Journal of Computational and Applied Mathematics 233 (2009) 482–491

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Optimality of the barrier strategy in de Finetti's dividend problem for spectrally negative Lévy processes: An alternative approach

Chuancun Yin [a,*], Chunwei Wang [a,b]

[a] School of Mathematical Sciences, Qufu Normal University, Shandong 273165, China

[b] School of Sciences, Henan University of Science and Technology, Henan 471003, China

ARTICLE INFO

Article history:
Received 23 February 2009
Received in revised form 25 June 2009

MSC:
60G51
40J50

Keywords:
Spectrally negative Lévy process
Optimal dividend problem
Scale function
Log-convexity
Complete monotonicity
Convexity
Barrier strategy

ABSTRACT

The optimal dividend problem proposed in de Finetti [1] is to find the dividend-payment strategy that maximizes the expected discounted value of dividends which are paid to the shareholders until the company is ruined. Avram et al. [9] studied the case when the risk process is modelled by a general spectrally negative Lévy process and Loeffen [10] gave sufficient conditions under which the optimal strategy is of the barrier type. Recently Kyprianou et al. [11] strengthened the result of Loeffen [10] which established a larger class of Lévy processes for which the barrier strategy is optimal among all admissible ones. In this paper we use an analytical argument to re-investigate the optimality of barrier dividend strategies considered in the three recent papers.

1. Introduction

This paper considers the classical optimal dividend control problem for a company. The idea is that the company wants to pay some its surplus to the shareholders as dividends, the problem is to find a dividend-payment strategy that maximizes the expected discounted value of all payments until the company's capital is negative for the first time. This optimization problem goes back to [1], who considered a discrete time random walk with step sizes ± 1 and proved that the optimal dividend strategy is a barrier strategy. Optimal dividend problem has recently gained a lot of attention in actuarial mathematics. It has been studied extensively in the diffusion process setting, see [2–5]. It is well known that under some reasonable assumptions, the optimality in the diffusion process setting is achieved by using a barrier strategy (see [4,5]). However, in the Cramér–Lundberg setting this is not the case; it was shown in [6] that the optimal dividend strategy is of so-called band type. This results was re-derived by means of viscosity theory in [7]. In particular, for exponentially distributed claim sizes this optimal strategy simplifies to a barrier strategy. The summary of Finetti and Gerber's work can be found in [8]. Recently, Avram et al. [9] considered the case where the risk process is given by a general spectrally negative Lévy process and gave a sufficient condition involving the generator of the Lévy process for the optimal strategy to consist of a barrier strategy. In [10], Loeffen defined an optimal barrier level which is slightly different than the one given in [9] and proved the remarkable fact that, if the q-scale function $W^{(q)}$ is convex in the interval (a^*, ∞), where

$$a^* = \sup\{a \geq 0 : W^{(q)\prime}(a) \leq W^{(q)\prime}(y) \text{ for all } y \geq 0\} < \infty,$$

* Corresponding author.
E-mail address: ccyin@mail.qfnu.edu.cn (C. Yin).

0377-0427/$ – see front matter © 2009 Elsevier B.V. All rights reserved.
doi:10.1016/j.cam.2009.07.051

then the barrier strategy at a^* is an optimal strategy among all admissible strategies. Moreover, it is shown that when the Lévy measure Π of X has a completely monotone density, then $W^{(q)\prime}$ is strictly convex on $(0,\infty)$ for all $q>0$. Consequently, the barrier strategy at a^* is an optimal strategy. In a very recent work [11], the authors prove a more general result: Suppose that the Lévy measure Π of X has a non-increasing density which is logconvex, then for $q>0$ the scale function $W^{(q)}$ is convex in the interval (a^*,∞). As a consequence, the barrier strategy at a^* is an optimal strategy. In the other recent paper, Albrecher and Thonhausera [12] discussed the maximization problem in a generalized setting including a constant force of interest in the Cramér–Lundberg risk model. The value function is identified in the set of viscosity solutions of the associated Hamilton–Jacobi–Bellman equation and the optimal dividend strategy in this risk model is derived, which in the general case is again of band type and for exponential claim sizes collapses to a barrier strategy.

In this paper, it is assumed that the surplus process is a general spectrally negative Lévy process, we provide an analytical study of the solution to the classical dividend control problem due to [9,11,10].

The rest of the paper is organized as follows. In Section 2, we recall some preliminaries on the spectrally negative Lévy process and state the problem. In Section 3, we will review some basic results on the logconvexity and complete monotonicity of the functions that will be needed later on. In Section 4 we discuss the convex solutions for two kinds of integro-differential equations and in Section 5 we present the main results and prove them by using the results of Section 4 and some earlier results from [9,11,10]. Finally, some remarks are included in Section 6.

2. The model

Suppose that $X=(X(t):t\ge 0)$ is a spectrally negative Lévy process with probabilities $\{P_x : x\in\mathbb{R}\}$ such that $X(0)=x$ with probability one, where we write $P=P_0$. Let E_x be the expectation with respect to P_x and write $E=E_0$. That is to say X is a real valued stochastic process whose paths are almost surely right continuous with left limits and whose increments are stationary and independent. Let $\{\mathcal{F}_t : t\ge 0\}$ be the natural filtration satisfying the usual assumptions. Since the jumps of a spectrally negative Lévy process are all non-positive, the moment generating function $E(\mathrm{e}^{\theta X(t)})$ exists for all $\theta\ge 0$ and is given by $E(\mathrm{e}^{\theta X(t)})=\mathrm{e}^{t\psi(\theta)}$ for some function $\psi(\theta)$, which is called the Laplace exponent of X. From the Lévy–Khintchin formula [13,14], it is known that

$$\psi(\theta)=a\theta+\frac{1}{2}\sigma^2\theta^2-\int_0^\infty\left(1-\mathrm{e}^{-\theta x}-\theta x\mathbf{1}_{\{0<x<1\}}\right)\Pi(\mathrm{d}x) \tag{2.1}$$

where $a\in\mathbb{R}$, $\sigma\ge 0$ and Π is a measure on $(0,\infty)$ satisfying $\int_0^\infty(1\wedge x^2)\Pi(\mathrm{d}x)<\infty$ and is called the Lévy measure. ψ is strictly convex on $(0,\infty)$ and satisfies $\psi(0+)=0$, $\psi(\infty)=\infty$ and $\psi'(0+)=EX(1)$. Further, ψ is strictly increasing on $[\phi(0),\infty)$, where $\phi(0)$ is the largest root of $\psi(\theta)=0$ (there are at most two). We shall denote the right-inverse function of ψ by ϕ: $[0,\infty)\to[\phi(0),\infty)$. If $\sigma^2>0$ and $\Pi=0$, then the process is a Brownian motion; When $\sigma^2=0$ and $\int_0^\infty\Pi(\mathrm{d}x)<\infty$, the process is a compound Poisson process; when $\sigma^2=0$, $\int_0^\infty\Pi(\mathrm{d}x)=\infty$ and $\int_0^\infty(1\wedge x)\Pi(\mathrm{d}x)<\infty$, the process has an infinite number of small jumps, but is of finite variation; when $\sigma^2=0$, $\int_0^\infty\Pi(\mathrm{d}x)=\infty$ and $\int_0^\infty(1\wedge x)\Pi(\mathrm{d}x)=\infty$, the process has infinitely many jumps and is of unbounded variation. In short, such a Lévy process has bounded variation if and only if $\sigma=0$ and $\int_0^1 x\Pi(\mathrm{d}x)<\infty$. In this case the Lévy exponent can be re-expressed as

$$\psi(\alpha)=b\alpha-\int_0^\infty(1-\mathrm{e}^{\alpha x})\Pi(\mathrm{d}x),$$

where $b=a-\int_0^1 x\Pi(\mathrm{d}x)$ is known as the drift coefficient. If $\sigma^2>0$, X is said to have a Gaussian component.

For θ such that $\psi(\theta)$ is finite we denote by P_x^θ an exponential tilting of the measure P_x with a Radom–Nikodym derivative with respect to P_x given by

$$\left.\frac{\mathrm{d}P_x^\theta}{\mathrm{d}P_x}\right|_{\mathcal{F}_t}=\exp(\theta(X(t)-x)-\psi(\theta)t).$$

Under the measure P_x^θ the process X is still a spectrally negative Lévy process with Laplace exponent ψ_θ given by

$$\psi_\theta(\eta)=\psi(\eta+\theta)-\psi(\theta),\quad \eta\ge-\theta.$$

We recall from [15,13], that for each $q\ge 0$ there exits a continuous and increasing function $W^{(q)}:\mathbb{R}\to[0,\infty)$, called the q-scale function, defined in such a way that $W^{(q)}(x)=0$ for all $x<0$, and on $[0,\infty)$ its Laplace transform is given by

$$\int_0^\infty \mathrm{e}^{-\theta x}W^{(q)}(x)\mathrm{d}x=\frac{1}{\psi(\theta)-q},\quad \theta>\phi(q). \tag{2.2}$$

For convenience we shall write W in place of $W^{(0)}$ and call this the scale function rather than the 0-scale function.

The following facts about the smoothness of the scale functions are taken from [11]. If X has paths of bounded variation then, for all $q\ge 0$, $W^{(q)}|_{(0,\infty)}\in C^1(0,\infty)$ if and only if Π has no atoms. In the case that X has paths of unbounded variation, it is known that, for all $q \geq 0$, $W^{(q)}|_{(0,\infty)} \in C^1(0,\infty)$. Moreover if $\sigma > 0$ then $C^1(0,\infty)$ may be replaced by $C^2(0,\infty)$. Further, if the Lévy measure has a density, then the scale functions are always differentiable. In particular, if π is completely monotone then $W^{(q)}|_{(0,\infty)} \in C^\infty(0,\infty)$.

Spectrally negative Lévy processes have been considered recently in [16–24], among others, in the context of insurance risk models. It is assumed that, in the absence of dividends, the surplus of a company at time t is $X(t)$. We assume now that the company pay dividends to its shareholders according to some strategy. Let $\pi = \{L_t^\pi : t \geq 0\}$ be a dividend strategy consisting of a left-continuous non-negative non-decreasing process adapted to the filtration $\{\mathcal{F}_t : t \geq 0\}$ of X. L_t^π represents the cumulative dividends paid out up to time t under the control π by the insurance company whose risk process is modelled by X. We define the controlled risk process $U^\pi = \{U_t^\pi : t \geq 0\}$ by $U_t^\pi = X(t) - L_t^\pi$. Let $\tau^\pi = \{t > 0 : U_t^\pi < 0\}$ be the ruin time when the dividend payments are taken into account. Define the value function of a dividend strategy π by

$$V_\pi(x) = E_x\left(\int_0^{\tau^\pi} \mathrm{e}^{-qt}\mathrm{d}L_t^\pi\right),$$

where $q > 0$ is the discounted rate.

A dividend strategy is called admissible if $L_{t+}^\pi - L_t^\pi \leq U_t^\pi$ for $t < \tau^\pi$, in other words the lump sum dividend payment is smaller than the size of the available capital. Let Ξ be the set of all admissible dividend policies. De Finetti's dividend problem consists of solving the following stochastic control problem:

$$V_*(x) = \sup_{\pi\in\Xi} V_\pi(x),$$

and, if it exists, to find a strategy $\pi^* \in \Xi$ such that $V_{\pi^*}(x) = V_*(x)$ for all $x \geq 0$.

If the dividends are paid according to barrier strategies with parameter $a > 0$. That is, when the controlled surplus reaches the level a, the overflow will be paid as dividends, if the surplus is less than a, no dividends are paid out. We denote by $\pi_a = \{L_t^a : t \geq 0\}$ the barrier strategy. Note that $\pi_a \in \Xi$. Let $V_a(x)$ denote the dividend-value function if a barrier strategy with level a is applied. It is well known that $V_a(x)$ can be expressed in terms of scale functions as following: If $W^{(q)}(x)$ is continuously differential on $(0,\infty)$, then

$$V_a(x) = \frac{W^{(q)}(x)}{W^{(q)\prime}(a)}, \quad \text{if } x \leq a; \qquad x - a + \frac{W^{(q)}(a)}{W^{(q)\prime}(a)}, \quad \text{if } x > a.$$

For details, see [9,25].

3. Preliminaries on logconvex functions and related functions

In this section we review definitions and some properties of logconvex functions and completely monotone functions. We refer the reader to [26,27] for more details.

Definition 3.1 ([26]). Suppose $g(x) : \mathbb{R} \to \mathbb{R}^+$ is a measurable function with $B = \{x \in \mathbb{R} : g(x) > 0\}$. $g(x)$ is logconvex in B if

$$g(\lambda x_1 + (1-\lambda)x_2) \leq [g(x_1)]^\lambda [g(x_2)]^{1-\lambda},$$

for all $x_1, x_2 \in B$ and all $\lambda \in [0, 1]$.

Lemma 3.1 ([26]). *For any $a > -\infty$, logconvexity of g in (a,∞) is equivalent to*

$$g(x_1 + \delta)g(x_2) \leq g(x_2 + \delta)g(x_1),$$

for all $a < x_1 < x_2$ and all $\delta > 0$.

Lemma 3.2 ([26]). *Let $g(x) : \mathbb{R} \to \mathbb{R}^+$ be a measurable function. Suppose $\{x : g(x) > 0\} = (a, b)$ and g is twice differentiable in (a, b). Then $g(x)$ is logcovex in (a, b) if and only if*

$$g''(x)g(x) - [g'(x)]^2 \geq 0, \quad \text{for all } x \in (a, b).$$

Lemma 3.3 ([27]). *Let g be a continuously-differentiable function, mapping the interval (a, b) into the positive real numbers. Let $G(x) = \int_a^x g(z)\mathrm{d}z$ and $\overline{G}(x) = \int_x^b g(z)\mathrm{d}z$ for x in (a, b), and define $g(a) = \lim_{x\to a} g(x)$ and $g(b) = \lim_{x\to b} g(x)$. Then*

(1) If g is logconvex on (a, b) and $g(a) = 0$, then G is also logconvex on (a, b).
(2) If g is logconvex on (a, b) and $g(b) = 0$, then $\overline{G}$ is also logconvex on (a, b).

Lemma 3.4. *If $\int_0^t (1-F(x))\mathrm{d}x$ is logconvex on $(0,\infty)$, and F has a density f. Then $c+\sigma^2\rho-\lambda\hat{f}(\rho)\int_0^t(1-F(x))\mathrm{d}x$ is logconvex on $(0,\infty)$, where c, σ, ρ and λ are positive constants. Here and henceforth, $\hat{f}$ denotes the Laplace transform of f.*

Proof. The logconvexity of $\int_0^t (1-F(x))\mathrm{d}x$ implies that

$$-f(t)\int_0^t (1-F(x))\mathrm{d}x-(1-F(t))^2\geq 0,$$

which implies that

$$\left[\log\left(c+\sigma^2\rho-\lambda\hat{f}(\rho)\int_0^t(1-F(x))\mathrm{d}x\right)\right]''$$
$$=\frac{\lambda\hat{f}(\rho)f(t)\left(c+\sigma^2\rho-\lambda\hat{f}(\rho)\int_0^t(1-F(x))\mathrm{d}x\right)-\lambda\hat{f}(\rho)(1-F(t))^2}{\left(c+\sigma^2\rho-\lambda\hat{f}(\rho)\int_0^t(1-F(x))\mathrm{d}x\right)^2}\geq 0. \quad \square$$

Definition 3.2 (*[28]*). Let $f\in C^\infty(0,\infty)$ with $f\geq 0$. We say f is completely monotone if $(-1)^n f^{(n)}\geq 0$ for all $n\in\mathbb{N}$ and a Bernstein function if $(-1)^n f^{(n)}\leq 0$ for all $n\in\mathbb{N}$.

Lemma 3.5 (*[28]*).

(1) *A function g is completely monotone on* $(0,\infty)$ *if and only if there exists a measure* μ *on* $[0,\infty)$ *for which*

$$g(x)=\int_0^\infty \exp(-sx)\mu(\mathrm{d}s).$$

(2) *A function f is a Bernstein function if and only if it has the representation*

$$f(x)=a+bx+\int_0^\infty(1-\exp(-yx))\Lambda(\mathrm{d}y)$$

for all $x>0$, *where* $a,b\geq 0$ *and* Λ *is a measure on* $(0,\infty)$ *satisfying* $\int_0^\infty(y\wedge 1)\Lambda(\mathrm{d}y)<\infty$.

Note that the combinations of completely monotone functions with positive coefficients are also completely monotone; if f is completely monotone and $0<a\leq 1$, then f^a is again completely monotone. Further, the class of logconvex functions contains the class of completely monotone functions. In fact, any completely monotone function is both nonincreasing and logconvex. The logconvexity of a function h implies that the right and left derivatives of h exist everywhere and satisfy that $h'^{-}(x)\leq h'^{+}(x)$. Following the steps in [11] one can construct logconvex functions which are not completely monotone in general: Suppose that f_i, $i=1,2,\ldots,n(n\geq 2)$ are n logconvex functions on $(0,\infty)$ satisfying that for some fixed $x_1\leq x_2\leq\cdots\leq x_{n-1}$ we have $f_i(x_i)=f_{i+1}(x_i)$ and $(f_i(x_i))'^{-}<(f_{i+1}(x_i))'^{+}$, $i=1,2,\ldots,n-1$. Then the functions of the forms

$$g(x)=f_1(x)\mathbf{1}_{\{0<x<x_1\}}+f_2(x)\mathbf{1}_{\{x_1\leq x<x_2\}}+\cdots+f_n(x)\mathbf{1}_{\{x_{n-1}\leq x<\infty\}}$$

are logconvex but not completely monotone. If, moreover, g is the density of a Lévy measure on $(0,\infty)$, then it is necessary to need the condition

$$\sum_{j=1}^n\int_{x_{j-1}}^{x_j}(1\wedge x^2)f_j(x)\mathrm{d}x<\infty,$$

where $x_0=0$ and $x_n=\infty$.

For example, $g(x)=2\mathrm{e}^{-2x}\mathbf{1}_{\{0<x<\ln 2\}}+\mathrm{e}^{-x}\mathbf{1}_{\{x\geq\ln 2\}}$ is a decreasing, logconvex function which is not completely monotone.

Several important examples of spectrally negative Lévy processes with completely monotone densities (see [10,29]) are:

- α-stable process with Lévy density: $\pi(x)=\lambda x^{-1-\alpha}$, $x>0$ with $\lambda>0$ and $\alpha\in(0,1)\cup(1,2)$.
- One-sided tempered stale process (particular cases include the gamma process ($\alpha=0$) and the inverse Gaussian process ($\alpha=\frac{1}{2}$)) with Lévy density: $\pi(x)=\lambda x^{-1-\alpha}\mathrm{e}^{-\beta x}$, $x>0$ with $\beta,\lambda>0$ and $-1\leq\alpha<2$.
- The associated parent process with Lévy density: $\pi(x)=\lambda_1 x^{-1-\alpha}\mathrm{e}^{-\beta x}+\lambda_2 x^{-2-\alpha}\mathrm{e}^{-\beta x}$, $x>0$ with $\lambda_1,\lambda_2>0$ and $-1\leq\alpha<1$.

Note that they all satisfy $\int_0^\infty\pi(x)\mathrm{d}x=\infty$.

Some distributions with logconvex density functions (see [27,10]) are:

- Weibull distribution with density: $f(x)=crx^{r-1}\mathrm{e}^{-cx^r}$, $x>0$, with $c>0$ and $0<r<1$.
- Pareto distribution with density: $f(x)=\alpha(1+x)^{-\alpha-1}$, $x>0$, with $\alpha>0$.
- Mixture of exponential densities: $f(x)=\sum_{i=1}^n A_i\beta_i\mathrm{e}^{-\beta_i x}$, $x>0$, with $A_i>0$, $\beta_i>0$ for $i=1,2,\ldots,n$, and $\sum_{i=1}^n A_i=1$.
- Gamma distribution with density: $f(x)=\frac{x^{c-1}\mathrm{e}^{-x/\beta}}{\Gamma(c)\beta^c}$, $x>0$, with $\beta>0$, $0<c\leq 1$.

4. Convex solutions for integro-differential equations

In this section, we will discuss the convex solutions for two kinds of integro-differential equations. We first consider the following second order integro-differential equation:

$$\frac{\sigma^2}{2}v''(x)+cv'(x)+\lambda\int_0^x v(x-z)f(z)\mathrm{d}z=(\lambda+\delta)v(x),\quad x>0, \tag{4.1}$$

with $v(0)\geq 0$ and $v'(0)>0$. Here $\sigma, c, \lambda, \delta$ are positive constants.

Theorem 4.1. *Let F be a distribution function on $(0,\infty)$ with a logconvex density f. If v is an increasing function on $(0,\infty)$ with $v(0)\geq 0$, and satisfies the integro-differential equation (4.1), then there exists $x_0>0$ such that $v''(x)>0$ a.e. for all $x>x_0$.*

Proof. Note that v can be written as the form $v(x)=\mathrm{e}^{\rho x}\Phi_\rho(x)$, where $\rho=\rho(\delta)$ is the unique positive root of Lundberg's fundamental equation (note that $c-\lambda\int_0^\infty xf(x)\mathrm{d}x$ is not necessarily positive):

$$\lambda\hat{f}(s)=\lambda+\delta-cs-\frac{1}{2}\sigma^2s^2,$$

and $\Phi_\rho(x)$ satisfies the following second order integro-differential equation:

$$\frac{\sigma^2}{2}\Phi_\rho''(x)+(c+\sigma^2\rho)\Phi_\rho'(x)+\lambda\hat{f}(\rho)\int_0^x\Phi_\rho(x-z)\tilde{f}(z)\mathrm{d}z=\lambda\hat{f}(\rho)\Phi_\rho(x),\quad x>0, \tag{4.2}$$

with $\Phi_\rho(0)\geq 0$ and $\Phi_\rho'(0)>0$, where $\tilde{f}$ is the Esscher-transformed density function

$$\tilde{f}(x)=\frac{\mathrm{e}^{-\rho x}f(x)}{\hat{f}(\rho)},\quad x>0.$$

By integrating equation (4.2) twice, we transform it into a Volterra integral equation of the second kind

$$\Phi_\rho(x)+\int_0^x a(x-z)\Phi_\rho(z)\mathrm{d}z=N(x),\quad x>0, \tag{4.3}$$

where

$$N(x)=\Phi_\rho(0)+\Phi_\rho'(0)x+\frac{2(c+\sigma^2\rho)\Phi(0)}{\sigma^2}x,$$

$$a(x)=\frac{c+\sigma^2\rho-\lambda\hat{f}(\rho)\int_0^x(1-\tilde{F}(z))\mathrm{d}z}{\frac{1}{2}\sigma^2},$$

with $\tilde{F}(x)=\int_0^x\tilde{f}(z)\mathrm{d}z$. Let

$$L(s):=\lambda\hat{f}(s)-\lambda-\delta+cs+\frac{1}{2}\sigma^2s^2.$$

It is easy to prove that $L(s)$ is a convex function and it follows that $L'(\rho)>0$, i.e. $\lambda\hat{f}'(\rho)+c+\sigma^2\rho>0$. Thus we have

$$a(x)\geq\frac{c+\sigma^2\rho-\lambda\hat{f}(\rho)\int_0^\infty(1-\tilde{F}(z))\mathrm{d}z}{\frac{1}{2}\sigma^2}=\frac{c+\sigma^2\rho+\lambda\hat{f}'(\rho)}{\frac{1}{2}\sigma^2}>0.$$

Furthermore, $a\in L^1(0,1)$, $a\in C^1(0,\infty)$, and

$$a'(x)=-\frac{\lambda\hat{f}(\rho)(1-\tilde{F}(x))}{\frac{1}{2}\sigma^2}$$

is negative and increasing on $(0,\infty)$. If f is logconvex, then $1-\tilde{F}$ is logconvex, where $\tilde{F}(x)=(\hat{f}(\rho))^{-1}\int_0^x\mathrm{e}^{-\rho z}p(z)\mathrm{d}z$. By Lemma 3.1 we know that the logconvexity of $1-\tilde{F}$ on $(0,\infty)$ is equivalent to that for any $T>0$ the function $\frac{a'(x)}{a'(x+T)}$ is a nonincreaing function of x on $(0,\infty)$. We remark that the logconvexity of $1-\tilde{F}$ on $(0,\infty)$ also implies the logconvexity of $(c+\sigma^2\rho-\lambda\hat{f}(\rho)\int_0^x(1-\tilde{F}(z))\mathrm{d}z)$ on $(0,\infty)$, which is equivalent to that for any $T>0$ the function $\frac{a(x)}{a(x+T)}$ is a nonincreasing function of x on $(0,\infty)$. It follows from [30, Theorem 2] that the following Volterra integral equation

$$r(x)+\int_0^x a(x-z)r(z)\mathrm{d}z=a(x),\quad x\geq 0, \tag{4.4}$$

has a unique, continuous, positive and nonincreasing solution on $(0, \infty)$. Moreover, from [31, Theorem 2] (see also [30]) $0 \le r(x) \le a(x)$ on $(0, \infty)$ and $\int_0^\infty r(x)\mathrm{d}x \le 1$. It follows that for $x > 0$,

$$\Phi_\rho(x) = \mathrm{e}^{-\rho x}v(x) > 0,$$
$$\Phi'_\rho(x) = \left(\Phi'_\rho(0) + \frac{2(c+\sigma^2\rho)}{\sigma^2}\Phi_\rho(0)\right)\left(1 - \int_0^x r(z)\mathrm{d}z\right) - \Phi_\rho(0)r(x),$$

and

$$\Phi''_\rho(x) = -\left(\Phi'_\rho(0) + \frac{2(c+\sigma^2\rho)}{\sigma^2}\Phi_\rho(0)\right)r(x) - \Phi_\rho(0)r'(x), \quad \text{a.e. } x > 0.$$

As $\lim_{x\to\infty} r(x) = 0$, $r'(x) \le 0$, a.e., we have

$$\lim_{x\to\infty} \Phi'_\rho(x) = \left(\Phi'_\rho(0) + \frac{2(c+\sigma^2\rho)}{\sigma^2}\Phi_\rho(0)\right)\left(1 - \int_0^\infty r(z)\mathrm{d}z\right) \ge 0,$$
$$\lim_{x\to\infty} \Phi''_\rho(x) \ge -\left(\Phi'_\rho(0) + \frac{2(c+\sigma^2\rho)}{\sigma^2}\Phi_\rho(0)\right)\lim_{x\to\infty} r(x) = 0.$$

Thus, there exists $x_0 > 0$ such that for all $x > x_0$ we have

$$v''(x) = \mathrm{e}^{\rho x}\left(\rho^2\Phi_\rho(x) + 2\rho\Phi'_\rho(x) + \Phi''_\rho(x)\right) > 0, \quad \text{a.e.},$$

which ends the proof of Theorem 4.1. □

Corollary 4.1. *Let F be a distribution function on $(0, \infty)$ with a completely monotone density f. If v is an increasing function on $(0, \infty)$ with $v(0) = 0$ and satisfies the integro-differential equation (4.1). Then v' is strictly convex on $(0, \infty)$.*

Proof. If f is completely monotone on $(0, \infty)$, then $a \in L^1(0, 1)$ is completely monotone on $(0, \infty)$. Friedman [32] showed that if $a \in L^1(0, 1)$ is completely monotone on $(0, \infty)$, then r is also completely monotone. From the proof of Theorem 4.1 we see that $\Phi'_\rho(x)$ is completely monotone. Thus by Lemma 3.5, it has the representation

$$\Phi_\rho(x) = a + bx + \int_0^\infty (1 - \exp(-yx))\lambda(\mathrm{d}y)$$

for all $x > 0$, where $a, b \ge 0$ and λ is a measure on $(0, \infty)$ satisfying $\int_0^\infty (y \wedge 1)\lambda(\mathrm{d}y) < \infty$. Therefore, it follows from $v(x) = \mathrm{e}^{\rho x}\Phi_\rho(x)$ that (see [10]),

$$v'''(x) = (\mathrm{e}^{\rho x}(a + bx))''' + \int_0^\infty \left(\rho^3\mathrm{e}^{\rho x} + (y-\rho)^3\mathrm{e}^{-x(y-\rho)}\right)\lambda(\mathrm{d}y) > 0,$$

as desired. □

Theorem 4.2. *Assume that F and v satisfy the same conditions as in Theorem 4.1. Then the function $v(x)$ is strictly convex in (a^*, ∞), where*

$$a^* = \sup\{a \ge 0 : v'(a) \le v'(x) \text{ for all } x \ge 0\}.$$

Proof. Because $v'(x)$ tends to infinity as x tends to ∞, it follows that $a^* < \infty$. Using the same argument as that in [11] we will prove that $v(x)$ is strictly convex in (a^*, ∞). To do so, let $\alpha_1 < \alpha_2$ be points at which $v'(x)$ reaches local minima. Then $v''(\alpha_1) = v''(\alpha_2) = 0$, which implies

$$-\rho(v'(\alpha_1) - v'(\alpha_2)) = \rho\mathrm{e}^{\rho\alpha_1}\Phi'_\rho(\alpha_1) + \mathrm{e}^{\rho\alpha_1}\Phi''_\rho(\alpha_1) - \left(\rho\mathrm{e}^{\rho\alpha_2}\Phi'_\rho(\alpha_2) + \mathrm{e}^{\rho\alpha_2}\Phi''_\rho(\alpha_2)\right). \tag{4.5}$$

We claim that the right hand side of (4.5) is nonpositive. In fact, let $Y(x) = \mathrm{e}^{\rho x}(1 - \int_0^x r(s)\mathrm{d}s)$, for simplicity we let $\Phi_\rho(0) = 0$, then it follows from (4.4) that $Y(x)$ satisfies the following Volterra integral equation

$$Y(x) = \mathrm{e}^{\rho x} - \int_0^x a_1(x-s)Y(s)\mathrm{d}s,$$

where $a_1(x) = \mathrm{e}^{\rho x}a(x)$. Taking the derivative with respect to x twice and rearranging it yields

$$\mathrm{e}^{-\rho x}Y''(x) = K(x) - \int_0^x a(x-s)\mathrm{e}^{-\rho s}Y''(s)\mathrm{d}s \tag{4.6}$$

where $K(x) = \rho^2 - 2\rho a(x) - a'(x) + a(0)a(x)$. The solution of (4.6) has the form

$$e^{-\rho x} Y''(x) = K(x) - \int_0^x r(x-s)K(s)\mathrm{d}s, \tag{4.7}$$

where the resolvent kernel r is the solution of the Eq. (4.4). It is easy to show that

$$K(x) = \rho^2 + \frac{2c}{\sigma^2}\frac{c + \rho\sigma^2 - \lambda\hat{f}(\rho)\int_0^x (1-\tilde{F}(z))\mathrm{d}z}{\frac{\sigma^2}{2}} + \frac{\lambda\hat{f}(\rho)(1-\tilde{F}(x))}{\frac{\sigma^2}{2}} > 0, \tag{4.8}$$

$$K(0) = \rho^2 + \frac{2c}{\sigma^2}\left(2\rho + \frac{2c}{\sigma^2}\right) + \frac{2\lambda\hat{f}(\rho)}{\sigma^2} > 0,$$

$$K'(x) = \frac{2\lambda\hat{f}(\rho)}{\sigma^2}\left(-\tilde{f}(x) - \frac{2c}{\sigma^2}(1-\tilde{F}(x))\right) < 0.$$

Differentiating (4.6) and using the mean value theorem of integrals we find that there exists $x_0 \in (0, x)$ such that

$$\left(e^{-\rho x} Y''(x)\right)' = K'(x) - r(x_0)K(x) + K(0)(r(x_0) - r(0)) < 0, \quad x > 0,$$

since r is positive and nonincreasing on $(0, \infty)$. Thus $e^{-\rho x} Y''(x)$ is a decreasing function on $(0, \infty)$. From (4.7) and (4.8) it follows that

$$\begin{aligned}\lim_{x\to\infty} e^{-\rho x} Y''(x) &= \lim_{x\to\infty}\left(1 - \int_0^x r(z)\mathrm{d}z\right)\lim_{x\to\infty} K(x) \\ &= \left(1 - \int_0^\infty r(z)\mathrm{d}z\right)\left(\rho^2 + \frac{2c}{\sigma^2}\frac{c + \rho\sigma^2 - \lambda\hat{f}(\rho)\int_0^\infty (1-\tilde{F}(z))\mathrm{d}z}{\frac{\sigma^2}{2}}\right) \geq 0.\end{aligned}$$

Therefore $e^{-\rho x} Y''(x) \geq 0$, $x > 0$, and hence $Y''(x) \geq 0$ on $(0, \infty)$. Note that

$$Y(x) = e^{\rho x}\frac{\Phi'_\rho(x)}{\Phi'_\rho(0)}, \qquad Y'(x) = \frac{1}{\Phi'_\rho(0)}\left(\rho e^{\rho x}\Phi'_\rho(x) + e^{\rho x}\Phi''_\rho(x)\right).$$

Thus $\rho e^{\rho x}\Phi'_\rho(x) + e^{\rho x}\Phi''_\rho(x)$ is an increasing function, so the claim is valid and hence $v'(\alpha_1) \geq v'(\alpha_2)$. This implies that the last place where v' reaches a local minimum is also the last place where it hits its global minimum. For any $x > a^*$, we claim $v''(x) > v''(a^*)$. Otherwise, if there exist $x^* > a^*$ such that $v''(x^*) = v''(a^*)$, because $v''(x) = \rho v'(x) + Y'(x)$ and $Y'(x)$ is increasing function, we include that $v'(a^*) \geq v'(x^*)$, which is a contradiction to the fact that a^* is the largest value where v' attains its global minimum. It follows that $v'(x)$ is strictly increasing for $x > a^*$. This proves Theorem 4.2. □

Next, we consider the following first order integro-differential equation:

$$ch'(x) + \lambda\int_0^x h(x-z)f(z)\mathrm{d}z = (\lambda + \delta)h(x), \quad x > 0, \tag{4.9}$$

with $h(0) > 0$. Here c, λ, δ are positive constants.

Theorem 4.3. *Let F be a distribution function on $(0, \infty)$ with density f. Assume that $f(x)$ is nonincreasing and logconvex on $(0, \infty)$. If h is an increasing function on $(0, \infty)$ with $h(0) > 0$, and satisfies the integro-differential equation (4.9). Then h' is convex on $(0, \infty)$.*

Proof. Note that the technique used in proof of Theorem 4.1 does not work in this case. Here, we give a more direct proof. Integrating both sides of (4.9) from 0 to u and noting that

$$\int_0^u \int_0^y h(y-z)\mathrm{d}F(z)\mathrm{d}y = \int_0^u F(u-z)h(z)\mathrm{d}z$$

yield

$$h(u) = h(0) + \int_0^u h(u-y)k(y)\mathrm{d}y,$$

where

$$k(y) = \frac{\delta + \lambda(1 - F(y))}{c}.$$

Because $f(x)$ is nonincreasing and logconvex, it can be verified that k is logconvex. The result follows from Theorem 3.2 in [33], since logconvexity is a stronger property than convexity. □

5. The optimality of the barrier strategy

In this section we will prove the main result of [11] by using an alternative argument.

Theorem 5.1. *Suppose that X has a Gaussian component $\sigma > 0$ and that the Lévy measure Π of X has a density π which is logconvex, then for $q > 0$ the scale function $W^{(q)}$ is convex in the interval (a^*, ∞), where a^* is the largest value at which $W^{(q)\prime}$ attains its global minimum. As a consequence, the barrier strategy at a^* is an optimal strategy.*

Corollary 5.1. *For the Cramér–Lundberg model that is perturbed by Brownian motion, suppose that the claim sizes have a common distribution with a logconvex density. Then the barrier strategy at a^* is an optimal strategy.*

Theorem 5.2. *Suppose that X has no Gaussian component and that the Lévy measure Π of X has a density π which is nonincreasing and logconvex, then for $q > 0$ the scale function $W^{(q)}$ is convex in the interval (a^*, ∞), where a^* is the largest value at which $W^{(q)\prime}$ attains its global minimum. As a consequence, the barrier strategy at a^* is an optimal strategy.*

Corollary 5.2. *For the Cramér–Lundberg model, suppose that the claim sizes have a common distribution with a non-increasing and logconvex density, then the barrier strategy at a^* is an optimal strategy.*

Remark 5.1. For spectrally negative Lévy process with the Laplace exponent given by (2.1) if the density π of Lévy measure Π is any one of densities listed at the end of Section 3, then a barrier strategy will form an optimal strategy.

The proofs of these two theorems follow the same scheme. So we will only describe the proof of Theorem 5.1.

Proof of Theorem 5.1. We first assume that

$$\lambda := \int_0^\infty \Pi(\mathrm{d}x) < \infty, \qquad \int_0^1 x\Pi(\mathrm{d}x) < \infty.$$

Set

$$c = a + \int_0^1 x\Pi(\mathrm{d}x), \qquad f(x) = \pi(x)/\lambda,$$

then f is a probability density on $(0, \infty)$. π is logconvex if and only if f is logconvex. Consider the integro-differential equation

$$\frac{\sigma^2}{2}v''(x) + cv'(x) + \lambda\int_0^x v(x-z)f(z)\mathrm{d}z = (\lambda+q)v(x), \quad x > 0, \tag{5.1}$$

with $v(0) = 0$ and $v'(0) = \frac{2}{\sigma^2}$. The Laplace transform $\hat{v}$ for v can be easily determined from Eq. (5.1) as

$$\hat{v}(\theta) = \frac{\frac{\sigma^2}{2}v'(0) + \frac{\sigma^2}{2}\theta v(0) + cv(0)}{\frac{\sigma^2}{2}\theta^2 + c\theta + \lambda\hat{f}(\theta) - \lambda - q} = \frac{1}{\psi(\theta) - q}, \quad \theta > 0. \tag{5.2}$$

Comparing (5.2) with (2.2) we get that $v = W^{(q)}$. It follows from Theorem 4.2 that for $q > 0$ the scale function $W^{(q)}$ is convex in the interval (a^*, ∞), where a^* is the largest value at which $W^{(q)\prime}$ attains its global minimum. Hence by Theorem 2 in [10], the barrier strategy at a^* is an optimal strategy.

Now, we assume that

$$\lambda := \int_0^\infty \Pi(\mathrm{d}x) = \infty.$$

Let Π_n be measures on $(0, \infty)$:

$$\Pi_n(\mathrm{d}x) = \Pi(\mathrm{d}x)\mathbf{1}_{\left(\frac{1}{n},\infty\right)}(x), \quad n \geq 1.$$

Then we have

$$\begin{aligned}\lambda_n &:= \int_0^\infty \Pi_n(\mathrm{d}x) = \int_{\frac{1}{n}}^1 \Pi(\mathrm{d}x) + \int_1^\infty \Pi(\mathrm{d}x)\\ &\leq n^2\int_{\frac{1}{n}}^1 x^2\Pi(\mathrm{d}x) + \int_1^\infty (1\wedge x^2)\Pi(\mathrm{d}x) < \infty.\end{aligned}$$

Consider the integro-differential equation

$$\frac{\sigma^2}{2}v_n''(x) + c_n v_n'(x) + \lambda_n \int_0^x v_n(x-z)f_n(z)\mathrm{d}z = (\lambda_n + q)v_n(x), \quad x > 0 \tag{5.3}$$

with $v_n(0) = 0$ and $v_n'(0) = \frac{2}{\sigma^2}$, where

$$c_n = a + \int_0^1 x\Pi_n(\mathrm{d}x), \qquad f_n(x) = \pi_n(x)/\lambda_n.$$

Here π_n is the density of Π_n.

It is easy to see that f_n is a probability density on $(0, \infty)$, π_n is logconvex if and only if f_n is logconvex. The Laplace transform $\hat{v}_n$ for v_n can be easily determined from Eq. (5.3) as

$$\hat{v}_n(\theta) = \frac{1}{\psi_n(\theta) - q}, \quad \theta > 0, \tag{5.4}$$

where

$$\psi_n(\theta) = a\theta + \frac{1}{2}\sigma^2\theta^2 - \int_0^\infty \left(1 - \mathrm{e}^{-\theta x} - \theta x \mathbf{1}_{\{0<x<1\}}\right)\Pi_n(\mathrm{d}x).$$

Since $\lim_{n\to\infty}\Pi_n = \Pi$, we obtain $\lim_{n\to\infty}\psi_n(\theta) = \psi(\theta)$, and thus for $x > 0$, by (2.2) and (5.4), $W^{(q)}(x) = \lim_{n\to\infty} v_n(x)$. It follows from Theorem 4.2 that v_n is convex in the interval (a_n^*, ∞), where a_n^* is the largest value at which v_n' attains its global minimum. Therefore, for $q > 0$ the scale function $W^{(q)}$ is convex in the interval (a^*, ∞) since $\lim_{n\to\infty} a_n{}^* = a^*$. Hence by Theorem 2 in [10], the barrier strategy at a^* is an optimal strategy. This ends the proof of Theorem 5.1. □

From the proof of Theorems 5.1 and 5.2, we get

Corollary 5.3. *Suppose that X has a Gaussian component $\sigma > 0$, then for $q \geq 0$ the scale function $W^{(q)}$ satisfies $\Gamma W^{(q)}(x) = qW^{(q)}(x)$, $x > 0$, with $W^{(q)}(0) = 0$ and $W^{(q)\prime}(0) = \frac{2}{\sigma^2}$, where*

$$\Gamma g(x) = \frac{1}{2}\sigma^2 g''(x) + ag'(x) + \int_0^\infty \left[g(x-y) - g(x) + g'(x)y\mathbf{1}_{(0<y<1)}\right]\Pi(\mathrm{d}y).$$

Corollary 5.4. *Suppose that X has no Gaussian component and Π has no atoms. Then for $q \geq 0$ the scale function $W^{(q)}$ satisfies $\Gamma_1 W^{(q)}(x) = qW^{(q)}(x)$, $x > 0$, with*

$$W^{(q)}(0) = \frac{1}{c}, \quad \text{if } \int_0^1 x\Pi(\mathrm{d}x) < \infty;\ 0,\ \text{if } \int_0^1 x\Pi(\mathrm{d}x) = \infty,$$

where $c = a + \int_0^1 x\Pi(\mathrm{d}x)$, and

$$\Gamma_1 g(x) = ag'(x) + \int_0^\infty \left[g(x-y) - g(x) + g'(x)y\mathbf{1}_{(0<y<1)}\right]\Pi(\mathrm{d}y).$$

6. Concluding remarks

In this paper, we provided an analytical study of the classical dividend control problem of de Finetti for spectrally negative Lévy process. Related optimal dividend problems were also established in [9,34,35].

Barrier strategies are simple and convenient for operating, so they are of particular interest, even in cases where the optimal dividend strategy is not of this form. The problem of finding the optimal dividend-payment barrier has been discussed extensively. See, among others, [36] for the classical risk model, [37] for a Brownian motion model, [38] for the stationary Markovian model. The references therein are rich sources for the literature on this subject. For most risk models even for particular cases of spectrally negative Lévy processes, an explicit expression for the optimal barrier a^* can not be obtained. The recent paper of [39] has given several methods for estimating the optimal dividend barrier.

Acknowledgements

We would like to thank the anonymous referee who gave us many constructive suggestions and valuable comments on the previous version of this paper. This research was supported by the National Natural Science Foundation of China (No. 10771119) and the research Fund for the Doctoral Program of Higher Education (No. 20093705110002).

References

[1] B. De Finetti, Su un'impostazion alternativa dell teoria collecttiva del rischio, Transactions of the XVth International Congress of Actuaries 2 (1957) 433–443.
[2] S. Asmussen, B. Høgaard, M. Taksar, Optimal risk control and dividend distribution policies. Example of excess of loss reinsurance for an insurance corporation, Finance Stoch. 4 (2000) 299–324.
[3] J.D. Décamps, S. Villeneuve, Optimal dividend policy and growth option, Finance Stoch. 11 (2007) 3–27.
[4] J. Paulsen, Optimal dividend payouts for diffusions with solvency constraints, Finance Stoch. 7 (2003) 457–473.
[5] S.E. Shreve, J.P. Lehoczky, D.P. Gaver, Optimal consumption for general diffusions with absorbing and reflecting barriers, SIAM J. Control Optim. 22 (1) (1984) 55–75.
[6] H.U. Gerber, Entscheidungskriterien für den zusammengesetzten Poisson-Prozess, Mitteilungen der Vereinigung Schweizerischer Versicherungsmathematiker 69 (1969) 185–227.
[7] P. Azcue, N. Muler, Optimal reinsurance and dividend distribution policies in the Cramér–Lundberg model, Math. Finance 15 (2005) 261–308.
[8] H. Bühlmann, Mathematical Methods in Risk Theory, Springer-Verlag, Heidelberg, 1970.
[9] F. Avram, Z. Palmowski, M.R. Pistorius, On the optimal dividend problem for a spectrally negative Lévy process, Ann. Appl. Probab. 17 (2007) 156–180.
[10] R. Loeffen, On optimality of the barrier strategy in de Finetti's dividend problem for spectrally negative Lévy processes, Ann. Appl. Probab. 18 (5) (2008) 1669–1680.
[11] A.E. Kyprianou, V. Rivero, R. Song, Convexity and smoothness of scale functions with applications to de Finetti's control problem, http://arxiv.org/PS-cache/arxiv/pdf/0801/0801.1951v3.pdf.
[12] H. Albrecher, S. Thonhausera, Optimal dividend strategies for a risk process under force of interest, Insurance: Math. Econom. 43 (2008) 134–149.
[13] J. Bertoin, Lévy Processes, Cambridge University Press, Cambridge, 1996.
[14] A.E. Kyprianou, Introductory lectures on fluctuations of Lévy processes with applications, in: Universitext, Springer-Verlag, Berlin, 2006.
[15] J. Bertoin, Exponential decay and ergodicity of completely asymmetric Lévy processes in a finite interval, Ann. Appl. Probab. 7 (1997) 156–169.
[16] A.D. Doney, A.E. Kyprianou, Overshoots and undershoots of Lévy processes, Ann. Appl. Probab. 16 (1) (2006) 91–106.
[17] H. Furrer, Risk processes perturbed by α-stable Lévy motion, Scand. Actuar. J. 1 (1998) 59–74.
[18] J. Garrido, M. Morales, On the expected discounted penalty function for Lévy risk processes, N. Am. Actuar. J. 10 (2006) 196–218.
[19] M.M. Huzak, M. Perman, H. Šikić, Z. Vondraček, Ruin probabilities for competing claim processes, J. Appl. Probab. 41 (2004) 679–690.
[20] M.M. Huzak, M. Perman, H. Šikić, Z. Vondraček, Ruin probabilities and decompositions for general perturbed risk processes, Ann. Appl. Probab. 14 (3) (2004) 1378–1397.
[21] C. Klüppelberg, A.E. Kyprianou, On extreme ruinous behaviour of Lévy insurance risk processes, J. Appl. Probab. 43 (2) (2006) 594–598.
[22] C. Klüuppelberg, A.E. Kyprianou, R.A. Maller, Ruin probabilities and overshoots for general Lévy insurance risk processes, Ann. Appl. Probab. 14 (4) (2004) 1766–1801.
[23] A.E. Kyprianou, Z. Palmowski, Distributional study of de Finetti's dividend problem for a general Lévy insurance risk process, J. Appl. Probab. 44 (2) (2007) 428–443.
[24] H.L. Yang, L. Zhang, Spectrally negative Lévy processes with applications in risk theory, Adv. Appl. Probab. 33 (2001) 281–291.
[25] J.F. Renaud, X. Zhou, Distribution of the dividend payments in a general Lévy risk model, J. Appl. Probab. 44 (2) (2007) 420–427.
[26] Y. An, Logconcavity versus logconvexity: A complete characterization, J. Econom. Theory 80 (1998) 350–369.
[27] M. Bagnoli, T. Bergstrom, Log-concave probability and its applications, Econom. Theory 26 (2005) 445–469.
[28] D. Applebaum, Lévy processes and Stochastic Calculus, Cambridge University Press, Cambridge, 2004.
[29] F. Hubalek, A.E. Kyprianou, Old and new examples of scale functions for spectrally negative Lévy processes, 2007, arXiv:0801.0393v1[math.PR].
[30] G. Gripenberg, On positive, nonincreasing resolvents of Volterra equations, J. Differential Equations 30 (1978) 380–390.
[31] R.K. Miller, On Volterra integral equations with nonnegative integrable resolvents, J. Math. Anal. Appl. 22 (1968) 319–340.
[32] A. Friedman, On integral equations of Volterra type, J. Anal. Math. 11 (1963) 381–413.
[33] B.G. Hansen, J.B.G. Frenk, Some monotonicity properties of the delayed renewal function, J. Appl. Probab. 28 (1991) 811–821.
[34] R. Loeffen, Stochastic Control for Spectrally Negative Lévy Processes, Ph.D. Thesis of University of Bath, 2008.
[35] R. Loeffen, An optimal dividends problem with a terminal value for spectrally negative Lévy processes with a completely monotone jump density, J. Appl. Probab. 46 (1) (2009) 85–98.
[36] H.U. Gerber, E.S.W. Shiu, On the time value of ruin, N. Am. Actuar. J. 2 (1) (1998) 48–78.
[37] H.U. Gerber, E.S.W. Shiu, Optimal dividends: Analysis with Brownian motion, N. Am. Actuar. J. 8 (2004) 1–20.
[38] H.U. Gerber, X.S. Lin, H. Yang, A note on the dividends-penalty identity and the optimal dividend barrier, Astin Bull. 36 (2006) 489–503.
[39] H.U. Gerber, E.S.W. Shiu, N. Smith, Methods for estimating the optimal dividend barrier and the probability of ruin, Insurance: Math. Econom. 42 (2008) 243–254.