Partial Steiner Triple Systems with Equal-Sized Holes

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The existence of group divisible designs of type \( u'1' \) with block size three is completely settled for all values of \( u, r, \) and \( t \). © 1995 Academic Press, Inc.

1. INTRODUCTION

Let \( \mathcal{X} \) be a finite set of \( x \) elements, and let \( G = \{ G_1, G_2, \ldots, G_s \} \) be a partition of \( \mathcal{X} \) into subsets called groups. Let \( \mathcal{B} \) be a collection of subsets of \( \mathcal{X} \) called blocks, and let set \( \mathcal{K} = \{ |B| : B \in \mathcal{B} \} \), be the set of block sizes. If \( (\mathcal{X}, \mathcal{B}) \) has the property that every pair of elements either appears in exactly one block or in exactly one group, it is a group divisible design and is denoted by \( \mathcal{K}-\text{GDD} \). The type of the GDD is denoted by \( g_1^{t_1}g_2^{t_2}g_3^{t_3} \ldots g_s^{t_s} \) when the number of groups of size \( g_i \) is \( t_i \). In this paper we establish necessary and sufficient conditions for the existence of a \( \{3\}\)-GDD of type \( u'1' \). When \( r = 0 \), such a \( \{3\}\)-GDD is just a Steiner triple system of order \( t \), and when \( t = 0 \), such a \( \{3\}\)-GDD has type \( u' \). In both cases, existence has been settled (see [3]), so we assume that \( r \) and \( t \) are positive throughout. In particular, we prove the following result.

**MAIN THEOREM.** Let \( u, r, \) and \( t \) be positive integers. Then there exists a \( \{3\}\)-GDD of type \( u'1' \) if and only if the following conditions are satisfied:

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(i) \( u \equiv 1 \) (mod 2);
(ii) \( r + t \equiv 1 \) (mod 2);
(iii) if \( r = 1 \), \( t \geq u + 1 \);
(iv) if \( r = 2 \), \( t \geq u \);
(v) \( \binom{u}{2} + rt \equiv \binom{u}{2} u^2 \equiv 0 \) (mod 3).

We can establish necessity as follows. Since \( t \geq 1 \), consider an element in a group of size 1. It must appear in \( \frac{1}{2} (ru + t - 1) \) triples and, hence, \( ru + t - 1 \equiv 0 \) (mod 2). Since \( r \geq 1 \), consider an element in a group of size \( u \). It appears in \( \frac{1}{2} (u(r - 1) + t) \) triples and, hence, \( ru + t - u \equiv 0 \) (mod 2). For both to hold, \( u \) must be odd; when \( u \) is odd, since \( ru + t - 1 \equiv r + t - 1 \) (mod 2), \( r + t \) must also be odd. For (iii), any element in a singleton group must appear in a triple with each element of the group of size \( u \); the \( u \) third elements of such triples are all distinct, and none appear in the large group. Thus \( t \geq u + 1 \). For (iv), consider an element in one of the groups of size \( u \). Consider the \( u \) triples in which it appears with the elements of the other group of size \( u \). The third elements of such triples form \( u \) distinct elements, none of which appear in one of the large groups; thus \( t \geq u \). Finally, for (v), the number of pairs occurring in triples must be divisible by three for the pairs to be partitioned into triples. These conditions are summarized in Table I. The main effort is in establishing sufficiency. We rely heavily on a theorem of Colbourn, Hoffman, and Rees [3].

**Theorem 1.1.** Let \( g, t, \) and \( u \) be nonnegative integers. There exists a \( \{3\} \)-GDD of the type \( g^t u^1 \) if and only if the following conditions are all satisfied:

(i) if \( g > 0 \), then \( t \geq 3 \), or \( t = 2 \) and \( u = g \), or \( t = 1 \) and \( u = 0 \), or \( t = 0 \);
(ii) \( u \leq g(t - 1) \) or \( gt = 0 \);
(iii) \( g(t - 1) + u \equiv 0 \) (mod 2) or \( gt = 0 \);
(iv) \( gt \equiv 0 \) (mod 2) or \( u = 0 \);
(v) \( \frac{1}{2} g^2 t(t - 1) + gtu \equiv 0 \) (mod 3).

**Table I**

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Corollary 1.2. The conditions of the Main Theorem are sufficient when \( r = 1 \).

When \( u \in \{1, 3\} \), the conditions of the Main Theorem are also sufficient, using known results on resolvable and almost resolvable Steiner triple systems (see [7], for example). Henceforth we treat only the cases when \( u \geq 5 \).

In general, our strategy in proving sufficiency for the conditions in the Main Theorem is to develop recursive constructions that suffice, provided that a finite number of small cases can be produced by direct techniques. To obtain the small \( \{3\}\)-GDD needed, we then apply a variant of Stinson’s hill-climbing algorithm, as described in [2]. Since the actual \( \{3\}\)-GDD produced by hill-climbing exhibit no particular structure of interest, we simply state the cases settled by hill-climbing and do not display each solution.

Lemma 1.3. The conditions of the Main Theorem are sufficient when \( ur + t \leq 60 \), and also for types \( 9714, 9716, 114139, 114145, 11716, 134111, 174157, 174163, 174169, 2371t \) for \( t \in \{6, 12, 18\} \).

Proof. Sufficiency for \( ur + t \leq 60 \) follows from the main computational result in [2]. For the remaining cases, hill-climbing is used (solutions are available from the first author).

2. Extending Resolvable \( \{3\}\)-GDD

In this section we develop a construction that is useful for small numbers of groups. First we require some definitions. A parallel class in a GDD is a set of disjoint blocks that contain each element of the GDD exactly once. A GDD is resolvable if all of its blocks can be partitioned into parallel classes. A holey parallel class is a set of disjoint blocks that contain each element of the GDD, except those of a single group, once; no elements of the special group appear in any block of the set.

Lemma 2.1. Suppose that there exists a resolvable \( \{3\}\)-GDD of type \( g \), having \( p = g(r - 1)/2 \) parallel classes. Let \( w \) be an integer, and let

\[ \mathcal{A}(w) = \{ x : \exists \{3\}\text{-GDD of type } w^3x^1 \}. \]

Suppose further that there is a \( \{2, 3\}\)-GDD of type \( w^3 \) in which the blocks of size two have a partition into \( r - 3 + e \) parallel classes and, for each of the three groups, one holey parallel class that misses precisely that group. Finally write \( p = m + l \), suppose that \( a_1, \ldots, a_l \in \mathcal{A}(w) \) and that a \( \{3\}\)-GDD of type \( m^1 \sum_{i=1}^m a_i \) exists. Then a \( \{3\}\)-GDD of type \( (w + m)^r \sum_{i=1}^m a_i \) exists.
Proof. Let $P_1, \ldots, P_p$ be the parallel classes of the resolvable $\{3\}$-GDD of type $g'$. We give every point of this resolvable $\{3\}$-GDD weight $w$. For each parallel class $P_i$, $1 \leq i \leq l$, we add $a_i$ additional elements; then for each block of $P_i$, on the $3w$ corresponding points, together with the $a_i$ additional points, we place the blocks of a $\{3\}$-GDD of type $w^3a^i_i$. For the parallel classes $P_{i+1}, \ldots, P_p$, we proceed differently. For each, we add one new point to each of the $r$ groups and $e$ extra new points. Then for each block of the parallel class, we proceed as follows. We place the $\{2, 3\}$-GDD of type $w^3$ on the $3w$ points arising from this block. The $r-3+e$ parallel classes of this GDD are used to form triples with each of the $e$ extra points and each of the $r-3$ new points added to groups not met by the chosen block. Finally, the three holey 1-factors are employed to form triples with the three new points added to the groups that are met by the chosen block. Once all parallel classes have been treated, $m = p-l$ additional elements have been added to each of the $r$ groups. On these, together with the $em$ extra elements and the $\sum_{i=1}^{l} a_i$ additional elements, we place the blocks of a $\{3\}$-GDD of type $\sum m^i + \sum a_i$.

The existence of resolvable $\{3\}$-GDD has been recently settled.

**Lemma 2.2** [5]. There exists a resolvable $\{3\}$-GDD of type $g'$ if and only if $g(r-1) \equiv 0 \pmod{2}$, $gr \equiv 0 \pmod{3}$, and $(g, r) \notin \{(2, 3), (2, 6), (6, 3)\}$.

Now there exists a $\{2, 3\}$-GDD of type $2^3$ having three holey 1-factors and two triples. Specifically, on groups $\{\{0, 1\}, \{2, 3\}, \{4, 5\}\}$, form triples $\{\{0, 2, 4\}, \{1, 3, 5\}\}$ and three holey parallel classes $\{\{2, 5\}, \{3, 4\}\}$, $\{\{0, 5\}, \{1, 4\}\}$, and $\{\{0, 3\}, \{1, 2\}\}$. This allows the use of $w=2$ and $r-3+e=0$ in Lemma 2.1. Thus we obtain the following.

**Corollary 2.3.** For $u \equiv 1 \pmod{2}$, $t \equiv 0, 4 \pmod{6}$, and $t \leq 2u-6$ there exists a $\{3\}$-GDD of type $u^31^t$.

**Proof.** If $u \in \{1, 3, 5, 13\}$, the required $\{3\}$-GDD are from Lemma 1.3. Otherwise we apply Lemma 2.1 with $g = (u-1)/2$, $w=2$, $r=3$, $m=1$, and $e=0$. A resolvable $\{3\}$-GDD of type $g^3$ has $g$ parallel classes. Theorem 1.1 gives $\mathcal{A}(2) = \{0, 2, 4\}$. Thus $t = \sum_{i=1}^{l} a_i$, where each $a_i \in \mathcal{A}(2)$ if and only if $t \leq 2u-6$. Furthermore, the $\{3\}$-GDD of type $m^i + \sum a_i$ required by Lemma 2.1 is, after substitution, a Steiner triple system of order $3+t$ which exists for orders congruent to 1, 3 (mod 6). Since $t \equiv 0, 4 \pmod{6}$ we obtain the desired result.

There is a $\{2, 3\}$-GDD of type $4^3$ having three holey parallel classes of pairs, two parallel classes of pairs, and eight triples. For example, on $Z_4 \times \{0, 1, 2\}$, the three groups are $Z_4 \times \{i\}$, for $i \in \{0, 1, 2\}$ (we use the
common notation \( i_j \) for \((i, j) \in \mathbb{Z}_4 \times \{0, 1, 2\}\). The eight triples are \(\{i_0, i_1, i_2\}\) and \(\{i_0, (i+1)_1, (i+2)_2\}\), for \(i \in \mathbb{Z}_4\), arithmetic modulo 4. The three holey parallel classes are \(\{(i, (i+1)_2) : i \in \mathbb{Z}_4\}\), \(\{(i_0, (i+1)_2) : i \in \mathbb{Z}_4\}\), and \(\{(i_0, (i+3)_1) : i \in \mathbb{Z}_4\}\). The 12 pairs that remain uncovered form a 12-gon, which can be factored into two perfect matchings to give the two parallel classes. Now using \(w = 4\) and \(r = 3 + e = 2\) in Lemma 2.1 we obtain the following.

**Corollary 2.4.** Suppose that the Main Theorem holds for \(r = 5\) and \(u \in \{7, 9\}\). If the conditions of the Main Theorem are met for \(u, r, t, r = 5\) and \(t < u\), then there exists a \(\{3\}\)-GDD of type \(u^5\).

**Proof.** Write \(m \equiv u \pmod{12}\), \(0 < m < 12\). Applying Lemma 2.1 with \(g = (u - m)/4\), \(w = 4\), \(r = 5\), and \(e = 0\) gives the required \(\{3\}\)-GDD. (Details are similar to Corollary 2.3). \(\square\)

There is a \(\{2, 3\}\)-GDD of type \(4^3\) having three holey parallel classes and four parallel classes of pairs. The construction is similar to that given above. Hence we obtain the following.

**Corollary 2.5.** Let \(u, r, t, r = 7\), satisfy the conditions of the Main Theorem and let \(t < u\). Let \(m \equiv u \pmod{12}\) and \(0 < m < 12\). Suppose that the Main Theorem holds for \(m^7\). Then there is a \(\{3\}\)-GDD of type \(u^7\).

**Proof.** If \(u < 12\), the statement holds by the assumption in the statement. If \(u = 23\), the required \(\{3\}\)-GDD are from Lemma 1.3. Otherwise, apply Lemma 2.1 with \(g = (u - m)/4\), \(w = 4\), \(r = 7\), and \(e = 0\) to obtain the required \(\{3\}\)-GDD. \(\square\)

Finally, we present a construction for \(r \in \{4, 8\}\). We require a result of Rees [4].

**Lemma 2.6.** Let \(r \geq 1\) and \(0 \leq f \leq 2r\), \((r, f) \neq (1, 2)\) or \((3, 6)\). There exists a \(\{2, 3\}\)-GDD of type \((2r)^3\) which is resolvable into \(f\) parallel classes of blocks of size 3 and \(x = 4r - 2f\) parallel classes of blocks of size 2.

**Lemma 2.7.** For any \(u \equiv 1 \pmod{6}\), \(u \geq 19\), \(t \equiv 5 \pmod{6}\) and \(t < u\), there exists a \(\{3\}\)-GDD of type \(u^1\) for \(r = 4\) and 8.

**Proof.** Let \(\gamma = 1\) if \(r = 8\), and \(\gamma = 3\) if \(r = 4\). Let \(z = (u - 1)/2\). Now choose even integers \(f_1, \ldots, f_{z-1}\) so that \(0 \leq f_i \leq 2r - 2\) for \(1 \leq i \leq z - 1\), and \(\gamma + \sum_{i=1}^{z-1} f_i = t\).

Using Lemma 2.2, form a resolvable \(\{3\}\)-GDD \(G\) of type \(2^r\) on elements \(\{x_i : 1 \leq i \leq 2z\}\) with groups \(\{x_{2i-1}, x_{2i} : 1 \leq i \leq z\}\). Then give each point of \(G\) weight \(r\)—that is, form a set of \(2rz\) elements \(\{x_{ij} : 1 \leq i \leq 2z, 1 \leq j \leq r\}\).
Now $G$ has $z - 1$ parallel classes of triples $T_1, \ldots, T_{z - 1}$. Associate with each parallel class $T_i$ a set $F_i$ of $f_i$ new elements. Using Lemma 2.6, form a \{2, 3\}-GDD $G_i$ of type $r^3$ having $f_i$ parallel classes of edges and $(2r - f_i)/2$ parallel classes of triples. For each triple $\{x_{a}, x_{b}, x_{c}\} \in T_i$, place a copy of $G_i$ on groups $\{\{x_{i,j}: 1 \leq j \leq r\}: i \in \{a, b, c\}\}$, omitting a parallel class of triples on $\{\{x_{a_i,j}, x_{b_i,j}, x_{c_i,j}\}: 1 \leq j \leq r\}$. Now use the points of $F_i$ to extend the $f_i$ parallel classes of edges in each copy of $G_i$ to triples.

Add next a further set of new elements $R = \{y_1, \ldots, y_r\}$, and a set $I$ of $\gamma$ final new elements. Then for each $1 \leq i \leq z$, place on $\{x_{2i-1,j}, x_{2i,j}: 1 \leq j \leq r\} \cup R \cup I$ the triples of a \{3\}-GDD of type $(r + \gamma)^{1/2}r$, leaving the hole on $R \cup I$. In this placement, omit the set of triples $\{\{x_{2i-1,j}, x_{2i,j}, y_j\}: 1 \leq j \leq r\}$. Finally, observe that $r + \gamma + \sum_{i=1}^{r-1} f_i \equiv 1, 3 \pmod{6}$, since $r + \gamma + \sum_{i=1}^{r-1} f_i \equiv 5 \pmod{6}$. Hence we can place the triples of a Steiner triple system on $R \cup I \cup \bigcup_{i=1}^{r-1} F_i$. This completes the construction, leaving $r$ holes of size $u$ on $\{x_{i,j}: 1 \leq i \leq 2z\} \cup \{y_j\}$, for $1 \leq j \leq r$.

3. The Case $u = 1, 3 \pmod{6}$

We employ Theorem 1.1, together with a simple lemma, to settle the majority of the cases.

**Lemma 3.1.** Let $0 \leq s < r$. If a \{3\}-GDD of type $u^{r-s}(su + t)^{1}$ and a \{3\}-GDD of type $u^{1}$ exist, then a \{3\}-GDD of type $u^{1}$ exists.

**Proof.** Fill the group of size $su + t$ with the \{3\}-GDD of type $u^{1}$.

**Theorem 3.2.** There exists a \{3\}-GDD of type $u^{1}$ whenever $u \equiv 1, 3 \pmod{6}$ and the conditions of the Main Theorem are met.

**Proof.** First we establish that if the necessary conditions in the Main Theorem are sufficient for $u \equiv 1, 3 \pmod{6}$ when $t < u$, they are sufficient for $u \equiv 1, 3 \pmod{6}$ and all $t$. If $t \geq u$, write $s = \lfloor t/u \rfloor$. Form a \{3\}-GDD of type $u^{s+1}t - su$. Fill $s$ groups of size $u$ of the \{3\}-GDD with the triples of a \{3\}-GDD of type $u^{1}$ to obtain the required \{3\}-GDD of type $u^{1}$. We suppose henceforth that $t < u$.

If $(r, u, t)$ (mod 6, 6, 6) is one of $(0, 1, 1), (0, 1, 3), (0, 3, 1), (0, 3, 3), (2, 1, 1), (2, 3, 1), (2, 3, 3), (4, 1, 3), (4, 3, 1), (4, 3, 3)$, apply Lemma 3.1 with $s = 0$ (Theorem 1.1 produces the required \{3\}-GDD).

If $r = 3$, we apply Corollary 2.3. So suppose that $r \geq 4$. If $(r, u, t)$ (mod 6, 6, 6) is one of $(1, 1, 0), (1, 3, 0), (1, 3, 4), (3, 1, 0), (3, 1, 4), (3, 3, 0), (3, 3, 4), (5, 1, 4), (5, 3, 0), and $(5, 3, 4)$, and $r \geq 8$, apply Lemma 3.1 with $s = 3$ (the \{3\}-GDD of type $u^{r-3}(3u + t)^{1}$ is from Theorem 1.1).
If \( r = 4 \), \( u \equiv 1 \pmod{6} \), \( u > 13 \), and \( t \equiv 5 \pmod{6} \), apply Lemma 2.7. If instead \( u \in \{7, 13\} \), apply Lemma 1.3. This completes the last case when \( r = 4 \).

If \( r = 5 \), the cases with \( u \in \{7, 9\} \) and \( t < u \) are from Lemma 1.3. Now apply Corollary 2.4 to settle all cases with \( u > 12 \) and \( t < u \).

If \((r, u, t) \equiv (6, 6, 6)\) is \((2, 1, 5)\) or \((4, 1, 5)\) and \( r \geq 10 \), apply Lemma 3.1 with \( s = 4 \), using Theorem 1.1 to provide the \( \{3\} \)-GDD of type \( u' - 4(4u + t)^1 \).

If \((r, u, t) \equiv (6, 6, 6)\) is \((1, 1, 2)\) or \((5, 1, 2)\), and \( r \geq 12 \), apply Lemma 3.1 with \( s = 5 \), using Theorem 1.1 to provide the \( \{3\} \)-GDD of type \( u - 5(5u + t)^1 \).

It remains to treat all cases with \( r = 7 \), the case when \( r = 8 \), \((u, t) \equiv (1, 5) \pmod{6, 6} \), and the case when \( r = 11 \), \((u, t) \equiv (1, 2) \pmod{6, 6} \) (in each case under the restriction that \( t < u \)). For \( r = 7 \), all \( \{3\} \)-GDD with \( u \in \{7, 9\} \) and \( t < u \) are from Lemma 1.3. Now apply Corollary 2.5 to settle the remaining cases with \( t < u \) and \( u > 12 \). For \( r = 8 \), apply Lemma 2.7. For \( r = 11 \), we proceed as follows. Form a resolvable \( \{3\} \)-GDD of type \((u - 1)/2^1\) on groups \( G_1, ..., G_{11} \) with parallel classes \( P_1, ..., P_{5(u - 1)/2} \). Write \( \alpha = t/2 \). Form a set \( T \) of \( \alpha \) elements \( \{e_1, ..., e_{\alpha}\} \) disjoint from \( G_1, ..., G_{11} \). Now define a new set of groups: \( G'_i = (G_i \times \{1, 2\}) \cup \{f_i\} \) for \( 1 \leq i \leq 11 \), and \( T' = T \times \{1, 2\} \). For \( 1 \leq i \leq \alpha \), and for each triple \( \{x, y, z\} \in P_i \), place a \( \{3\} \)-GDD on \( \{x, y, z\} \times \{1, 2\} \cup \{f(x), f(y), f(z)\} \), missing one triple on \( \{f(x), f(y), f(z)\} \). For \( \alpha + 2 \leq i \leq 5(u - 1)/2 \), and each triple \( \{x, y, z\} \in P_i \), place a \( \{3\} \)-GDD of type \( 2^3 \) on \( \{x, y, z\} \times \{1, 2\} \). Finally, on the elements \( T'' \cup \{f_i: 1 \leq i \leq 11\} \), place a \( \{3\} \)-GDD of type \( 1^{2\alpha + 11} \). This completes the proof.

In particular Theorem 3.2 shows that the necessary conditions are sufficient when \( r = 2 \).

4. The Case \( u \equiv 5 \) modulo 6 and \( r \equiv 0 \) modulo 3

First we give a construction for \( r = 3 \).

**Lemma 4.1.** Let \( u \equiv 5 \pmod{6} \), \( t \equiv 0 \pmod{6} \), and \( 2u - 4 \leq t < 6u \). Then there exists a \( \{3\} \)-GDD of type \( u^31' \).

**Proof.** If \( u \in \{5, 11\} \), the required \( \{3\} \)-GDD are from Lemma 1.3. Otherwise let \( x = (u - 2)/3 \) and form a \( \{5\} \)-GDD of type \( x^5 \) (i.e., a transversal design \( \text{TD}(5, x) \)). Choose one special block \( B \). In the first three groups, give all points not in the special block weight 3; and in the last two groups, give all points not in the special block weight 3 or 9. Give the points of the special block in the first three groups weight 5 and give the
last two points weight 3 and weight 3 or 7. Using \{3\}-GDD of type \(5^33^2, \ 5^13^4, \ 5^13^19^1, \ 5^13^29^2, \ 3^5, \ 3^49^1, \ 3^39^2, \ 7^15^23^1, \ 9^17^13^3, \) and \(3^47^1\) (all from Lemma 1.3), we obtain a \{3\}-GDD of type \(u^3v^1w^1\) in which \(v\) and \(w\) are groups whose sizes are 1 or 3 (mod 6). Filling in the groups of sizes \(v\) and \(w\) with Steiner triple systems constructs the required \{3\}-GDD.

Now we complete the proof of the Main Theorem when \(r \equiv 0 \pmod{3}\).

**Theorem 4.2.** If \(u^1v^1 \) meets the conditions of the Main Theorem, \(u \equiv 5 \pmod{6}\) and \(r \equiv 0 \pmod{3}\), then there exists a \{3\}-GDD of type \(u^1v^1w^1\).

**Proof.** If \(r \equiv 3\) and \(u \leq 2u - 6\), apply Lemma 2.3. If \(r \equiv 3\) and \(2u - 4 \leq t \leq 3u\), apply Lemma 4.1. If \(r \equiv 6\) and \(t < 3u\), apply Lemma 3.1 with \(s = 0\). In all other cases, form a \{3\}-GDD of type \((3u)^r/3 \ i^t\) by Theorem 3.2 and fill each group with a \{3\}-GDD of type \(u^1\). 

5. **The Case \(u \equiv 5\) modulo 6 and \(r \equiv 1\) modulo 3**

We determine some new \{3\}-GDD here of type \(u^1v^1w^1\), adapting an old construction due to Rosa [6].

**Lemma 5.1.** A \{3\}-GDD of type \(u^1v^1w^1\) exists if and only if

\[(u, v) \equiv (1, 1), (3, 1), (3, 3), (3, 5), (5, 1) \pmod{6, 6}\]

and \(v \leq u\).

**Proof.** If \(u \equiv 3 \pmod{6}\), let \(\mathcal{X} = \{x_1, ..., x_u\}\) be a set of \(u = 2s + 1\) points, and let \(P_1, ..., P_s\) be the parallel classes of a Kirkman triple system on \(\mathcal{X}\). Let \(\mathcal{Y} = \{y_1, ..., y_u\}\) be a set of \(u\) additional elements. For \(1 \leq i \leq (u-v)/2\), replace each triple \(\{x_a, x_b, x_c\}\) of \(P_i\) by the triples \(\{y_a, x_b, x_c\}\), \(\{x_a, y_b, x_c\}\), and \(\{x_a, x_b, y_c\}\). The remaining pairs containing an element of \(\mathcal{X}\) and an element of \(\mathcal{Y}\) form a \(v\)-regular bipartite graph. Form a 1-factorization \(F_1, ..., F_v\) of this graph. Add \(v\) new elements \(\{z_1, ..., z_v\}\), and for each edge \(\{x, \beta\}\) of \(F_i\), form the triple \(\{z_i, \alpha, \beta\}\). The result is a \{3\}-GDD of type \(u^1v^1w^1\).

When \(u \equiv 1, 5 \pmod{6}\), form instead a partial cyclic Steiner triple system on \(\mathcal{X}\) having \((v-1)/6\) full orbits of triples (using Lemma 3.2 of [3]). Let \(d_1, ..., d_{(u-v)/2}\) be the remaining differences on \(\mathcal{X}\). For each such difference \(d\), let \(f\) satisfy \(2f \equiv d \pmod{u}\) and form the set of \(u\) triples \(\{x_i, x_{i+d}, y_{i+f}\} : 0 \leq i < u\}\), subscripts modulo \(u\). Now on \(\mathcal{X} \cup \mathcal{Y}\), what remains is a \(v\)-regular bipartite graph, so we proceed as before.

We also give a construction for \(r = 4\).
LEMMA 5.2. Let \( u \equiv 5 \pmod{6}, u \geq 23, t \equiv 3 \pmod{6} \) and \( 3u - 6 \leq t < 9u - 18 \). Then there exists a \( \{3\} \)-GDD of type \( u^4 t \).

Proof. Write \( x = (u - 2)/3 \), whence \( x \) is odd. A TD(7, \( x \)) exists for \( x \) odd except possibly when \( x \in \{1, 3, 5, 15, 21, 33, 35, 39, 45, 51\} \) \[1\]. (A TD(7, \( x \)) is a \( \{7\} \)-GDD of type \( x^7 \)). In one block apply weights 5, 5, 5, 5, 3, 3, and 3. Give all remaining points in the first four groups weight 3, and all remaining in the last three groups weights 3 or 9. Using \( \{3\} \)-GDD of type \( 5^43^3, 5^13^6, 5^13^59^1, 5^13^49^2, 5^13^39^3, 3^7, 3^69^1, 3^59^2, \) and \( 3^49^3 \) (all from Lemma 1.3), we obtain the required \( \{3\} \)-GDD for the specified values of \( t \) except when \( u \in \{47, 65, 101, 107, 119, 147, 155\} \). If \( u \in \{47, 65, 101, 119, 147, 155\} \), write \( x = (u - 8)/3 \) and proceed as before using weight 11 in place of weight 5. This leaves \( u = 107 \). Employ a TD(7, 31), weights 11 and 3 on the special block, and choose one disjoint block with weights 9, 9, 9, 3 or 9, 3 or 9, 3 or 9. In each case we obtain a \( \{3\} \)-GDD with four groups of size \( u \) and three more groups each having 3 (mod 6) elements. These three groups can be filled with \( \{3\} \)-GDD having all groups of size 1.

THEOREM 5.3. If \( u^r l^t \) meets the conditions of the Main Theorem, \( u \equiv 5 \pmod{6} \) and \( r \equiv 4 \pmod{6} \), there exists a \( \{3\} \)-GDD of type \( u^r t^l \).

Proof. If \( t \leq (r - 1)u \), apply Lemma 3.1 with \( s = 0 \). If \( t \geq ru + 7 \), form a \( \{3\} \)-GDD of type \( (ru + 3)^1 1^{(r - 3)u} \) (from Theorem 1.1) and fill the hole with a \( \{3\} \)-GDD of type \( u^r 3^l \). When \( r = 4 \), apply Lemma 5.2 to complete the determination when \( u \geq 23 \) and apply Lemma 1.3 for \( u < 17 \). For \( r \geq 10 \), it remains to treat the cases when \( (r - 1)u < t \leq ru + 1 \). In these cases, use Lemma 5.1 to form a \( \{3\} \)-GDD of type \( ((r - 3)u)^1 (t - (r - 6)u)^1 1^{(r - 3)u} \)

and fill the first group with a \( \{3\} \)-GDD of type \( u^{r - 3} \) and the second with a \( \{3\} \)-GDD of type \( u^3 1^{-(r - 3)u} \) (from Theorem 4.2).

THEOREM 5.4. If \( u^r l^t \) meets the conditions of the Main Theorem, \( u \equiv 5 \pmod{6} \) and \( r \equiv 1 \pmod{6} \), there exists a \( \{3\} \)-GDD of type \( u^r t^l \).

Proof. If \( t > ru \), form a \( \{3\} \)-GDD of type \( (ru)^1 l^t \), and fill the hole with a \( \{3\} \)-GDD of type \( u^r \). If \( (r - 3)u \leq t \leq ru \), when \( r = 7 \) use Lemma 5.1 to form a \( \{3\} \)-GDD of type \( (4u + 6m + 3)^1 (3u + 6l + 4)^1 1^{4u + 6m + 3} \), where \( m \) and \( l \) satisfy \( t = 4u + 10 + 12m + 6l, 6m + 3 \leq 3u, 6l + 4 \leq 2u, \) and \( 6l + 4 \leq u + 6m + 3 \). Fill its two large groups with \( \{3\} \)-GDD of types \( u^4 1^{6m + 3} \) and
$u^31^{6t+4}$ to obtain the desired $\{3\}$-GDD. For $r \geq 13$, use Theorem 5.1 to form a $\{3\}$-GDD of type 

$((r-4)u)^1 (t-(r-8)u)^1 1^{(r-4)u}$

and fill the holes with $\{3\}$-GDD of type $u^{r-4}$ and $u^4 t^{3-(r-4)u}$.

When $u < t \leq (r-3)u$, apply Lemma 3.1 with $s = 1$. For $t < u$, if $r \geq 13$ apply Lemma 3.1 with $s = 3$; for $r = 7$, first settle the cases $u \in \{5, 11\}$ and $t < u$ by Lemma 3.1, and then apply Corollary 2.5.

6. CONCLUDING REMARKS

The Main Theorem dictates when a Steiner triple system can have disjoint subsystems or holes of equal size. Two generalizations would be of particular interest. First, the problem of prescribed subsystems meeting in a point or a triple would be of interest. Second, the Main Theorem here could be viewed as the next step in determining when $\{3\}$-GDD of type $u^tg^t$ exist.

REFERENCES