^*) = o(1)$ for $j \geqslant 5$.

Let A be the event that $Z_k = 0$ and $X_5^* = 0$. Let $\boldsymbol{H}_A^{(3)}(n, p)$ be the conditional probability space for this event. For $H \in \boldsymbol{H}_A^{(3)}(n, p)$, let $\mu_1, \mu_2, \ldots, \mu_l$ be the 4-sets in $F(H)$. For each $i \in \{1, 2, \ldots, l\}$ choose at random (independently for each μ_i), one of the six $\tau_{\mu_i} \in D(H)$ such that $\tau_{\mu_i} \subset \mu_i$. Call this vertex pair $\tau_{\mu_i}^*$. It is possible that we will choose, for some $i \neq j$, vertex pairs $\tau_{\mu_i}^* = \tau_{\mu_j}^*$. Now define a sequence of hyperedge sets $E_0, E_1, E_2, \ldots, E_l$, by setting $E_0 = E(H)$ and, having defined E_{i-1}, setting

$$E_i = E_{i-1} \backslash \{\sigma: \sigma \in E_{i-1} \text{ and } \tau_{\mu_i}^* \subset \sigma\}.$$

Let H_i be the sub-hypergraph of the hypergraph H with edge set $E(H_i) = E_i$. This gives us a sequence of hypergraphs

$$H = H_0, H_1, H_2, \ldots, H_l = H^*.$$

Call H^* a 'derived hypergraph' of H. Let $\boldsymbol{H}_A^{(3)}(n, p)^*$ be the probability space of such hypergraphs. For $H^* \in \boldsymbol{H}_A^{(3)}(n, p)^*$, define the graph G_{H^*} on the vertex set V which has an edge joining vertex i to vertex j if $\{i, j\}$ is contained in a hyperedge of H^*. Clearly such a graph G_{H^*} is K^4-free.

For $\epsilon > 0$, let $m = n^{(7/10)+\epsilon}$ and define Y (strictly $Y_{\epsilon,\delta}$) to be that random variable on $\boldsymbol{H}_A^{(3)}(n, p)^*$ such that $Y(H^*)$ is the number of m-sets in V containing no hyperedge of H^*.

Lemma 4. *Let* $0 < \delta < \epsilon$ *and* $p = n^{-(7/5)-\delta}$. *Then* $E(Y) = o(1)$.

Proof. Consider the probability space $\boldsymbol{H}_A^{(3)}(n, p)$. For a random hypergraph

$H_p \in \boldsymbol{H}_A^{(3)}(n, p)$ and an edge $\sigma \in E(H_p)$, let B_σ be the event that the hyperedge σ is removed when creating H_p^* (i.e., $\sigma \in E(H_p) \backslash E(H_p^*)$). Each vertex pair in $D(H_p)$ is in at most k of the K^4's in $G(H_p)$; and a given hyperedge is removed from $E(H_p)$ only if one of the three vertex pairs contained in it is removed. For $\tau \in D(H_p)$, let C_τ be the event that the vertex pair τ is not removed. If $\sigma^{(2)} = \{\tau, \tau', \tau''\}$ then

$$P(C_\tau) \geq (1 - \tfrac{1}{6})^k, \quad P(C_{\tau'} \mid C_\tau) \geq (1 - \tfrac{1}{5})^k, \quad P(C_{\tau''} \mid C_\tau \cap C_{\tau'}) \geq (1 - \tfrac{1}{4})^k.$$

Thus

$$P(C_\tau \cap C_{\tau'} \cap C_{\tau''}) \geq ((1 - \tfrac{1}{6})(1 - \tfrac{1}{5})(1 - \tfrac{1}{4}))^k,$$

so

$$P(B_\sigma) = 1 - P(\overline{B_\sigma}) = 1 - P(C_\tau \cap C_{\tau'} \cap C_{\tau''})$$
$$\leq 1 - ((1 - \tfrac{1}{6})(1 - \tfrac{1}{5})(1 - \tfrac{1}{4}))^k = c < 1.$$

Since vertex pairs are chosen in H_p independently, it follows that if $\{\sigma_1, \sigma_2, \ldots, \sigma_l\}$ is a set of hyperedges of H_p such that $|\sigma_i \cap \sigma_j| \leq 1$ for all $i, j \in \{1, 2, \ldots, l\}$, then

$$P\Big(\bigcap_{i=1}^{l} B_{\sigma_i}\Big) \leq \prod_{i=1}^{l} P(B_{\sigma_i}) \leq c^l.$$

What is the corresponding probability for an arbitrary set $T = \{\sigma_1, \sigma_2, \ldots, \sigma_t\} \subset E(H_p)$? From the definition of event A, each vertex pair is in at most four hyperedges, thus each σ_i meets at most nine other σ_j's in two vertices, so there is a set $T' \subset T$, with $|T'| \geq \frac{1}{10}|T|$, such that no two hyperedges in T' have more than one vertex in common. Thus

$$P\Big(\bigcap_{\sigma \in T} B_\sigma\Big) \leq P\Big(\bigcap_{\sigma \in T'} B_\sigma\Big) \leq c^{t/10}. \tag{1}$$

Let $p_\alpha(i)$ be the probability that a given m-set in V contains exactly i hyperedges. Let $p_\beta(i)$ be the maximum, over all set of i hyperedges, of the probability that set of i hyperedges is removed from a hypergraph $H_p \in \boldsymbol{H}_A^{(3)}(n, p)$ to generate a derived hypergraph H_p^*. From inequality (1) we see that $p_\beta(i) \leq c^{i/10}$.

Now consider the probability space $\boldsymbol{H}_A^{(3)}(n, p)^*$. Set $M = \binom{m}{3}$. Then

$$E(Y) \leq \binom{n}{m} \sum_{i=0}^{M} p_\alpha(i) p_\beta(i).$$

To assist the calculation, the summation is evaluated in two parts. Recall that $m = n^{(7/10)+\epsilon}$ and set $L = n^{(7/10)+2\epsilon}$. Let

$$T_1 = \binom{n}{m} \sum_{i=0}^{L} p_\alpha(i) p_\beta(i)$$

and

$$T_2 = \binom{n}{m} \sum_{i=L+1}^{M} p_\alpha(i) p_\beta(i).$$

In showing T_1 to be small, we are essentially showing that in a random hypergraph in $\boldsymbol{H}_A^{(3)}(n, p)$ the expected number of m-sets containing fewer than L hyperedges is small. We thus note that $p_\beta(i) < 1$, so

$$T_1 \leqslant \binom{n}{m} \sum_{i=0}^{L} p_\alpha(i).$$

In showing T_2 to be small, we are essentially showing that given an m-set of vertices containing more than L hyperedges of a hypergraph in $\boldsymbol{H}_A^{(3)}(n, p)$, the probability that all are removed when forming a derived hypergraph is small. Thus noting that $p_\alpha(i) < 1$, we get

$$T_2 \leqslant \binom{n}{m} \sum_{i=L+1}^{M} p_\beta(i).$$

A 3-set is chosen to be a hyperedge of $H_p \in \boldsymbol{H}^{(3)}(n, p)$ independently of the choice for any other 3-set; and since $P(\bar{A}) = \mathrm{o}(1)$ (by Lemma 3 and the remark following it), so $P(A) \geqslant 1 - \mathrm{o}(1)$. It follows that

$$p_\alpha(i) \leqslant \binom{M}{i} p^i (1-p)^{M-i} (1 + \mathrm{o}(1)).$$

Thus, for sufficiently large n,

$$T_1 \leqslant 2\binom{n}{m} \sum_{i=0}^{L} \binom{M}{i} p^i (1-p) M^{-i}. \tag{2}$$

Recalling that $p_\beta(i) \leqslant c^{i/10}$, we see that

$$T_2 \leqslant \binom{n}{m} \sum_{i=L+1}^{M} C^{i/10}. \tag{3}$$

The remainder of the proof involves establishing that expressions on the right-hand sides of the inequalities (2) and (3) are both o(1).

The bound for T_1 given in (2) is $2\binom{n}{m} P(S_{M,p} \leqslant L)$, where $S_{M,p}$ is the random variable having binomial distribution with parameters M and p. Here we have

$$pM = n^{(7/10)+3\epsilon-\delta}$$

and

$$L = n^{(7/10)+2\epsilon} \leqslant \tfrac{11}{12} pM$$

for sufficiently large n, so from a theorem of Bollobás (see [18, p. 13, Theorem 7(i)]) with $\epsilon = \frac{1}{12}$,

$$P(S_{M,p} \leqslant L) \leqslant P(|S_{M,p} - pM| \geqslant \tfrac{1}{12} pM) \leqslant \frac{12}{\sqrt{pM}} \exp\left\{ -\frac{1}{432} n^{(7/10)+3\epsilon-\delta} \right\}.$$

Since

$$\binom{n}{m} \leqslant (en^{(3/10)-\epsilon})^{n^{(7/10)+\epsilon}} = \exp\{n^{(7/10)+\epsilon} \log(en^{(3/10)-\epsilon})\},$$

it follows that $T_1 = \mathrm{o}(1)$.

Turning to T_2, and recalling that $0 \leq c < 1$, it follows that

$$T_2 \leq \binom{n}{m} \sum_{i=L+1}^{M} c^{i/10} \leq M \binom{n}{m} c^{L/10}.$$

Using the inequality $\binom{a}{b} \leq \left(\frac{ea}{b}\right)^b$,

$$T_2 \leq M\left(\frac{en}{m}\right)^m c^{L/10} = M \exp\left\{m \log\left(\frac{en}{m}\right) - \frac{L}{10}|\log c|\right\},$$

but $m = n^{(7/10)+\epsilon}$ and $L = n^{(7/10)+2\epsilon}$, so

$$T_2 \leq M \exp\{n^{(7/10)+\epsilon} \log(en^{(3/10)-\epsilon}) - \tfrac{1}{10}n^{(7/10)+2\epsilon} |\log c|\};$$

so $T_2 = o(1)$. Thus

$$E(Y) \leq T_1 + T_2 = o(1). \quad \square$$

This theorem shows that as n gets large the expected number of vertex subsets of V of size $m = n^{(7/10)+\epsilon}$ not containing a hyperedge of a derived hypergraph tends to zero. In the following theorem we show (formally) that this implies the existence of a graph G with $h_3(G) \leq n^{(7/10)+\epsilon}$.

Theorem 5. *For $\epsilon > 0$ and sufficiently large n, $f_{3,4}(n) \leq n^{(7/10)+\epsilon}$.*

Proof. We seek to show the existence of a 3-uniform hypergraph H^* such that each m-set of vertices contains a hyperedge and such that each 4-set of vertices contains a pair of vertices for which there is no hyperedge in $E(H^*)$ containing both vertices.

Let $0 < \delta < \epsilon$ and let $p = n^{-(7/5)-\delta}$. Consider the probability space $\boldsymbol{H}^{(3)}(n, p)$ with probability measure P. The event A is the event that $Z_k = 0$ and no vertex pair is contained in five or more hyperedges, i.e.,

$$A(Z_k = 0) \cap (X_5^* = 0).$$

Thus, from Lemma 3 and the subsequent remark,

$$P(A) = P((Z_k = 0) \cap (X_5^* = 0)) \geq 1 - P(Z_k \geq 1) - P(X_5^* \geq 1)$$
$$\geq 1 - E(Z_k) - E(X_5^*) = 1 - o(1) - o(1) = 1 - o(1).$$

Now consider the probability space $\boldsymbol{H}_A^{(3)}(n, p)^*$ with probability measure P_A^*. Recall that Y was defined to be that random variable on $\boldsymbol{H}_A^{(3)}(n, p)^*$ such that $Y(H^*)$ is the number of m-sets in V containing no hyperedge of H^*. Since $P(A) = 1 - o(1)$, it is sufficient to show that $P_A^*(Y = 0) = 1 - o(1)$.

Using the result of Lemma 4,

$$P_A^*(Y = 0) = 1 - P_A^*(Y \geq 1) \geq 1 - E(Y) = 1 - o(1).$$

Thus, for sufficiently large n, it follows that there exists a hypergraph $H^* \in \boldsymbol{H}_A^{(3)}(n, p)^*$ such that $Y(H^*) = 0$. Define $G^* = G_{H^*}$ to be the graph on V in which a vertex i is joined to a vertex j if some hyperedge of H^* contains $\{i, j\}$, Then, by the construction of H^*, the graph G^* does not contain a K^4 and G^* satisfies the condition $h_3(G^*) \leq n^{(7/10)+\epsilon}$. The theorem follows. □

The results above give bounds for the function $f_{3,4}(n)$, however the methods of proof used can be generalized to bound $f_{s-1,s}(n)$ for $s \geq 4$. The proof of Theorem 6 is an extension of the proof for Theorem 1 and provides a general lower bound for the function $f_{r,s}(n)$.

Theorem 6. *Let* $3 \leq r < s$ *and* $n \geq 1$, *then*

$$f_{r,s}(n) \geq n^{1/(s-r+1)}.$$

Proof. Let G be a graph of order n such that $\mathrm{cl}(G) \leq s-1$. We define a sequence of graphs

$$G = G_0, G_1, \ldots, G_{s-r},$$

by putting

$$G_{i+1} = G_i[\Gamma(v_i)]$$

for $i = 0, 1, 2, \ldots, s-r-1$, where v_i is a vertex of maximal degree in G_i. As G does not contain a K^s, it follows that G_i does not contain a K^{s-i} for $i = 1, 2, \ldots, s-r$.

Let $\alpha = 1/(s-r+1)$. If

$$\Delta(G_i) < |G_i|\, n^{-\alpha}$$

for an $i \in \{0, 1, \ldots, s-r-1\}$, then $\chi(G_i) < |G_i|\, n^{-\alpha} + 1$. Choosing W_1 and W_2 to be colour classes of a $\chi(G_i)$-vertex colouring of G_i such that $|W_1 \cup W_2|$ is maximal, it follows (crudely) that $|W_1 \cup W_2| > n^{\alpha}$. The subgraph of G_i (and thus of G) induced by $W_1 \cup W_2$ does not contain a K^3, proving the result in this case.

Thus we may assume that for each $i \in \{0, 1, \ldots, r-s-1\}$,

$$\Delta(G_i) \geq |G_i|\, n^{-\alpha}.$$

Therefore

$$|G_{i+1}| \geq |G_i|\, n^{-\alpha}$$

for each $i \in \{0, 1, \ldots, r-s-1\}$, and so

$$|G_{s-r}| \geq |G_0|(n^{-\alpha})^{(s-r)}.$$

Since G_{s-r} does not contain a K^r, and

$$|G_{s-r}| \geq n \cdot n^{-(s-r)/(s-r+1)} = n^{1/(s-r+1)},$$

the result follows. □

We generalize the results of Theorem 5 to obtain an upper bound for $f_{s-1,s}(n)$, then the trivial fact that $f_{r,s}(n) \leq f_{r',s}(n)$ for $3 \leq r \leq r' < s$ allows us to deduce an upper bound for $f_{r,s}(n)$ for $3 \leq r < s$.

In the remainder of this paper we shall assume that $s \geq 4$. Let $\boldsymbol{H}^{(s-1)}(n, p)$ be the probability space of $(s-1)$-uniform hypergraphs on vertex set $V = [n]$, where $(s-1)$-sets are chosen to be hyperedges with probability p and independently of the choice for other $(s-1)$-sets. Define $\boldsymbol{G}^{(s-1)}(n, p)$ from $\boldsymbol{H}^{(s-1)}(n, p)$ in a manner similar to the definition of $\boldsymbol{G}^{(3)}(n, p)$ from $\boldsymbol{H}^{(3)}(n, p)$.

For $H \in \boldsymbol{H}^{(s-1)}(n, p)$ define

$$D^{(s-1)}(H) = \{\tau \subset V: |\tau| = 2, \tau \subset \sigma \text{ for some } \sigma \in E(H)\}$$

and

$$F^{(s-1)}(H) = \{\mu \subset V: \ |\mu| = s \text{ and } \mu^{(2)} \subset D(H)\}.$$

Define a function $g_H^{(s-1)}: D^{(s-1)}(H) \to \mathbb{N}$ by putting

$$g_H^{(s-1)}(\tau) = |\{\mu \in F^{(s-1)}(H): \tau \subset \mu\}|$$

and define the random variable $Z_k^{(s-1)}$ on $\boldsymbol{H}^{(s-1)}(n, p)$ by putting

$$Z_k^{(s-1)}(H) = |\{\tau \in D^{(s-1)}(H): g_H^{(s-1)}(\tau) \geq k\}|.$$

Thus $Z_k(H)$ is the number of edges of G_H, each of which is an edge of at least k distinct K^s's. Then we get the following lemma.

Lemma 7. *Let* $0 < \delta, p = n^{-(s-3)-2/(s+1)-\delta}$ *and* $k \geq \max\{\lceil 4s/(s+1)^2(s-2)\delta \rceil, 3\}$ *then* $E(Z_k^{(s-1)}) = o(1)$.

Proof. (Analogous to the proof of Lemma 3.) Let us consider the probability space $\boldsymbol{H}^{(s-1)}(n, p)$. For a vertex pair $\tau = \{i, i'\} \in V^{(2)}$, let A_τ be the event that $\tau \in D^{(s-1)}(H)$ and there exist k sets $\mu_1, \mu_2, \ldots, \mu_k \in F^{(s-1)}(H)$ such that $\tau \subset \mu_j$ for each $j \in \{1, 2, \ldots, k\}$. Then

$$E(Z_k^{(s-1)}) = \binom{n}{2} P(A_\tau).$$

Let $B_\tau(i_1, \ldots, i_l)$ be the event that $\tau \in D^{(s-1)}(H)$ and there exist k sets $\mu_1, \mu_2, \ldots, \mu_k \in F^{(s-1)}(H)$ such that $\tau \subset \mu_j$ for each $j \in \{1, 2, \ldots, k\}$ and $\bigcup_{j=1}^k \mu_j = \{i_1, \ldots, i_l\} \cup \{i, i'\}$. We may assume $l \leq (s-2)k$, since $P(B_\tau(i_1, \ldots, i_l)) = 0$ for $l > (s-2)k$; so

$$P(A_\tau) \leq \sum_{l=0}^{(s-2)k} \binom{n}{l} P(B_\tau(i_1, i_2, \ldots, i_l)).$$

If $H \in B_\tau(i_1, \ldots, i_l)$, then for each $j \in \{1, 2, \ldots, l\}$ the vertex pairs $\{i, i_j\}$ and $\{i', i_j\}$ must be in $D^{(s-1)}(H)$. Set $\lambda(l) = 0$ if $l = (s-2)k$ and $\lambda(l) = s-3$ if $(s-2)k > l$. Since each vertex in $\{i_1, i_2, \ldots, i_l\}$ is in at least one $\mu_t \in F^{(s-1)}(H)$ (and if $l < (s-2)k$ at least one vertex is in two $\mu_t \in F^{(s-1)}(H)$), at least $l(s-3)/2 + \lambda(l)$ vertex pairs of the form $\{i_j, i_{j'}\}$ must be in $D^{(s-1)}(H)$.

Defining an event C_τ in a manner similar to that for in Lemma 3, we evaluate $P(B_\tau(i_1, \ldots, i_l)$ and thus $P(A_\tau)$ to get

$$P(A_\tau) \leq \sum_{l \leq (s-2)k} \binom{n}{l} n^{(-2/(s+1)-\delta)(2l+l(s-3)/2+\lambda(l)+1)}.$$

Thus

$$P(A_\tau) = O(n^{k(s-2)+(-2/(s+1)-\delta)(2k(s-2)+k(s-2)(s-3)/2+1)})$$
$$= O(n^{-\delta(k(s+1)(s-2)/2+1)-2/(s+1)})$$

and

$$E(Z_k^{(s-1)}) = O(n^{-2/(s+1)-\delta(k(s+1)(s-2)/2+1)}n^2) = O(n^{2s/(s+1)-\delta(k(s+1)(s-2)/2+1)})$$

thus with $k \geq \lceil 4s/(s+1)^2(s-2)\delta \rceil$,

$$E(Z_k^{s-1}) = o(1). \qquad \square$$

Define $X_j^{(s-1)}$ to be the random variable on $\boldsymbol{H}^{(s-1)}(n, p)$ such that $X_j^{(s-1)}(H)$ is the number of vertex pairs in $V^{(2)}$ each of which is in exactly j hyperedges of H. Then $E(X_j^{(s-1)}) = o(1)$ for $j \geq s+1$. Further, if we define $X_j^{(s-1)} = \sum_{i \geq j} X_i^{(s-1)}$, then $E(X_j^{*(s-1)}) = o(1)$ for $j \geq s+1$.

Consider the probability space $\boldsymbol{H}^{(s-1)}(n, p)$ and define A to be the event that $Z_k^{(s-1)} = 0$ and $X_{s+1}^{*(s-1)} = 0$. Each random element in $\boldsymbol{H}^{(s-1)}(n, p)$ which is in A is an $(s-1)$-uniform hypergraph such that each vertex pair in $V^{(2)}$ is in at most s hyperedges. Furthermore, if H is in A, then no edge of G_H is in more than k distinct induced subgraphs isomorphic to K^s.

Let $\boldsymbol{H}_A^{(s-1)}(n, p)$ be the conditional probability space associated with the event A. For each $H \in \boldsymbol{H}_A^{(s-1)}(n, p)$, define a derived hypergraph, H^*, associated with H in a manner analogous to the definition of a derived hypergraph for a hypergraph in $\boldsymbol{H}_A^{(3)}(n, p)$. Let $\boldsymbol{H}_A^{(s-1)}(n, p)^*$ be the probability space associated with the derived hypergraphs.

For $\epsilon > 0$, let $m = n^{(s-3)/(s-2)+2/(s+1)(s-2)+\epsilon}$ and define $Y_{\epsilon,\delta}^{(s-1)}$ to be the random variable on $\boldsymbol{H}_A^{(s-1)}(n, p)^*$ such that $Y_{\epsilon,\delta}^{(s-1)}(H^*)$ is the number of m-sets in V not containing a hyperedge of H^*.

Lemma 8. *Let* $0 < \delta < \epsilon$ *and* $p = n^{-(s-3)-2/(s+1)-\delta}$. *Then*

$$E(Y_{\epsilon,\delta}^{(s-1)}) = o(1).$$

Proof. The proof is essentially the same as that for Lemma 4, with the following changes. In inequality (1) the number 10 must be replaced by $\binom{s-1}{2}(s-1)+1$.

The parameter

$$M = \binom{m}{s-1},$$

where

$$m = n^{(s-3)/(s-2)+2/(s+1)(s-2)+\epsilon},$$

and the parameter

$$L = n^{(s-3)/(s-2)+2/(s+1)(s-2)+2\epsilon}.$$

It follows that

$$E(Y_{\epsilon,\delta}^{(s-1)}) \leq O(\exp\{-cn^{(s-3)/(s-2)+2/(s+1)(s-2)+2\epsilon}\}) = o(1). \qquad \square$$

With the results of Lemmas 7 and 8 and using a simple modification of the proof of Theorem 5, we obtain the general result.

Theorem 9. *Let $\epsilon > 0$ and n be sufficiently large, then*

$$f_{s-1,s}(n) \leq n^{(s-3)/(s-2)+2/(s+1)(s-2)+\epsilon}.$$

Corollary 10. *Let $\epsilon > 0$ and n be sufficiently large, then if $3 \leq r < s$,*

$$f_{r,s}(n) \leq n^{(s-3)/(s-2)+2/(s+1)(s-2)+\epsilon}.$$

While the results presented in this paper improve those of Erdős and Rogers, it is still not clear what the actual order of the function $f_{r,s}(n)$ is. An improvement in the lower bound for $f_{r,s}(n)$ would be of particular interest.

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