1. INTRODUCTION

This is a sequel to “Generalized quadrangles of order \((s, s^2)\), I”, J. Combin. Theory Ser. A 67 (1994), 140–160.

For terminology, notation, results, etc., concerning finite generalized quadrangles and not explicitly given here, see the monograph of Payne and Thas [1984] denoted FGQ.

Let \(\mathcal{S} = (P, B, I)\) be a (finite) generalized quadrangle (GQ) of order \((s, t)\), \(s \geq 1, t \geq 1\). So \(\mathcal{S}\) has \(v = |P| = (1 + s)(1 + st)\) points and \(b = |B| = (1 + t)(1 + st)\) lines. If \(s \neq 1 \neq t\), then \(t \leq s^2\) and, dually, \(s \leq t^2\); also \(s + t\) divides \(st(1 + s)(1 + t)\).

There is a point-line duality for GQ (of order \((s, t)\)) for which in any definition or theorem the words “point” and “line” are interchanged and the parameters \(s\) and \(t\) are interchanged. Normally, we assume without further notice that the dual of a given theorem or definition has also been given.

Given two (not necessarily distinct) points \(x, x'\) of \(\mathcal{S}\), we write \(x \sim x'\) and say that \(x\) and \(x'\) are collinear, provided that there is some line \(L\) for which \(x \not\in L\) \(\&\ x' \not\in L\); hence \(x \not\sim x'\) means that \(x\) and \(x'\) are not collinear. Dually, for \(L, L' \in B\), we write \(L \sim L'\) or \(L \nmid L'\) according as \(L\) and \(L'\) are concurrent or nonconcurrent. When \(x \sim x'\) we also say that \(x\) is orthogonal or perpendicular to \(x'\); similarly for \(L \sim L'\). The line incident with distinct collinear points \(x\) and \(x'\) is denoted \(xx'\), and the point incident with distinct concurrent lines \(L\) and \(L'\) is denoted either \(L \& L'\) or \(L \cap L'\).

For \(x \in P\) put \(x^\perp = \{x' \in P \mid x \sim x'\}\), and note that \(x \in x^\perp\). The trace of a pair \(\{x, x'\}\) of distinct points is defined to be the set \(x^\perp \cap x'^\perp\) and is denoted either \(tr(x, x')\) or \(\{x, x'\}^\perp\); then \(|\{x, x'\}^\perp| = s + 1\) or \(t + 1\) according as \(x \sim x'\) or \(x \not\sim x'\). More generally, if \(A \subset P\), \(A\) “\(\perp\)” is defined by

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$A^+=\cap \{x^+ \mid x \in A\}$. For $x \neq x'$, the span of the pair $\{x, x'\}$ is $\text{sp}(x, x') = \{x, x'\}^{++} = \{u \in P \mid u \in z^+ \text{ for all } z \in x^+ \cap x'^+\}$. When $x \neq x'$, then $\{x, x'\}^{++}$ is also called the hyperbolic line defined by $x$ and $x'$, and $|\{x, x'\}^{++}| = s+1$ or $|\{x, x'\}^{++}| \leq t+1$ according as $x \sim x'$ or $x \nsim x'$.

2. REGULARITY, DUAL NETS, AND THE AXIOM OF VEBLEN

Let $\mathcal{G}=(P, B, 1)$ be a finite GQ of order $(s, t)$. If $x \sim x', x \neq x'$, or if $x \neq x'$ and $|\{x, x'\}^{++}| = t+1$, where $x, x' \in P$, we say the pair $\{x, x'\}$ is regular. The point $x$ is regular provided $\{x, x'\}$ is regular for all $x' \in P, x' \neq x$. Regularity for lines is defined dually.

A (finite) net of order $k(\geq 2)$ and degree $r(\geq 2)$ is an incidence structure $\mathcal{N}=(P, B, 1)$ satisfying

(i) each point is incident with $r$ lines and two distinct points are incident with at most one line;
(ii) each line is incident with $k$ points and two distinct lines are incident with at most one point;
(iii) if $x$ is a point and $L$ is a line not incident with $x$, then there is a unique line $M$ incident with $x$ and not concurrent with $L$.

For a net of order $k$ and degree $r$ we have $|P|=k^2$ and $|B|=kr$.

**Theorem 2.1** (1.3.1 of Payne and Thas [1984]). Let $x$ be a regular point of the GQ $\mathcal{G}=(P, B, 1)$ of order $(s, t), s > 1$. Then the incidence structure with pointset $x^+ \setminus \{x\}$, with lineset the set of spans $\{y, z\}^{++}$, where $y, z \in x^+ \setminus \{x\}, y \neq z$, and with the natural incidence, is the dual of a net of order $s$ and degree $t+1$. If in particular $s = t > 1$, there arises a dual affine plane of order $s$. Also, in the case $s = t > 1$ the incidence structure $\pi_1$ with pointset $x^+ \setminus \{x\}$, with lineset the set of spans $\{y, z\}^{++}$, where $y, z \in x^+, y \neq z$, and with the natural incidence, is a projective plane of order $s$.

Now we introduce the Axiom of Veblen for duals nets $\mathcal{N}^*=(P, B, 1)$.

**Axiom of Veblen.** If $L_1 \cap L_2, L_1 \neq L_2, M_1 \cap M_2$, and if $L_i$ is concurrent with $M_j$ for all $i, j \in \{1, 2\}$, then $M_i$ is concurrent with $M_2$.

The only known dual net $\mathcal{N}^*$ which is not a dual affine plane and which satisfies the Axiom of Veblen is the dual net $H_q^n, n > 2$, which is constructed as follows: the points of $H_q^n$ are the points of $\text{PG}(n, q)$ not in a given subspace $\text{PG}(n-2, q) \subset \text{PG}(n, q)$, the lines of $H_q^n$ are the lines of $\text{PG}(n, q)$ which have no point in common with $\text{PG}(n-2, q)$, the incidence in $H_q^n$ is the natural one. By the following theorem these dual nets $H_q^n$ are characterized by the Axiom of Veblen.
THEOREM 2.2 (Thas and De Clerck [1977]). Let $N^*$ be a dual net with $s+1$ points on any line and $t+1$ lines through any point, where $t+1 > s$. If $N^*$ satisfies the Axiom of Veblen, then $N^* \succeq H_q^n$ with $n > 2$ (hence $s = q$ and $t + 1 = q^{n-1}$).

3. TRANSLATION GENERALIZED QUADRANGLES, PROPERTY (G), AND THE AXIOM OF VEBLEN

Let $S = (P, B, I)$ be a GQ of order $(s, t)$, $s \neq 1$, $t \neq 1$. A collineation $\theta$ of $S$ is an elation about the point $p$ if $\theta = \text{id}$ or if $\theta$ fixes all lines incident with $p$ and fixes no point of $P - p^\circ$. If there is a group $H$ of elations about $p$ acting regularly on $P - p^\circ$, we say $S$ is an elation generalized quadrangle (EGQ) with elation group $H$ and base point $p$. Briefly, we say that $(S^{(p)}, H)$ or $S^{(p)} = (S, p)$ is an EGQ. If the group $H$ is abelian, then we say that the EGQ $(S^{(p)}, H)$ is a translation generalized quadrangle. For any TGQ $(S^{(p)}, H)$ the point $p$ is coregular, that is, each line incident with $p$ is regular. Hence the parameters $s$ and $t$ of a TGQ satisfy $s < t$; see 8.2 of FGQ. Also, by 8.5.2 of FGQ, for any TGQ with $s \neq t$ we have $s = q^m$ and $t = q^{m+1}$, with $q$ a prime power and $a$ an odd integer; if $s$ (or $t$) is even then by 8.6.1(iv) of FGQ either $s = t$ or $s^2 = t$.

In $PG(2m + m - 1, q)$ consider a set $O(n, m, q)$ of $q^{m+1}$ $(n-1)$-dimensional subspaces $PG^{(0)}(n-1, q), PG^{(1)}(n-1, q), \ldots, PG^{(q^m)}(n-1, q)$, every three of which generate a PG$(3n-1, q)$ and such that each element PG$^{(i)}(n+1, q)$ of $O(n, m, q)$ is contained in a PG$^{(i)}(n+m - 1, q)$ having no point in common with any PG$^{(j)}(n-1, q)$ for $j \neq i$. It is easy to check that PG$^{(i)}(n+m - 1, q)$ is uniquely determined, $i = 0, 1, \ldots, q^m$. The space PG$^{(n+m-1)}$ is called the tangent space of $O(n, m, q)$ at PG$^{(0)}(n, m, q)$. For $n = m$ such a set $O(n, n, q)$ is called a generalized oval or an $[n-1]$-oval of PG$(3n-1, q)$; a generalized oval of PG$(2, q)$ is just an oval of PG$(2, q)$. For $n \neq m$ such a set $O(n, m, q)$ is called a generalized ovoid or an $[n-1]$-ovoid or an egg of PG$(2m + m - 1, q)$; a $[0]$-ovoid of PG$(3, q)$ is just an ovoid of PG$(3, q)$.

Now embed PG$(2m + m - 1, q)$ in a PG$(2n + m, q)$, and construct a point-line geometry $T(n, m, q)$ as follows.

Points are of three types:

(i) the points of PG$(2n + m, q) - PG(2n + m - 1, q)$;

(ii) the $(n+m)$-dimensional subspaces of PG$(2n + m, q)$ which intersect PG$(2n + m - 1, q)$ in one of the PG$^{(i)}(n+m - 1, q)$;

(iii) the symbol ($\infty$).
Lines are of two types:

(a) the $n$-dimensional subspaces of $\text{PG}(2n+m, q)$ which intersect $\text{PG}(2n+m-1, q)$ in a $\text{PG}^{(i)}(n-1, q)$;

(b) the elements of $O(n, m, q)$.

Incidence in $T(n, m, q)$ is defined as follows. A point of type (i) is incident only with lines of type (a); here the incidence is that of $\text{PG}(2n+m, q)$. A point of type (ii) is incident with all lines of type (a) contained in it and with the unique element of $O(n, m, q)$ contained in it. The point ($\infty$) is incident with no line of type (a) and with all lines of type (b).

**Theorem 3.1** (8.7.1 of Payne and Thas [1984]). $T(n, m, q)$ is a TGQ of order $(q^n, q^m)$ with base point ($\infty$). Conversely, every TGQ is isomorphic to a $T(n, m, q)$. It follows that the theory of TGQ is equivalent to the theory of the sets $O(n, m, q)$.

**Corollary 3.2.** The following hold for any $O(n, m, q)$:

(i) $n = m$ or $n(a+1) = ma$ with $a$ odd;

(ii) if $q$ is even, then $n = m = 2n$. Let $O(n, 2n, q)$ be an egg of $\text{PG}(4n-1, q)$. We say that $O(n, 2n, q)$ is good at the element $\text{PG}(i)(n-1, q)$ of $O(n, 2n, q)$ if any $\text{PG}(3n-1, q)$ containing $\text{PG}^{(i)}(n-1, q)$ and at least two other elements of $O(n, 2n, q)$, contains exactly $q^n+1$ elements of $O(n, 2n, q)$. In such a case the corresponding TGQ $T(n, 2n, q)$ contains at least $q^{3n}+q^{2n}$ translation sub-quadrangles of order $q^n$ (see Thas [1994]).

**Theorem 3.3** (Thas and Van Maldeghem [1995]). Let $S^p$ be a TGQ of order $(s, s^2)$, $s \neq 1$, with base point $p$. Then the dual net $\Lambda^p$ defined by the regular line $L$, with $p \not\in L$, satisfies the Axiom of Veblen if and only if the egg $O(n, 2n, q)$ which corresponds to $S^p$ is good at its element $\text{PG}^{(i)}(n-1, q)$ which corresponds to $L$.

Let $O = O(n, 2n, q)$ be an egg in $\text{PG}(4n-1, q)$. By 8.7.2 of FGQ the $q^n+1$ tangent spaces of $O$ form an $O^* = O^*(n, 2n, q)$ in the dual space of $\text{PG}(4n-1, q)$. So in addition to $T(n, 2n, q) = T(O)$ there arises a TGQ $T(O^*)$ with the same parameters. The TGQ $T(O^*)$ is called the translation dual of the TGQ $T(O)$. Examples are known for which $T(O) \cong T(O^*)$, and examples are known for which $T(O) \not\cong T(O^*)$; see Thas [1994].

Let $S = (P, B, I)$ be a GQ of order $(s, s^2)$, $s \neq 1$. Let $x_1, y_1$ be distinct collinear points. We say that the pair $\{x_1, y_1\}$ has Property $(G)$, or that $S$
has Property (G) at \( \{ x_1, y_1 \} \), if every triple \( \{ x_1, x_2, x_3 \} \) of points, with \( x_1, x_2, x_3 \) pairwise noncollinear and \( y_1 \in \{ x_1, x_2, x_3 \} \), is 3-regular; for the definition of 3-regularity see 1.3 of FGQ. The GQ \( \mathcal{S} \) has Property (G) at the line \( L \), or the line \( L \) has Property (G), if each pair of points \( \{ x, y \}, x \neq y \) and \( x \not\in L \), \( y \) has Property (G). If \( (x, L) \) is a flag, that is, if \( x \not\in L \), then we say that \( \mathcal{S} \) has Property (G) at \( (x, L) \), or that \( (x, L) \) has Property (G), if every pair \( \{ x, y \}, x \neq y \) and \( y \not\in L \), has Property (G). Property (G) was introduced in Payne [1989] in connection with generalized quadrangles of order \((q^2, q)\) arising from flocks of quadratic cones in \( PG(3, q) \).

**Theorem 3.4** (Thas and Van Maldeghem [1995]). Let \( \mathcal{S} = (P, B, I) \) be a GQ of order \((q^2, q), s \) even, satisfying Property (G) at the point \( x \). Then \( x \) is regular in \( \mathcal{S} \) and the dual net \( \mathcal{N}^*_x \) defined by \( x \) satisfies the Axiom of Veblen. Consequently \( \mathcal{N}^*_x \cong H^* \).

**Theorem 3.5** (Thas [1994]). A TGQ \( T(n, 2n, q) = T(O) \) satisfies Property (G) at \( \{ \infty, \xi \} \), with \( \xi \) a point of type (ii) incident with the line \( \xi \) of type (b) (or, equivalently, at the flag \( (\infty, \xi) \)) if and only if for any two elements \( \xi_i, \xi_j (i \neq j) \) of \( O(n, 2n, q) = \{ \xi \} \) the \((n - 1)\)-dimensional space \( \mathcal{P}(n - 1, q) = \tau \cap \tau_i \cap \tau_j \), with \( \tau_i, \tau_j, \tau_j \) the respective tangent spaces of \( O(n, 2n, q) \) at \( \xi, \xi_i, \xi_j \), is contained in \( q^n + 1 \) tangent spaces of \( O(n, 2n, q) \).

**Theorem 3.6** (Thas and Van Maldeghem [1995]). Let \( \mathcal{S}(p) \) be a TGQ of order \((s, s^2), s \neq 1, \) with base point \( p \). Then the dual net \( \mathcal{N}^*_p \) defined by the regular line \( L \), with \( p \not\in L \), satisfies the Axiom of Veblen if and only if the translation dual \( \mathcal{S}(p') \) of \( \mathcal{S}(p) \) satisfies Property (G) at the flag \( (p', L) \), where \( L' \) corresponds to \( L \), in the even case \( \mathcal{N}^*_p \) satisfies the Axiom of Veblen if and only if \( \mathcal{S}(p) \) satisfies Property (G) at the flag \( (p, L) \).

**Theorem 3.7.** Let \( \mathcal{S}(p) \) be a TGQ of order \((s, s^2), s \) odd and \( s \neq 1, \) with base point \( p \). If the dual net \( \mathcal{N}^*_p \) defined by the regular line \( L \), with \( p \not\in L \), satisfies the Axiom of Veblen, then \( \mathcal{S}(p) \) contains at least \( s^3 + s^2 \) classical subquadrangles \( Q(4, s) \).

Proof. This follows immediately from Theorem 3.6 and Theorem 4.3.4 of Thas [1994].

4. FLOCK GENERALIZED QUADRANGLES AND THE AXIOM OF VEBLEN

Let \( F \) be a flock of the quadratic cone \( K \) with vertex \( x \) of \( PG(3, q) \), that is, a partition of \( K \setminus \{ x \} \) into \( q \) disjoint irreducible conics. Then, by Thas
with \( F \) there corresponds a GQ \( \mathcal{S}(F) \) of order \((q^2, q)\). In Payne [1989] it was shown that \( \mathcal{S}(F) \) satisfies Property \((G)\) at its point \((\infty)\).

**Theorem 4.1** (Thas and Van Maldeghem [1995]). For any GQ \( \mathcal{S}(F) \) of order \((q^2, q)\) arising from a flock \( F \), the point \((\infty)\) is regular. If \( q \) is even, then the dual net \( \mathcal{N}^\ast_{(\infty)} \) always satisfies the Axiom of Veblen and so \( \mathcal{N}^\ast_{(\infty)} \supseteq H_3^\ast \). If \( q \) is odd, then the dual net \( \mathcal{N}^\ast_{(\infty)} \) satisfies the Axiom of Veblen if and only if \( F \) is a Kantor flock.

**Corollary 4.2.** Suppose that the TGQ \( T(O) \), with \( O = O(n, 2q) \) and \( q \) odd, is a flock GQ \( \mathcal{S}(F) \) where the point \((\infty)\) of \( \mathcal{S}(F) \) corresponds to the line \( \zeta \) of type \((b)\) of \( T(O) \). Then \( T(O) \) is good at the element \( \zeta \) if and only if \( F \) is a Kantor flock.

**Proof.** This follows immediately from Theorems 4.1 and 3.3.

### 5. VERONESE VARIETIES

In Section 6 we shall see that there is a strong connection between TGQ satisfying Property \((G)\) (or, equivalently, satisfying the Axiom of Veblen) and the Veronesean \( \mathcal{V}^\ast \) of all conics of \( \text{PG}(2, q) \). So we include a short section on Veronese varieties; a good reference is Chapter 25 of Hirschfeld and Thas [1991].

The **Veronese variety** of all quadrics of \( \text{PG}(n, K) \), \( n \geq 1 \) and \( K \) any commutative field, is the variety

\[
\mathcal{V} = \{ (x_0^2, x_1^2, \ldots, x_n^2, x_0x_1, x_0x_2, \ldots, x_0x_n, x_1x_2, \ldots, x_1x_n, \ldots, x_{n-1}x_n) \mid (x_0, x_1, \ldots, x_n) \text{ is a point of } \text{PG}(n, K) \}
\]

of \( \text{PG}(N, K) \) with \( N = n(n + 3)/2 \). The variety \( \mathcal{V} \) has dimension \( n \) and order \( 2^n \); for \( \mathcal{V} \) we also write \( \mathcal{V}^n_2 \) or \( \mathcal{V}^n_2 \). It is also called the **Veronesean** of quadrics of \( \text{PG}(n, K) \), or simply the **quadric Veronesean** of \( \text{PG}(n, K) \). It can be shown that the quadric Veronesean is absolutely irreducible and non-singular.

Let \( \text{PG}(n, K) \) consist of all points

\[
(\gamma_0, \gamma_1, \ldots, \gamma_{n-1}) \}
\]

for \( \gamma_i \) we also write \( \gamma_{ij} \). Let \( \zeta : \text{PG}(n, K) \rightarrow \text{PG}(N, K) \), with \( N = n(n + 3)/2 \) and \( n \geq 1 \), be defined by

\[
(x_0, x_1, \ldots, x_n) \mapsto (\gamma_0, \gamma_1, \ldots, \gamma_{n-1})
\]
with $y_{ij} = x_i x_j$. Then $\zeta$ is a bijection of $\mathcal{V}(n, K)$ onto the quadric Veronesean $\mathcal{V}$ of $\mathcal{V}(n, K)$. It then follows that the variety $\mathcal{V}$ is rational.

**Theorem 5.1.** The quadrics of $\mathcal{V}(n, K)$ are mapped by $\zeta$ onto all hyperplane sections of $\mathcal{V}$.

**Corollary 5.2.** No hyperplane of $\mathcal{V}(N, K)$ contains the quadric Veronesean $\mathcal{V}$.

**Theorem 5.3.** Any two distinct points of $\mathcal{V}$ are contained in a unique irreducible conic of $\mathcal{V}$.

If $K = GF(q)$, then it is clear that $\mathcal{V}_n$ contains $\theta(n) = q^n + q^{n-1} + \ldots + q + 1$ points. As no three points of $\mathcal{V}$ are collinear we have the following theorem.

**Theorem 5.4.** The quadric Veronesean $\mathcal{V}_n$ is a $\theta(n)$-cap of $\mathcal{V}(N, q)$, $N = n(n + 3)/2$.

Let $n = 2$. Then $\mathcal{V}$ is a surface of order 4 in $\mathcal{V}(5, K)$. Apart from the conic, the variety $\mathcal{V}_2$ is the quadric Veronesean which is most studied and characterized. Assume also that $K = GF(q)$. To the conics (irreducible or not) of $\mathcal{V}(2, q)$ there correspond all hyperplane sections of $\mathcal{V}_2$. The hyperplane is uniquely determined by the conic if and only if the latter is not a single point. If the conic of $\mathcal{V}(2, q)$ is one line, then the corresponding hyperplane of $\mathcal{V}(5, q)$ meets $\mathcal{V}_2$ in an irreducible conic; the surface $\mathcal{V}_2$ contains no other irreducible conics. If it follows that $\mathcal{V}_2$ contains exactly $q^2 + q + 1$ irreducible conics, that any two distinct points of $\mathcal{V}_2$ are contained in a unique irreducible conic, and that any two distinct irreducible conics on $\mathcal{V}_2$ meet in a unique point. If the conic $\mathcal{C}$ of $\mathcal{V}(2, q)$ is two distinct lines, then the corresponding hyperplane $\mathcal{V}(4, q)$ meets $\mathcal{V}_2$ in two irreducible conics with exactly one point in common; if $\mathcal{C}$ is irreducible, then $\mathcal{V}(4, q)$ meets $\mathcal{V}_2$ in a rational quartic curve. The planes of $\mathcal{V}(5, q)$ which meet $\mathcal{V}_2$ is an irreducible conic are called the conic planes of $\mathcal{V}_2$.

**Theorem 5.5.** Any two distinct conic planes $\pi$ and $\pi'$ of $\mathcal{V}_2$ have exactly one point in common, and this common point belongs to $\mathcal{V}_2$.

The tangent lines of the irreducible conics of $\mathcal{V}_2$ are called the tangents or tangent lines of $\mathcal{V}_2$. Since no point of the surface $\mathcal{V}_2$ is singular, all tangent lines of $\mathcal{V}_2$ at the point $p$ of $\mathcal{V}_2$ are contained in a plane $\pi(p)$. This plane $\pi(p)$ is called the tangent plane of $\mathcal{V}_2$ at $p$. Since $p$ is contained
Theorem 5.6. For any two distinct points $p_1$ and $p_2$ of $\mathcal{V}_4^2$, the tangent planes $\pi(p_1)$ and $\pi(p_2)$ have exactly one point in common.

Corollary 5.8. Suppose that $q$ is odd. Then for any three distinct points $p_1, p_2, p_3$ of $\mathcal{V}_4^2$, the intersection $\pi(p_1) \cap \pi(p_2) \cap \pi(p_3)$ of the tangent planes is empty.

Corollary 5.9. Suppose that $q$ is odd. Then each point of $\text{PG}(5,q) - \mathcal{V}_4^2$ is on 0 or 2 tangent planes of $\mathcal{V}_4^2$.

Theorem 5.10. Suppose that $\text{PG}(4,q)$ is a hyperplane for which $\text{PG}(4,q) \cap \mathcal{V}_4^2$ is a non-singular conic $C$. Then $\mathcal{V}_4^2$ contains exactly $q + 1$ tangent planes $\pi(p_0), \pi(p_1), \ldots, \pi(p_q)$. Also $\mathcal{C} = \{p_0, p_1, \ldots, p_q\}$, and if $\pi$ is the conic plane determined by $\mathcal{C}$, then the intersections $\pi \cap \pi(p_0), \pi \cap \pi(p_1), \ldots, \pi \cap \pi(p_q)$ are the tangent lines of the conic $\mathcal{C}$.

Proof. Let $\mathcal{C} = \{p_0, p_1, \ldots, p_q\}$ and consider the tangent plane $\pi(p_i)$. Then the plane $\pi(p_i)$ contains the tangent line of $\mathcal{C}$ at $p_i$. Let $\pi$, be the threedimensional space containing $\pi(p_i)$ and the plane $\pi$ of $\mathcal{C}$. Considering the hyperplanes containing $\pi$, we see that $q$ of them intersect $\mathcal{V}_4^2$ in two non-singular conics through $p_i$ (one of which is $\mathcal{C}$) while the remaining one intersects $\mathcal{V}_4^2$ in $\mathcal{C}$. So this last hyperplane is the hyperplane $\text{PG}(4,q)$ of the statement of the theorem. Consequently $\text{PG}(4,q)$ contains the tangent planes $\pi(p_0), \pi(p_1), \ldots, \pi(p_q)$. If $\pi(p)$ is any tangent plane in $\text{PG}(4,q)$, then, as $\text{PG}(4,q) \cap \mathcal{V}_4^2 = \mathcal{C}$, it is clear that $p \in \mathcal{C}$.

Theorem 5.11. Suppose that $\mathcal{C}$ is a non-singular conic on $\mathcal{V}_4^2$, that $\pi$ is the plane of $\mathcal{C}$ and that $p' \in \mathcal{V}_4^2 - \mathcal{C}$. Then $\pi(p') \cap \pi = \emptyset$.

Proof. Assume, by way of contradiction, that $r \in \pi(p') \cap \pi$. Then $rp'$ is the tangent line at $p'$ of some non-singular conic $\mathcal{C}'$ on $\mathcal{V}_4^2$. Hence the plane $\pi'$ of $\mathcal{C}'$ and the conic plane $\pi$ have a point $r \notin \mathcal{V}_4^2$ in common, contradicting Theorem 5.5.

Theorem 5.12. Suppose that $q$ is odd and that $\text{PG}(4,q)$ is a hyperplane for which $\text{PG}(4,q) \cap \mathcal{V}_4^2$ is a non-singular conic $\mathcal{C}$. If $p' \in \mathcal{V}_4^2 - \mathcal{C}$, then the
6. TRANSLATION GENERALIZED QUADRANGLES OF ORDER 
\((s, s')\), \(s \neq 1\) AND \(s\) ODD, SATISFYING THE AXIOM
OF VEBLEN AND VERONESE SURFACES

Consider a TGQ \(T(n, 2n, q) = T(O)\), with \(O = O(n, 2n, q)\). The egg \(O(n, 2n, q)\) is good at its element \(PG(n - 1, q)\) if and only if the dual net \(A^\#_q\) defined by the regular line \(L\) satisfies the Axiom of Veblen if and only if the translation dual \(T(O^*)\) of \(T(O)\) satisfies Property \((G)\) at the flag \((\infty, \tau)\), with \(\tau\) the tangent space of \(O\) at \(PG(n - 1, q)\); see Theorems 3.3 and 3.6.

**Notation.** If \(GF(q^s)\) is an extension of \(GF(q)\), then the corresponding extension of \(PG(m, q)\) will be denoted by \(PG(m, q^s)\) or \(PG(m, q)\).

**Theorem 6.1.** Consider a non-classical TGQ \(T(n, 2n, q) = T(O)\), \(q\) odd, with \(O = O(n, 2n, q) = \{PG(n - 1, q), PG^{1/2}(n - 1, q), ..., PG^{(s-1)/2}(n - 1, q)\}\) an egg in \(PG(4n - 1, q)\). If \(O\) is good at \(PG(n - 1, q)\), then one of the following two cases occurs

(a) There exists a \(PG(4, q^0)\) in \(PG(4n - 1, q^0)\) which intersects \(PG(n - 1, q^0)\) in a line \(M\) and which has exactly one point \(r\), in common with any space \(PG^{(i/2)}(n - 1, q^0)\), \(i = 1, 2, ..., q^{2m}\).

(b) We are not in Case (a) and there exists a \(PG(5, q^0)\) in \(PG(4n - 1, q^0)\) which intersects \(PG(n - 1, q^0)\) in a plane \(\mu\) and which has exactly one point \(r\), in common with any space \(PG^{(i/2)}(n - 1, q^0)\), \(i = 1, 2, ..., q^0\).

**Proof.** Consider a non-classical TGQ \(T(n, 2n, q) = T(O)\), \(q\) odd, with \(O = O(n, 2n, q) = \{PG(n - 1, q), PG^{1/2}(n - 1, q), ..., PG^{(s-1)/2}(n - 1, q)\}\) an egg in \(PG(4n - 1, q)\), and assume that \(O\) is good at \(PG(n - 1, q)\). Let \(PG(3n - 1, q)\) be a subspace of \(PG(4n - 1, q)\) which is skew to \(PG(n - 1, q)\), and let \(\langle PG(n - 1, q), PG^{1/2}(n - 1, q) \rangle \cap PG(3n - 1, q) = \pi_{1}, \pi = \pi_{1} \cup \pi_{2} \cup \cdots \cup \pi_{q^{2m}}\). The tangent space \(\tau\) of \(O\) at \(PG(n - 1, q)\) intersects \(PG(3n - 1, q)\) in a \((2n - 1)\)-dimensional space \(\pi\). Clearly \(PG(3n - 1, q) = \pi \cup \pi_{1} \cup \pi_{2} \cup \cdots \cup \pi_{q^{2m}}\).
By the proof of Theorem 4.3.4 in Thas [1994], in \( \text{PG}(3n-1, q) \) there are \( n \) planes \( \eta_1, \eta_2, \ldots, \eta_n \) over \( \text{GF}(q^n) \), which are conjugate with respect to the \( n \)th extension \( \text{GF}(q^n) \) of \( \text{GF}(q) \) and which generate \( \text{PG}(3n-1, q) \), such that each plane \( \eta_i \) has just one point in common with \( \pi_j \) and such that \( \eta_i \) has just one line in common with \( \pi_j \), \( j = 1, 2, \ldots, n \) and \( i = 1, 2, \ldots, q^{2n} \). Let \( \langle \pi, \eta_i \rangle = \rho_i, j = 1, 2, \ldots, n \); then \( \rho_i \) is \((n + 2)\)-dimensional. The space \( \pi \) has just one point in common with \( \eta_i \), and so \( \text{PG}(i, q^n) \) has exactly one point \( r_i \) in common with \( \rho_i \), with \( i = 1, 2, \ldots, q^{2n} \).

In \( \text{PG}(n-1, q) \) we choose a \( \text{PG}(n-2, q) \), and in \( \rho_1 \) we choose a \( \text{PG}(3, q^n) = \Phi_1 \) skew to \( \text{PG}(n-2, q^n) \). Then \( \Phi_1 \) and \( \text{PG}(n-1, q^n) \) have exactly one point \( s_i \) in common. Now we project the point \( r_i \) from \( \text{PG}(n-2, q^n) \) onto \( \Phi_1 \), and we obtain the point \( r_i' \), \( i = 1, 2, 3, \ldots, q^{2n} \).

Clearly \( r_i' \neq s_i \) and if \( r_i' = r_j', i \neq j \), then \( r_i r_j \) has a point in common with \( \text{PG}(n-1, q^n) \), a contradiction as \( O \) is an egg. Hence the set \( T_1 = \{s_1, s_2, r_2', \ldots, r_n'\} \) contains \( q^{2n} + 1 \) distinct points. If \( r_i r_j \) is odd, by Barlotti [1955] and Panella [1955], the ovoid \( T_1 \) is a plane \( \Theta_1 \). Then the plane \( r_i r_j r_k = \xi \) has at least one point \( l \) in common with \( \text{PG}(n-2, q^n) \). Without loss of generality we may assume that \( l \neq r_i, r_j \), so \( \xi \neq r_i r_j \). Any space \( \text{PG}(i, q^n) \) is generated by \( r_i \) and its conjugates with respect to \( \text{GF}(q^n) \), hence the space \( \psi = \langle \text{PG}(i, q^n), \text{PG}(i, q^n) \rangle \) is generated by the plane \( \xi = r_i r_j r_k = r_i r_j \) and its conjugates. As \( l \) belongs to \( \text{PG}(n-2, q^n) = \text{PG}(n-2, q^n) \), the space \( \psi \) is generated by a subspace of \( \text{PG}(n-2, q^n) \) and \( \langle \text{PG}(n-1, q^n), \text{PG}(n-1, q^n) \rangle \). Consequently \( \psi \) is at most \((3n-2)\)-dimensional, a contradiction as \( O \) is an egg. So we conclude that no three points of \( T_1 \) are collinear, that is, \( T_1 \) is an ovoid of the space \( \Phi_1 \).

The extension \( r \) of the tangent space \( \tau \) of \( O \) at \( \text{PG}(n-1, q) \) intersects \( \rho_1 \) in the \((n + 1)\)-dimensional space \( \text{PG}(n+1, q^n) = \langle \pi \cap \eta_i, \text{PG}(n-1, q^n) \rangle \). Clearly \( \text{PG}(n+1, q^n) \cap \Phi_1 \) is a plane \( \Theta_1 \). This plane \( \Theta_1 \) contains \( s_i \) but no one of the points \( r_1', r_2', \ldots, r_n' \). Hence \( \Theta_1 \) is the tangent plane of the ovoid \( T_1 \) at \( s_i \).

Since \( q^n \) is odd, by Barlotti [1955] and Panella [1955], the ovoid \( T_1 \) is an elliptic quadric of \( \Phi_1 \).

Each space \( \langle \text{PG}(n-1, q^n), r_i, r_j \rangle \), \( i \neq j \), contains exactly \( q^n \) points of \( \eta_i \cap \eta_j \); so \( \langle \text{PG}(n-1, q^n), r_i, r_j \rangle \) contains exactly \( q^n \) of the points \( r_i, r_j, \ldots, r_n' \), say \( r_i, r_{i_2}, \ldots, r_{i_j} \). Let \( \langle \text{PG}(n-1, q^n), \text{PG}(n-1, q^n) \rangle \cap \text{PG}(n+1, q^n) = \text{PG}(2n-1, q^n) \). Then \( \eta_i \cap \text{PG}(2n-1, q^n) \) is a line \( L_i \), \( j = 1, 2, \ldots, n \). Clearly the space \( \langle \text{PG}(n-1, q^n), r_i, r_j \rangle \) intersects \( \text{PG}(2n-1, q^n) \) in the line \( L_i \). By Theorem 4.3.4 of Thas [1994] the \( q^n + 1 \) spaces \( \text{PG}(n-1, q^n) \), \( \text{PG}(n-1, q^n) \), \( \text{PG}(n-1, q^n) \), \( \ldots \) define \( q^n \) TQG \( \mathcal{T}(n, q^n) \) isomorphic to the classical GQ \( Q(4, q^n) \); we remark that \( \text{PG}(n-1, q^n) \), \( \text{PG}(n-1, q^n) \), \( \ldots \) are the elements of \( O \) in \( \langle \text{PG}(n-1, q^n), q^n \rangle \).
Now we show that $r_5, r_5, \ldots$ together with some point of $\text{PG}(n-1, q^n)$, form a conic of $\text{GF}(q^n)$.

Let us consider one of the $q^n$ TGO $T(n, n, q)$. Let $u_4, u_5$ be non-collinear points of $T(n, n, q)$, with $u_4 \neq (\infty) \neq u_5$. Further consider all grids of $T(n, n, q)$ containing $u_4, u_5$ and two lines of type (b) of $T(n, n, q)$. As $T(n, n, q) \cong O(4, q^n \cong T(1, 1, q^n)$, all these grids have $q^n$ common points of type (i). For the sake of convenience we now simplify our notation: 

\[ \{ \text{PG}(n-1, q), \text{PG}(n-1, q), \ldots \} = \{ x_0, x_1, \ldots, x_{q^n} \} \].

Let $\beta = x_{i_0} x_{i_1} \cap x_{j_0} x_{j_1}$, with all indices distinct. We now choose $u_4, u_5$ in such a way that the line $u_4 u_5$ (in the projective space containing $T(n, n, q)$) contains a point $g$ of $\beta$. Then all grids of $T(n, n, q)$ containing $u_4, u_5$, and two lines of type (b) of $T(n, n, q)$ contain all $q^n$ points of \( \langle x_i, x_j, u_4 \rangle \cap \langle x_{i_0}, x_{i_1}, u_1 \rangle \) not in $\text{PG}(4n-1, q)$; clearly $\langle x_i, x_j, u_4 \rangle \cap \langle x_{i_0}, x_{i_1}, u_1 \rangle \cap \text{PG}(4n-1, q) = \beta$. It easily follows that if $\beta = x_{i_0} x_{i_1} \cap x_{j_0} x_{j_1}$, then it also contains $\beta$. If $|\{ \infty, u_1, u_2 \}| = 2$ in $T(n, n, q)$, then $\beta$ belongs to $(q^n - 1)/2$ of the spaces $x_{i_0} x_{i_1}$ (in such a case $g$ belongs to 0 tangent spaces of the $[n-1]$-oval $O(n, n, q)$). Assume that $g$ belongs to 2 tangent spaces $x_{i_0}$ and $x_{i_1}$ of $O(n, n, q)$, and let $g' \in \beta$. Interchanging roles of $g$ and $g'$ we see that $\langle g', x_{i_0} \rangle$ and $\langle g', x_{i_1} \rangle$ are contained in the tangent spaces of $O(n, n, q)$ at respectively $x_{i_0}$ and $x_{i_1}$. Consequently the tangent spaces of $O(n, n, q)$ at $x_{i_0}$ and $x_{i_1}$ contain $\beta$. It now follows that all spaces $x_{i_0} x_{i_1} \cap x_{j_0} x_{j_1}$ with all indices distinct, together with the elements of $O(n, n, q)$, constitute a $(n-1)$-spread $S$ in the space $\text{PG}(3n-1, q)$ of $O(n, n, q)$. Let $\beta_1, \beta_2$ be distinct elements of $S$. Let $y$ be a point of type (i) in $T(n, n, q)$.

Then the spaces $\langle y, \beta_1 \rangle$ and $\langle y, \beta_2 \rangle$ correspond to lines $M_1$ and $M_2$ in the space $\text{PG}(3, q^n)$ of the GQ $T(1, 1, q^n) \cong T(n, n, q)$. Let $\{ y' \} = M_1 \cap M_2$ and consider all lines $M_1, M_2, \ldots, M_{q^n+1}$ through $y'$ in the plane $\langle M_1, M_2 \rangle$. In the space $\text{PG}(3n, q)$ of $T(n, n, q)$, with these $q^n+1$ lines there correspond $q^n+1$ n-dimensional spaces $\langle y, \beta_1 \rangle = y_1, \langle y, \beta_2 \rangle = y_2, \ldots, \langle y, \beta_{q^n+1} \rangle = y_{q^n+1},$ with $\beta_1, \beta_2, \ldots, \beta_{q^n+1} \in S$. Let $\gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_{q^n+1} = W$; then $|W| = (q^{2n+1} - 1)/(q - 1)$. Let $w_1, w_2 \in W$, $w_1 \neq w_2$, with $w_1, w_2 \notin \text{PG}(3n, 1, q)$.

Considering again $T(1, 1, q^n)$, we see that there is a $\beta_i, i \in \{ 1, 2, \ldots, q^n + 1 \}$, such that $w_1, w_2$ has a point in common with $\beta_i$, and $\langle w_1, w_2, \beta_i \rangle \subset W$ (two lines in the plane $\langle M_1, M_2 \rangle$, but not in the plane $\text{PG}(2, q^n)$) of the oval $O(1, 1, q^n)$, which contain a common point of $\text{PG}(2, q^n)$ define the same element of $S$). So $w_1, w_2 \in W$. Now it easily follows that $W$ is a $2q$-dimensional space, hence $\beta_1, \beta_2, \ldots, \beta_{q^n+1}$ is a $(2n-1)$-dimensional space. We conclude that in all spaces $\beta_1, \beta_2$, with $\beta_1$ and $\beta_2$ distinct elements of $S$, a $(n-1)$-spread is induced by $S$. Consequently the spread $S$ is geometric; see 8.2 of Thas [1995]. Hence by a theorem of Segre, see Theorem 2 in 8.2 of Thas [1995], in $\text{PG}(3n-1, q)$.
there are \( n \) planes \( \delta_1, \delta_2, \ldots, \delta_n \) over \( GF(q^n) \), which are conjugate with respect to the \( n \)th extension \( GF(q^n) \) of \( GF(q) \) and which generate \( PG(3n-1, q) \), such that each plane \( \delta_i \) intersects each element \( \beta \), with \( \beta \in S \). In particular each such plane \( \delta_i \) intersects \( \xi_1, \xi_2, \ldots, \xi_r \). Projecting from \( PG(n-1, q^n) \) onto \( PG(2n-1, q^n) \), the planes \( \delta_1, \delta_2, \ldots, \delta_n \) are projected onto the lines \( L_1, L_2, \ldots, L_n \); notations can be chosen in such a way that \( \delta_i \) is projected onto \( L_i, i = 1, 2, \ldots, n \). The space \( \langle PG(n-1, q^n), L_1 \rangle \) intersects \( PG^{(i)}(n-1, q^n) \), \( PG^{(s)}(n-1, q^n) \), \( \ldots \) in the points common to \( PG^{(i)}(n-1, q^n) \), \( PG^{(s)}(n-1, q^n) \), \( \ldots \) and the plane \( \delta_i \). It follows that \( r_1, r_2, \ldots \) belong to \( \delta_i \). Now it is clear that the point \( \delta_i \cap PG(n-1, q^n) \) together with the \( q^n \) points \( r_1, r_2, \ldots \) form a conic \( \mathcal{C} \) over \( GF(q^n) \).

Now we choose \( \Phi_1 = PG(3, q^n) \) in such a way that it contains \( \delta_1 \), that is, in such a way that it contains \( \mathcal{C} \). The point of \( \mathcal{C} \) in \( PG(n-1, q^n) \) will be denoted by \( s_1 \). In \( PG(n-1, q) \) we now choose spaces \( PG(n-2, q) = \varepsilon, PG(n-2, q) = \varepsilon', \ldots \) the extensions of which to \( GF(q^n) \) do not contain \( s_1 \). Projecting all points \( r_1, r_2, \ldots \) from the extensions \( \varepsilon, \varepsilon', \ldots \) of the resp. spaces \( \varepsilon_1, \varepsilon_2, \ldots \) onto \( \Phi_1 \), we obtain points \( t_1, t_1, \ldots \) containing the conic \( \mathcal{C} \). The tangent plane of \( t_1, t_1, \ldots \) at \( s_1 \) is the intersection of \( \Phi_1 \) with the extension \( \tau \) of the tangent space \( \tau \) of \( O \) at \( PG(n+1, q) \). Hence the quadrics \( T_1, T_1, \ldots \) have in common the tangent plane at \( s_1 \).

Consider a point \( r_2 \notin \mathcal{C} \). If the 3-dimensional space defined by \( \mathcal{C} \) and \( r_2 \) contains a line \( PG(n-1, q^n) \), then by projecting \( \mathcal{C} \cup \{r_2\} \) from \( PG(n-2, q^n) \) onto \( \Phi_1 \), there arises a plane \( (q^n + 2) - \text{arc} \), a contradiction as \( q \) is odd. Hence we may choose \( \Phi_1 \) in such a way that it contains \( \mathcal{C} \) and \( r_2 \). With the notations of the preceding paragraph, there arise quadrics \( T_1, T_1, \ldots \) containing the conic \( \mathcal{C} \), the point \( r_2 \) and having at \( s_1 \) a common tangent plane.

First, assume that \( T_1, T_1, \ldots \) all coincide. By way of contradiction, let \( r_1 \notin T_1 \), so \( r_1 \notin \Phi_1 \). By projecting \( r_1 \) from \( \varepsilon_1, \varepsilon_2, \ldots \) onto \( \Phi_1 \), we obtain points of \( T_1 \) which all belong to the line \( \langle PG(n-1, q^n), r_1 \rangle \cap \Phi_1 \) through \( s_1 \). Hence \( T_1 \) contains at least three collinear points, a contradiction. Consequently \( r_1 \in T_1 \), and so \( T_1 \setminus \{s_1\} \) is the set of all points \( r_1, i = 1, 2, \ldots, q^n \). It follows that any space \( PG^{(i)}(n-1, q^n) \) has a point in common with \( \Phi_1 \). But then \( T(n, 2n, q) \cong T(1, 2, q^n) \), and so \( T(n, 2n, q) \) is isomorphic to the classical GQ of order \( (q^n, q^n) \), a contradiction. We conclude that the quadrics \( T_1, T_1, \ldots \) do not all coincide.

Next, assume that \( \Phi_1 \) contains a point \( r_1 \), with \( r_1 \notin r_2 \) and \( r_1 \notin \mathcal{C} \). Then the quadrics \( T_1, T_1, \ldots \) all contain \( \mathcal{C}, r_1, r_2 \) and have at \( s_1 \) a common tangent plane. Hence they all belong to a uniquely defined pencil of quadrics in \( \Phi_1 \). The base \( \mathcal{B} \) of the pencil \( \mathcal{P} \), that is, the quartic curve common to all elements of \( \mathcal{P} \), contains \( \mathcal{C} \) as a component, contains \( r_2 \), and has \( s_1 \) as multiple point. As \( \mathcal{P} \) contains elliptic quadrics, \( \mathcal{B} \) does not contain a line over \( GF(q^n) \) as component; if \( \mathcal{B} \) contains a line over \( GF(q^{2n}) \) through \( s_1 \).
as component, then also $s_1 r_n$ is a component, a contradiction. Consequently, either $\mathcal{B}$ consists of two non-singular conics $C, C'$ over $\text{GF}(q^n)$ intersecting in distinct points $s_1$ and $s_1'$, or $\mathcal{B}$ consists of two non-singular conics $C, C'$ over $\text{GF}(q^n)$ which are mutually tangent at $s_1$.

Assume that $\mathcal{B}$ consists of two non-singular conics $C, C'$ over $\text{GF}(q^n)$ which intersect in distinct points $s_1$ and $s_1'$. Let $\langle C, r_n \rangle$ and $\langle C', r_n' \rangle$, with $r_n \notin \Phi_1$, intersect $\Phi_1$ in $r_n'$ and $r_n''$. We choose $e$ and $e'$ in such a way that $r_n' \neq r_n''$. Then put $r_n r_n' \cap \text{PG}(n-1, q^n) = \{l'_n\}$ and $r_n r_n' \cap \text{PG}(n-1, q^n) = \{l''_n\}$. The lines $l'_n, l''_n$ have a point in common which belongs to $\Phi_1$ and $\text{PG}(n-1, q^n)$. Hence $r_n, r_n'$ contains $s_1$. Consider a point $c \in C - \{s_1, s_1'\}$. Assume $c$ is the projection from $e$ of a point $r_n$ with $r_n \notin \Phi_1$; so $c = r_n$. Choose $e'$ in such a way that $r_n' \neq r_n''$. Then $s_1, r_n, r_n'$ are collinear. As they all belong to $T_1$ we have a contradiction. Hence $C' - \{s_1\}$ consists of points $r_n$. It follows that $(C \cup C') - \{s_1\}$ is the set of all points $r_n$ in $\Phi_1$.

Next assume that $\mathcal{B}$ consists of two non-singular conics $C, C'$ over $\text{GF}(q^n)$ which are mutually tangent at $s_1$. As in the previous section one shows that $(C \cup C') - \{s_1\}$ is the set of all points $r_n$ in $\Phi_1$.

Any two points $r_i$ and $r_j$, $i \neq j$, define exactly one non-singular conic $C$ over $\text{GF}(q^n)$. The tangent line $U$ of $C$ at $s_1$ is contained in the extension of the tangent space of $O(n, n, q) \subset O(n, 2n, q)$, where $O(n, n, q)$ contains $\text{PG}(n-1, q), \text{PG}(n-1, q), \text{PG}(n-1, q), \text{PG}(n-1, q)$, at $\text{PG}(n-1, q)$, so is contained in $T$. Hence $\langle \text{PG}(n-1, q^n), U \rangle$ intersects $\text{PG}(2n-1, q^n)$ in the point $L_1 \cap T$. Let $V$ be the set of all conics $C$ in $\text{PG}(n+2, q^n) = \rho_1$. Any two distinct conics $C, C^* \in V$ either have a point of $\rho_1 - \text{PG}(n-1, q^n)$ in common, or do not have a point of $\rho_1 - \text{PG}(n-1, q^n)$ in common in which case $\langle \text{PG}(n-1, q^n), U \rangle = \langle \text{PG}(n-1, q^n), U^* \rangle$, where $U$ is the tangent line of $C$ at $s_1$ $(\{s_1\} = C \cap \text{PG}(n-1, q^n))$ and $U^*$ is the tangent line of $C^*$ at $s_1^*$. Consider again the conic $C'$, where $C$ and $C'$ are the components of the base $\mathcal{B}$ of the pencil $\mathcal{P}$ of quadrics. By projecting $C' - \{s_1\}$ from $\text{PG}(n-1, q^n)$ onto $\eta_1$ there arise $q^n$ points of a line, so it is clear that also the conic $C'$ belongs to $V$.

Let us now consider a point $r_n \in C - C'$ and a point $r_n' \in C' - C$. The conic $C'$ of $V$ through $r_n$ and $r_n'$ intersects $\Phi_1$ exactly in the points $r_n, r_n'$. Let $\Phi_1$ be the $4$-dimensional space containing $C, C', C''$. Further, let $r_n \notin C' \cup C'' \cup C'$. In the plane $\eta_1$ there correspond with $C, C', C''$ distinct lines $K, K', K''$, and with $r_n$ the point $r_n$. Now let $K''$ be a line of $\eta_1$ through $r_n$ which intersects $K, K', K''$ in distinct points of $\eta_1 - r_n$. With $K''$ there corresponds a conic $C''$ of $V$ through $r_n$ which intersects $C \cup C' \cup C''$ in at least three distinct points. Hence $C''$ belongs to $\Phi_1$, hence $r_n$ belongs to $\Phi_1$. We conclude that the $q^n$ points $r_n$ belong to $\Phi_1$.

Still assuming that $\Phi_1$ contains a point $r$, with $r \neq r_n$ and $r \notin C$, we now assume, by way of contradiction, that there exists a $3$-dimensional space...
PG(3, \mathbb{P}) consists of points \( \mathcal{P} \) and exactly one point \( r_\mathcal{P} \neq \mathcal{P} \). Suppose there is a second 3-dimensional space \( \mathcal{A} \) containing \( \mathcal{P} \) and exactly one point \( r_\mathcal{A} \neq \mathcal{P} \). Let \( r_\mathcal{A} \in \mathcal{P} \) and let \( \mathcal{P}' \neq \mathcal{P} \) be a conic of \( \mathcal{A} \) with \( r_\mathcal{P} \in \mathcal{P}' \) and \( r_\mathcal{A} \neq \mathcal{P}' \). As \( \mathcal{P}' \cap \mathcal{P} = \{ \mathcal{P} \} \) and \( \mathcal{P}' \cap \mathcal{A} \subset \mathcal{P} \), the conics \( \mathcal{P} \) and \( \mathcal{P}' \) necessarily have a common tangent line at \( r_\mathcal{P} \). Hence \( \mathcal{P} \) and \( \mathcal{P}' \) belong to a common 3-dimensional space \( \mathcal{A} \). Now by a preceding section either \( |\mathcal{P} \cap \mathcal{P}'| = 2 \) or \( \mathcal{P} \) and \( \mathcal{P}' \) are mutually tangent at some point in \( PG(n-1, q^n) \), a contradiction. Next we consider all 3-dimensional subspaces of \( \mathcal{P}' \) which contain \( \mathcal{P} \). Any such subspace distinct from \( \mathcal{A} \) either contains 2\( q^n \) or 2\( q^n-1 \) points \( r_\mathcal{A} \). Let \( \alpha \) be the number of such subspaces which contain 2\( q^n \) points \( r_\mathcal{A} \), and let \( \beta \) be the number of such subspaces which contain 2\( q^n-1 \) points \( r_\mathcal{A} \). Then we have

\[
\alpha(q^n-1) + \beta q^n + 1 + q^n = q^{2n}, \quad \text{and} \quad \alpha + \beta + 1 \leq q^n + 1.
\]

If \( \alpha + \beta + 1 = q^n + 1 \), then \( \alpha = q^n + 1 \), so \( \alpha + \beta + 1 > q^n + 1 \), a contradiction; if \( \alpha + \beta + 1 \leq q^n - 1 \), then \( \alpha \leq -q^n + 1 \), also a contradiction. Hence \( \alpha + \beta + 1 = q^n \), so \( \alpha = 1 \) and \( \beta = q^n - 2 \). So there arise \( q^n - 2 \) conics \( \mathcal{C}_1, \ldots, \mathcal{C}_{q^n-1} \), distinct from \( \mathcal{P} \), which are tangent to \( \mathcal{P} \) at the common point \( s_1 \) of \( \mathcal{P} \) and \( PG(n-1, q^n) \), and one conic \( \mathcal{C}_p \), which intersects \( \mathcal{P} \) in distinct points \( s_1 \) and \( r_\mathcal{A} \). With \( \mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_{q^n-1} \) there correspond in \( \eta_1, \eta_2, \ldots, \eta_{q^n-1} \) a unique pencil of quadrics defined in the 3-dimensional space containing \( \mathcal{C}_1 \) and \( \mathcal{C}_p \). Hence \( \eta_1, \eta_2, \ldots, \eta_{q^n-1} \) there correspond in \( \eta_1, \eta_2, \ldots, \eta_{q^n-1} \) a uniquely defined pencil of quadrics in any 3-dimensional space containing \( \mathcal{C}_1 \) and \( \mathcal{C}_p \). Consequently any of the \( q^n \) points \( r_\mathcal{A} \), a contradiction; hence the 3-dimensional space containing \( \mathcal{P} \) and \( \mathcal{C}_p \), coincides with the 3-dimensional space containing \( \mathcal{P} \) and \( \mathcal{C}_1 \), \( i = 1, 2, \ldots, q^n - 1 \), clearly a contradiction. We conclude that there exists no 3-dimensional space containing a conic \( \mathcal{P} \in V \) and just one point \( r_\mathcal{A} \neq \mathcal{P} \).

From the preceding section it follows that either all 3-dimensional spaces containing any conic \( \mathcal{P} \in V \) and a point \( r_\mathcal{A} \neq \mathcal{P} \) also contain a second point \( r_\mathcal{A} \neq \mathcal{P} \) (and then there arises a uniquely defined pencil of quadrics in any such 3-dimensional space), or no 3-dimensional space containing any conic \( \mathcal{P} \in V \) contains more than one point \( r_\mathcal{A} \neq \mathcal{P} \).

(a) First assume that all 3-dimensional spaces containing any conic \( \mathcal{P} \in V \) and a point \( r_\mathcal{A} \neq \mathcal{P} \) also contain a second point \( r_\mathcal{A} \neq \mathcal{P} \). Hence in each such 3-dimensional space a pencil \( \mathcal{P} \) of quadrics is defined. Let \( s_1 \) be the common point of \( \mathcal{P} \) and \( PG(n-1, q^n) \) and let \( r_\mathcal{A} \neq \mathcal{P} \). Further let \( \mathcal{P} \) be the base of the pencil \( \mathcal{P} \) of quadrics defined in the 3-dimensional space containing \( \mathcal{P} \) and \( r_\mathcal{A} \). Then \( \mathcal{P} \) consists of two conics \( \mathcal{C}_1, \mathcal{C}_1' \), through \( s_1 \), with \( r_\mathcal{A} \in \mathcal{C}_1 \) and \( \mathcal{C}_1' \in V \). Consequently any of the \( q^n \) points \( r_\mathcal{A} \) belongs to a conic of \( V \) through \( s_1 \). Now we consider all conics \( \mathcal{C}_1, \mathcal{C}_2, \ldots \) of \( V \) containing the point \( s_1 \) of \( PG(n-1, q^n) \). Let \( PG(n+1, q^n) \) be a hyperplane of \( \rho_1 \), which does not contain \( s_1 \) and project all points of \( \mathcal{C}_1 \) from \( s_1 \) onto...
GENERALIZED QUADRANGLES OF ORDER \((s, t^2)\)

If \(q^n\) points of some line \(R_i\), then the set of all these lines \(R_i\) will be denoted by \(\mathcal{H}\). Now we project the \(q^{2n}\) points in \(R_i\) onto \(\text{PG}(n+1, q^n)\); the projection of \(R_i\) is denoted by \(R_i^*\). The set of all points \(r^*_j\) is denoted by \(B\); we have \(|B| = q^{2n}\). Further, let \(A\) be the set of all intersections of \(\text{PG}(n+1, q^n)\) with the tangent lines of the conics \(\mathcal{C}_j\) at \(s_1\). The tangent line \(U_j\) of \(\mathcal{C}_j\) at \(s_1\) belongs to \(\mathcal{T}\), hence \(\mathcal{T} \cap U_j = \{s_1\}\). It follows that \(A \cap B = \emptyset\). Each line \(R_i \in \mathcal{H}\) contains \(q^n\) points of \(B\) and just one point of \(A\). Let \(\pi\) be the plane containing \(R_i\) and \(r^*_j\), \(r^*_j \notin R_i\). If we now take for \(\mathcal{H}_i\) the space \(\langle R_i, s_1, r^*_j \rangle\), then the base \(\mathcal{H}\) of the pencil \(\mathcal{P}\) consists of two conics \(\mathcal{C}_i\) and \(\mathcal{C}_j\) and \((\mathcal{C}_i \cup \mathcal{C}_j) - \{s_1\}\) is the set of all points \(r^*_j\) in \(\mathcal{H}_i\). It immediately follows that \(\pi\) contains the line \(R_i\), and that \(\pi\) contains no line \(R_i \cap \mathcal{H}\) with \(i \neq l \neq j\); also, \(\pi \cap B\) consists of all points of \(R_i \cap R_j\) in \(B\).

Suppose that \(r^*_j\) and \(r^*_j\) are points of \(B\) not on a common line of \(\mathcal{H}\). Considering all planes of \(\text{PG}(n+1, q^n)\), the previous section shows that \(r^*_j\) and \(r^*_j\) are on the same number of lines of \(\mathcal{H}\).

Now let \(r^*_j\) and \(r^*_j\), \(i \neq j\), be points of \(B\) on a common line of \(\mathcal{H}\). Let \(s_1 + 1\) be the number of lines of \(\mathcal{H}\) containing \(r^*_j\). By way of contradiction we assume that \(t^*_j \neq t_j\). Let \(r^*_j\) be a point of \(B\) not on \(r^*_j r^*_j\). The plane \(r^*_j r^*_j r^*_j\) contains exactly one line \(R_k\) of \(\mathcal{H}\) through \(r^*_j\). If \(r^*_j \notin R_k\) and \(r^*_j \notin R_k\), then \(r^*_j r^*_j r^*_j \notin \mathcal{H}\) and \(r^*_j r^*_j r^*_j \notin \mathcal{H}\), so \(t^*_j = t_j = t_j\), a contradiction. Hence either \(r^*_j \notin R_k\) or \(r^*_j \notin R_k\), say \(r^*_j \notin R_k\). Then \(r^*_j\) is on \(t^*_j + 1\) lines of \(\mathcal{H}\). Each point of \(r^*_j r^*_j - \{r^*_j\}\) not in \(A\), is on \(t_j + 1\) lines of \(\mathcal{H}\). Also, each point of \(r^*_j r^*_j - \{r^*_j\}\) not in \(A\), is on \(t_j + 1\) lines of \(\mathcal{H}\). Let \(r^*_j \in B\) be not on \(r^*_j r^*_j\) and not on \(r^*_j r^*_j\). Assume, by way of contradiction that \(r^*_j r^*_j \notin \mathcal{H}\). As \(r^*_j r^*_j \notin \mathcal{H}\), the point \(r^*_j\) is on \(t_j + 1\) lines of \(\mathcal{H}\). For any point \(r^*_j \in r^*_j r^*_j - \{r^*_j\}, \{r^*_j\}\) the line \(r^*_j r^*_j\) does not belong to \(\mathcal{H}\), and so \(r^*_j\) is on \(t_j + 1\) lines of \(\mathcal{H}\). Hence \(t_j = t_j\), a contradiction. Consequently \(r^*_j r^*_j \notin \mathcal{H}\). Interchanging roles of \(r^*_j\) and \(r^*_j\), we see that necessarily \(r^*_j r^*_j \notin \mathcal{H}\). It follows that all lines of \(\mathcal{H}\) contain \(r^*_j\). Hence \(t_j = 0\). So \((t_j + 1)(q^n - 1) + 1 = |B| = q^{2n}\), that is, \(t_j = q^n\). Consequently there are \(q^n + 1\) conics in \(V\) containing \(s_1\), and they all share a point \(r^*_j\). In \(\text{PG}(n - 1, q^n)\) we now consider a point \(s_2 \neq s_1\) on an element of \(V\). Let \(\mathcal{C}\) be a conic of \(V\) containing \(r^*_j\) and \(s_2\). Choose on \(\mathcal{C}\) a point \(r^*_j\) different from \(r^*_j\). Then there is a conic \(\mathcal{C}^*\) in \(V\) containing \(r^*_j\) and \(s_1\), and so \(\mathcal{C}^*\) contains \(r^*_j\). Consequently \(V\) contains two distinct conics \(\mathcal{C}\) and \(\mathcal{C}^*\) through \(r^*_j\) and \(r^*_j\), a contradiction. We conclude that the number of lines of \(\mathcal{H}\) through any point \(r^*_j \in B\) is independent of the index \(i\).

The constant number of lines of \(\mathcal{H}\) which contains \(r^*_j \in B\) will be denoted by \(t^*_j + 1\).

Fix a line \(R^j \in \mathcal{H}\) and count on two ways the number of ordered pairs \((r^*_j, R^j)\), with \(r^*_j \in B, r^*_j \notin R, R \cap R \neq \emptyset\) and \(r^*_j \in R^j\). We obtain

\[
q^n t(q^n - 1) + \beta q^n = q^{2n} - q^n,
\]
where $\beta + 1$ is the number of lines of $\mathcal{A}$ through $R \cap A$. Hence

$$(q^n - 1)(t - 1) + \beta = 0.$$ 

(1) $t = 0$ and $\beta = q^n - 1$.

The $q^n$ lines of $\mathcal{A}$ through $R \cap A$ form a partition of $B$, and as $t = 0$ the set $\mathcal{A}$ only contains these $q^n$ lines. It follows that $s_1$ belongs to exactly $q^n$ conics of $V$; also these conics are mutually tangent at $s_1$.

(2) $t = 1$ and $\beta = 0$.

Then $\mathcal{A}$ contains $q^{2n} \cdot 2 = 2q^n$ lines and $|A| = 2q^n$. So $s_1$ belongs to exactly $q^n$ conics of $V$. Let $R_{d_1}, R_{d_2}, \ldots, R_{d_\alpha}$ be the $q^n$ lines of $\mathcal{A}$ which intersect $R$ in $B$. The remaining $q^n$ lines of $\mathcal{A}$ necessarily intersect $R_{d_1}, R_{d_2}, \ldots, R_{d_\alpha}$. So the intersections $R_{d_i} \cap B$, $R_i \in \mathcal{A}$, form a grid $\mathcal{G}$ of order $(q^n - 1, 1)$. Clearly $\mathcal{G}$ is contained in a 3-dimensional space $\mathcal{P}$. $A$ consists of two distinct intersecting lines $N_1, N_2$ minus their common point, and $\mathcal{A} \cup \{N_1, N_2\}$ is the set of all lines of a hyperbolic quadric in $\mathcal{P}$.

In Case (2) $A \cup B$ is contained in a 3-dimensional space $\mathcal{P}$, hence all points $r_i$ are contained in a $PG(4, q^n)$.

Now we consider Case (1). Let $s_2 \neq s_1$ be a point of $PG(n - 1, q^n)$ on a conic $\mathcal{C} \subseteq V$. Suppose that $\mathcal{C}', \mathcal{C}''$ are distinct conics of $V$ containing $s_1$. The $q^n$ elements of $V$ through $s_1$ correspond with the $q^n$ lines of $\mathcal{C}$, distinct from $\eta_1 - \tau = \eta_1 - \tau$ and containing a common point $e$ on $\eta_1 \cap \tau$. Hence with $\eta_1$ there corresponds a line of $\eta_1$ not containing $e$. So $|\mathcal{C} \cap \mathcal{C}'| = |\mathcal{C} \cap \mathcal{C}''| = 1$. So $\mathcal{C}$ belongs to the space $PG(4, q^n)$ defined by $\mathcal{C}', \mathcal{C}''$ and $s_2$. It follows that all conics of $V$ containing $s_2$ belong to $PG(4, q^n)$. Consequently all points $r_i$ are contained in $PG(4, q^n)$.

Since $PG(4, q^n) \subseteq \langle PG(n - 1, q^n), \eta_1 \rangle = \varphi_1$ and $PG(n - 1, q^n)$ has exactly one point $r_j$ in common with $\varphi_1$, it has also one point $r_j$ in common with $PG(4, q^n)$.

As $\langle PG(4, q^n), PG(n - 1, q^n) \rangle = \varphi_1$, the space $PG(4, q^n)$ intersects $PG(n - 1, q^n)$ in a line $M$. Consequently all the common points of $PG(n - 1, q^n)$ and the conics of $V$ belong to $M$.

(b) Next assume that no 3-dimensional space containing any conic $\mathcal{C} \subseteq V$ contains more than one point $r_d \notin \mathcal{C}$. Let $\mathcal{W}$ be the set of all points $r_j$; then $|\mathcal{W}| = q^{2n}$.

Let $\mathcal{C} \subseteq V$ and $r_d \notin \mathcal{W} - \mathcal{C}$. Further, let $\mathcal{P}_1$ be the 3-dimensional space containing $\mathcal{C}$ and $r_d$. The common point of $\mathcal{C}$ and $PG(n - 1, q^n)$ will be denoted by $s$. If $\mathcal{C} = \{s, r_1, r_2, \ldots, r_\alpha\}$, then $r_j$ and $r_d$ are on exactly one conic $\mathcal{C}'$, with $j = 1, 2, \ldots, q^n$. Let $\mathcal{C}_j \cap PG(n - 1, q^n) = \{s_j\}$, $j = 1, 2, \ldots, q^n$. If $s = s_j$, with $j \neq j'$ and $j, j' \in \{1, 2, \ldots, q^n\}$, then the 3-dimensional space defined by $\mathcal{C}_j$ and $r_j$ contains $\mathcal{C}_j'$, so contains more than one
point \( r_j \in \mathcal{W} - \mathcal{G}_2 \), a contradiction. So the points \( s_1, s_2, \ldots, s_{q^n} \) are distinct. A same argument shows that \( s \neq s_j, j \neq 1, 2, \ldots, q^n \). Let \( \mathcal{G} = \{ s, s_1, s_2, \ldots, s_{q^n} \} \).

Let \( \Phi'_1 \) be the space generated by \( \mathcal{G}, \mathcal{G}_1, \mathcal{G}_2 \). Further, let \( r_d \in \mathcal{W} - (\mathcal{G} \cup \mathcal{G}_1 \cup \mathcal{G}_2) \). In the plane \( \eta_1 \) there correspond with \( \mathcal{G}, \mathcal{G}_1, \mathcal{G}_2 \) distinct lines \( K, K_1, K_2 \), and with \( r_d \) the point \( \overline{r}_d \). Now let \( K' \) be a line of \( \eta_1 \) through \( \overline{r}_d \) which intersects \( K, K_1, K_2 \) in distinct points of \( \eta_1 - \overline{r} \). With \( K' \) there corresponds a conic \( \mathcal{C}' \) of \( V \) through \( r_d \) which intersects \( \mathcal{G} \cup \mathcal{G}_1 \cup \mathcal{G}_2 \) in at least three distinct points. Hence \( \mathcal{C}' \) belongs to \( \Phi'_1 \), hence \( r_j \) belongs to \( \Phi'_1 \). We conclude that \( \mathcal{W}' \) is a subset of \( \Phi'_1 \).

Clearly \( \Phi'_1 \) is either 4-dimensional or 5-dimensional. If \( \Phi'_1 \) is 4-dimensional, then considering all hyperplanes of \( \Phi'_1 \) on \( \mathcal{G} \) we have that \( \mathcal{W}' \) contains at most \( 2q^n + 1 \) points, a contradiction. Consequently \( \Phi'_1 \) is 5-dimensional.

Assume, by way of contradiction, that \( \mathcal{G} \) is a line. Then \( \mathcal{W}' \) is contained in the 4-dimensional space generated by \( \mathcal{G}, s_1, r_d \), a contradiction. So \( \mathcal{G} \) is not a line, hence it also follows that \( \Phi'_1 \cap \text{PG}(n - 1, q^n) \) is at least 2-dimensional. If \( \text{PG}(5, q^n) \cap \text{PG}(n - 1, q^n) = \mu \) is at least 3-dimensional, then \( \mu \cap \Phi_1 \) is at least a line, which was shown to be impossible in the first half of the proof. It follows that \( \mu \) is a plane.

Finally, since \( \Phi'_1 \subset \langle \text{PG}(n - 1, q^n), \eta_1 \rangle = \mu_1 \) and \( \text{PG}(n - 1, q^n) \) contains exactly one point \( r \) in common with \( \mu_1 \), it has also one point \( r_d \) in common with \( \Phi'_1 \).

Let \( \mathcal{W}' \) be the set of all points \( r_i \); then \( |\mathcal{W}'| = q^{2n} \). Assume that we are in Case (a) of Theorem 6.1. Then the set \( \mathcal{W}' \) belongs to \( \text{PG}(4, q^n) \) and \( \text{PG}(4, q^n) \cap \text{PG}(n - 1, q^n) \) is a line \( M \). The following observations are taken from the proof of Theorem 6.1. The set \( \mathcal{W}' \) provided with the conics \( \mathcal{G} \) of \( V \) forms a \( 2 - (q^{2n}, q^n, 1) \) design, that is an affine plane of order \( q^n \). If \( r_d \in \mathcal{G} \), with \( \mathcal{G} \in V \), and if \( \{ s_1 \} = \mathcal{G} \cap \text{PG}(n - 1, q^n) \), then any point \( r_d \in \mathcal{W}' \) is on a conic of \( V \) through \( s_1 \). If the 3-dimensional space \( \Phi_1 \) contains \( \mathcal{G} \in V \) and a point of \( \mathcal{W}' \) not on \( \mathcal{G} \), then it contains two conics \( \mathcal{G}, \mathcal{G}' \) of \( V \). Also \( |\mathcal{G} \cap \mathcal{G}'| \leq 2 \) and the conics \( \mathcal{G}, \mathcal{G}' \) have a common point on \( M \). Further, \( (\mathcal{G} \cup \mathcal{G}') - \{ s_1 \} = \Phi_1 \cap \mathcal{W}' \).

Let \( s_1 \) be the common point of \( \mathcal{G} \in V \) and \( \text{PG}(n - 1, q^n) \). We distinguish two cases:

1. The point \( s_1 \) belongs to \( q^n \) elements of \( V \) and all these conics have a common tangent line \( U \) at \( s_1 \); this line \( U \) does not belong to \( \text{PG}(n - 1, q^n) \).

2. The point \( s_1 \) belongs to \( 2q^n \) elements of \( V \). The set of these \( 2q^n \) conics can be partitioned into sets \( \mathcal{A}_i \) and \( \mathcal{A}_j \), where each conic of \( \mathcal{A}_i \) has two distinct points in common with each conic of \( \mathcal{A}_j \) and where any two distinct conics of \( \mathcal{A}_i \) have just \( s_1 \) in common, \( i = 1, 2 \). The tangents at \( s_1 \) of
all conics of $\mathcal{A}$ are all distinct and belong to a plane $\theta_i, i=1, 2$; also, $\theta_1 \cap \theta_2$ is a line, distinct from these $2q^n$ tangent lines. With the $q^n$ conics of $\mathcal{A}$, there correspond $q^n$ lines of $\eta_i$, of $\tau_i \cap \eta_i, i=1, 2$. It follows that the tangent lines of the elements of $\mathcal{A}$ at $s_1$ all belong to a $n$-dimensional space through $\text{PG}(n-1, q^n)$, and so $\theta_i \cap \text{PG}(n-1, q^n)$ is a line $M_i, i=1, 2$. As $\theta_i \subset \text{PG}(4, q^n)$ and $\text{PG}(4, q^n) \cap \text{PG}(n-1, q^n) = M$ we necessarily have $M_1 = M_2 = M$, and so $\theta_1 \cap \theta_2 = M$.

Let $s_1$ be the common point of $\mathcal{C} \in V$ and $\text{PG}(n-1, q^n)$, and let $s_2$ be the common point of $\mathcal{C}' \in V$ and $\text{PG}(n-1, q^n)$, where $s_1 \neq s_2$. With $\mathcal{C}$, resp. $\mathcal{C}'$, there corresponds a line $N$, resp. $N'$, of $\eta_i$. From the preceding section follows that $N$ and $N'$ intersect the line $\eta_1 \cap \tau$ in distinct points. As $N \cap N'$ does not belong to $\eta_1 \cap \tau$, the conics $\mathcal{C}$ and $\mathcal{C}'$ have exactly one point (not in $\text{PG}(n-1, q^n)$) in common.

The set of all common points of $M$ with elements of $V$ will be denoted by $\mathcal{M}$.

**Theorem 6.2.** Consider a non-classical TGQ $\text{T}(n, 2n, q) = \text{T}(O), q$ odd, with $O = O(n, 2n, q) = \{\text{PG}(n-1, q), \text{PG}^{(1)}(n-1, q), \text{PG}^{(2)}(n-1, q), \ldots, \text{PG}^{(q^n)}(n-1, q)\}$. If $O$ is good at $\text{PG}(n-1, q^n)$ and if the set $\mathcal{W}$ is contained in a $\text{PG}(4, q^n)$, then the set $\mathcal{W} \cup \mathcal{M}$ of $\text{PG}(4, q^n)$ is the projection of a quadric Veronesean $\mathcal{V}^{\frac{q^n}{2}}$ from a point $p$ in a conic plane onto a hyperplane $\text{PG}(4, q^n)$.

**Proof.** We use the notations of Theorem 6.1 and its proof, and these of the sections preceding Theorem 6.2.

Let $\alpha_1$ be the number of points in $\mathcal{M}$ which belong to $q^n$ elements of $V$, and let $\alpha_2$ be the number of points in $\mathcal{M}$ which belong to $2q^n$ elements of $V$. Now we count on two ways the number of ordered pairs $(s, \mathcal{C})$, with $s \in \mathcal{M}$ and $\mathcal{C} \in V$ containing $s$. We obtain

$$\alpha_1 q^n + \alpha_2 (2q^n) = q^{2n} + q^n,$$

that is,

$$\alpha_1 + 2\alpha_2 = q^n + 1.$$

It follows that $\alpha_1$ is even. We distinguish two cases.

(i) \(\alpha_1 \geq 2\)

Let $s_1, s_2, \ldots, s_{\alpha_1}$ be the points of $\mathcal{M}$ which are contained in $q^n$ conics of $V$. All conics of $V$ through $s_i$ have a common tangent line $U_i$ at $s_i, i=1, 2, \ldots, \alpha_1$. As $\langle \text{PG}(n-1, q^n), U_i \rangle$ and $\langle \text{PG}(n-1, q^n), U_2 \rangle$ intersect $\eta_1$ in different points, the lines $U_1$ and $U_2$ do not intersect. Let $\mathcal{C}$ be a conic...
in $V$ not containing $s_1$ or $s_2$, and let $s$ be the point of $\mathcal{C}$ in $\text{PG}(n-1, q^n)$. Further, let $r_1 \in \mathcal{W}$. There is a conic $\mathcal{C}' \in V$ containing $r_1$ and $s_1$. Then $\mathcal{C}$ and $\mathcal{C}'$ have a point $r_1$ in common. As $\mathcal{C}'$ is tangent to $U_1$ at $s_1$, it is contained in the plane $\langle U_1, r_1 \rangle$. Hence $\mathcal{C}'$ belongs to the quadratic cone $\mathcal{Z}_1$ with vertex $U_1$, and base $\mathcal{C}$ (as $\langle \text{PG}(n-1, q^n), U_1 \rangle \cap \eta_1$ is not on the line of $\eta_1$ defined by $\mathcal{C}$, the line $U_1$ is skew to the plane $\sigma$ of $\mathcal{C}$). Consequently $\mathcal{W}$ belongs to $\mathcal{Z}_1$. Clearly also $M$ belongs to $\mathcal{Z}_1$. Analogously $\mathcal{W}$, and also $M$, belong to the quadratic cone $\mathcal{Z}_2$ with vertex $U_2$ and base $\mathcal{C}$.

Let $\xi$ be the 3-dimensional space $\langle U_1, U_2 \rangle$. As $\xi \subset \mathcal{C} \subset \text{PG}(4, q^n)$, we have $\xi \cap \text{PG}(4, q^n) = \xi$. Hence the tangent line $U$ of $\mathcal{C}$ at $s$ is contained in the subspace $\xi$.

Now let us choose coordinates in $\text{PG}(4, q^n)$. Let $s(0, 0, 1, 0, 0), (0, 1, 0, 0, 0) \in \mathcal{C}$, and let $s(1, 0, 0, 0, 0) \in U$ be the pole of the line $\langle (0, 0, 1, 0, 0), (0, 1, 0, 0, 0) \rangle$ with respect to $\mathcal{C}$. So the plane $\sigma$ of $\mathcal{C}$ has equation $X_3 = X_4 = 0$. Further assume that $s(0, 0, 1, 0, 0), (0, 1, 0, 0, 0) \in U$. On $U_1$ we choose the point $(0, 0, 0, 1, 1)$ in such a way that the line $\langle (1, 0, 0, 0, 0), (0, 0, 0, 0, 1) \rangle$ intersects $U_2$ (this is possible as $U, U_1, U_2$ are contained in the 3-dimensional space $\xi$). Then the hyperplane $\xi$ of $\text{PG}(4, q^n)$ has equation $X_1 = 0$. Let $e'$ be any point of $\mathcal{C} = \{s, (0, 1, 0, 0, 0)\}$ and let $e$ be the intersection of the planes $U_1 e'$ and $\langle U_2, (0, 1, 0, 0, 0) \rangle$. Choose $e$ as point $(1, 1, 1, 1, 1)$. Then $\langle e, (1, 0, 1, 0, 0) \rangle \cap \xi$ is the point $(1, 0, 1, 1, 1)$, and so $(1, 0, 1, 1, 1)$ is on $U_2$. It follows that $U_2 \cap M = \{(0, 0, 1, 1, 1)\}$ and $U_2 \cap \langle (0, 0, 0, 0, 1), (1, 0, 0, 0, 0) \rangle = \{(1, 0, 0, 0, 1)\}$. Further, $U_1 e \cap \sigma = \{e(1, 1, 1, 1, 1)\}$, with $\sigma$ the plane of $\mathcal{C}$. It follows that $\mathcal{C}$ has equations $X_3 = X_4 = X_0^2 - X_1 X_2 = 0$.

Consequently
\begin{align*}
\mathcal{Z}_1: & \quad X_0^2 = X_1 X_2, \\
\mathcal{Z}_2: & \quad (X_0 - X_4)^2 = X_1 (X_2 - X_3).
\end{align*}

So we have
\begin{equation*}
\mathcal{W} \subseteq \mathcal{Z}_1 \cap \mathcal{Z}_2 = \mathcal{V}(X_0^2 - X_1 X_2, (X_0 - X_4)^2 - X_1 (X_2 - X_3)).
\end{equation*}

Now we calculate the number of points of $\mathcal{Z}_1 \cap \mathcal{Z}_2$ not in $\xi$. So we put $x_1 = 1$. As $x_0$ and $x_4$ can be chosen and define uniquely $s_1$ and $s_3$, we have $\langle \mathcal{Z}_1 \cap \mathcal{Z}_2 \rangle = \mathcal{W}$. So $\mathcal{W} = \langle \mathcal{Z}_1 \cap \mathcal{Z}_2 \rangle$. As $\mathcal{Z}_1 \cap \mathcal{Z}_2 \cap \xi = M$, we also have $\mathcal{W} = \langle \mathcal{Z}_1 \cap \mathcal{Z}_2 \rangle - \xi$. Clearly $\mathcal{W} = \{m, 1, m^2, 2mn - n^2, n\} \cap \mathcal{V}$, $m, n \in \text{GF}(q^n)$.

Now we consider the quadric Veronesean $\mathcal{V}^-_4$ in $\text{PG}(5, q^n)$, where
\begin{equation*}
\mathcal{V}^-_4 = \{(ml, l^2, m^2, 2mn - n^2, ml, n^2) \mid (l, m, n) \text{ is a point of } \text{PG}(2, q^n)\}.
\end{equation*}
Assume that $\text{PG}(4, q^n)$ is the hyperplane $X_5 = 0$ of $\text{PG}(5, q^n)$. Project $\mathcal{X}^2$ from $(0, 0, 0, 0, 1) \neq \mathcal{X}^2$ onto $\text{PG}(4, q^n)$. Then we obtain the surface

$$I = \{ (ml, l^2, m^2, 2mn - n^2, nl, 0) \mid (l, m, n) \neq (0, 0, 0) \}.$$ 

For $l = 1$ we obtain the points $(m, 1, m^2, 2mn - n^2, n, 0)$, with $m, n \in GF(q^n)$. Hence $I - \xi = \mathcal{H} = \{ \xi \cap Z \} - M$. For $l = 0$ we obtain the points $(0, 0, m^2, 2mn - n^2, 0, 0)$, with $(m, n) \neq (0, 0)$. So $I \cap \xi \neq M$ and $|I \cap \xi| = (q^n + 3)/2$. The point $(0, 0, 0, 0, 0, 1)$ from which we project is in the conic plane $\varphi$: $X_0 = X_1 = X_4 = 0$ of $\mathcal{X}^2$, and is an exterior point of the conic $\varphi \cap \mathcal{X}^2$.

Let $p_1, p_2 \in \mathcal{H}$, $p_1 \neq p_2$, let $\mathcal{C}$ be the conic of $V$ containing $p_1$ and $p_2$, and let $\mathcal{B}$ be any non-singular conic of $\mathcal{I}$ containing $p_1$ and $p_2$. Assume, by way of contradiction that $\mathcal{C} \neq \mathcal{B}$. If $\sigma$ is the plane of $\mathcal{C}$ and if $p$ is a point of $\mathcal{B}$ not on $\sigma$ and not in $\xi$, then $\varphi_{\sigma} = (\sigma, p)$ contains two conics $\mathcal{C}$ and $\mathcal{C}'$ of $V$. Also $\mathcal{B} - \xi \neq (\mathcal{C} \cup \mathcal{C}') - \xi$. Consequently $\mathcal{C}' = \mathcal{B}$, a contradiction as $|\mathcal{C} \cap \mathcal{C}' - \xi| \leq 1$. Hence $\mathcal{C} = \mathcal{B}$. Now it follows easily that $V$ is the set of all non-singular conics on $\mathcal{I}$, and that $V$ is the set of the projections of all non-singular conics on $\mathcal{X}^2$ but not in the plane $\varphi$.

From the preceding section it also follows that the set $\mathcal{A}$ of all common points of $M$ with elements of $V$, is the projection from $(0, 0, 0, 0, 0, 1)$ onto $\text{PG}(4, q^n)$ of the conic $\varphi \cap \mathcal{X}^2$. Hence $|\mathcal{A}| = (q^n + 3)/2$. If $T_1, T_2$ are the tangent lines of $\varphi \cap \mathcal{X}^2$ through $(0, 0, 0, 0, 0, 1)$, then the points $T_1 \cap M$ and $T_2 \cap M$ are contained in $q^n$ conics of $V$; if $Z_t$ is any line of $V$ through $(0, 0, 0, 0, 1)$ which intersects $\varphi \cap \mathcal{X}^2$ in two distinct points, then $Z_t \cap M$ is contained in $2q^n$ conics of $V$. Hence $s_1 = 2$.

We conclude that $\mathcal{H} \cup \mathcal{A}$ is the projection of the quadric Veronesean $\mathcal{X}^2$ from $(0, 0, 0, 0, 0, 1)$ onto the space $\text{PG}(4, q^n)$.

(ii) $s_2 = 0$

In this case each point of $\mathcal{A}$ is contained in exactly $2q^n$ conics of $V$. Further $s_2 = |\mathcal{A}| = (q^n + 1)/2$.

Let $\mathcal{C}_1, \mathcal{C}_2$ be conics of $V$ intersecting $M$ in distinct points $s_1, s_2$. The tangent line of $\mathcal{C}_i$ at $s_i$ will be denoted by $U_i$, $i = 1, 2$. As $\mathcal{C}_1 \neq \mathcal{C}_2$, $U_1 \neq U_2$. As $\mathcal{C}_1 \cap \mathcal{C}_2 = \xi$, $U_1 \cap U_2 = \xi$. If $\eta_1$ and $\eta_2$ are the 3-dimensional spaces $\langle U_1 \rangle \cap \langle U_2 \rangle$, as $\mathcal{C}_1 \neq \mathcal{C}_2$, $\eta_1 \neq \eta_2$ and $\eta_1 \cap \eta_2 = \xi$. Hence the tangent line $U$ of any conic $\mathcal{C} \in V$ at $\xi \in M$ is contained in the subspace $\xi$.

Let $\mathcal{C} \in V, \mathcal{C} \neq \mathcal{C}'$, with $M \cap \mathcal{C} = M \cap \mathcal{C}'$, and assume that $\mathcal{C} \cap \mathcal{C}' = \{s, t\}$.

Now let us choose coordinates in $\text{PG}(4, q^n)$. Let the hyperplane $\xi$ of $\text{PG}(4, q^n)$ have equation $X_5 = 0$, and let the hyperplane $\pi$ of $\text{PG}(4, q^n)$
containing $\mathcal{C} \cup \mathcal{C}'$ have equation $X_3 = 0$. Further, let $X_0 = X_3 = 0$ be the plane of $\mathcal{C}$, and let $X_4 = X_3 = 0$ be the plane $\mathcal{C}'$ of $\mathcal{C}'$. The conics $\mathcal{C}$ and $\mathcal{C}'$ define distinct lines of $\eta_1$, and so $M$ is not contained in $\pi$. As $M$ is contained in $\zeta$ and intersects $\sigma \cap \sigma'$ we may choose $M$ to be the line $X_6 = X_1 = X_4 = 0$. Then $s(0, 0, 1, 0)$, and let $s(0, 0, 1, 0, 1) \in \mathcal{M}$. Also, let the pole of $sr$ with respect to $\mathcal{C}$ be the point $(0, 0, 0, 1, 1)$ and let the pole of $sr$ with respect to $\mathcal{C}'$ be the point $(0, 0, 0, 0, 0)$. Call $t$ a point of $\mathcal{C}' - \{s, r\}$. Now let $s_2 \in \mathcal{M} - \{s_1, s\}$ and let $w$ be the intersection of $st$ and $\langle r, (0, 0, 0, 0, 1) \rangle$. Further, let $t'$ be the common point of $s_2w$ and $st$. Finally let $e$ be a point of $\langle (1, 0, 0, 0, 0), t' \rangle - \{(1, 0, 0, 0, 0), r'\}$, and choose $e$ as point $(1, 1, 1, 1, 1)$. Then $t(0, 1, 1, 1, 1)$, $s(0, 1, 0, 0, 0), w(0, 1, 0, 0, 1), s_2(0, 1, 1, 0, 0)$.

Consequently, $\mathcal{C}$: $X_3 = X_1X_2 - X_4^2 = 0$.

$\mathcal{C}'$: $X_3 = X_1X_2 - vX_0^2 = 0$, $v \neq 0$.

Let $\psi$ be the pencil of quadrics in $X_3 = 0$ with base curve $\mathcal{C} \cup \mathcal{C}'$. Then $\psi$ is the set of quadrics $\Phi_\psi$ with $\Phi_\psi$: $X_3 = 2hX_0X_4 + (X_1X_2 - vX_0^2 - X_4^2) = 0$, $h \in GF(q^n) \cup \{\infty\}$.

Clearly $\sigma \cup \sigma'$ is an element of $\psi$. If $\gamma_1$ is the number of cones (with a point vertex) in $\psi$, $\gamma_2$ is the number of non-singular elliptic quadrics in $\psi$, and $\gamma_3$ is the number of non-singular hyperbolic quadrics in $\psi$, then, counting in two ways the number of pairs $(p, \Phi_\psi)$, with $p$ a point of $\pi$ not on $\mathcal{C} \cup \mathcal{C}'$ and $p \in \Phi_\psi \in \psi$, we obtain

$$(2q^{2n} + q^n + 1 - 2q^n) + \gamma_1(q^{2n} + q^n + 1 - 2q^n) + \gamma_2(q^{2n} + 1 - 2q^n) + \gamma_3((q^n + 1)^2 - 2q^n) = q^{3n} + q^{2n} + q^n + 1 - 2q^n.$$ 

Also $\gamma_1 + \gamma_2 + \gamma_3 = q^n$. The singular points of $\Phi_\psi$, $h \neq \infty$, are determined by the equations $X_3 = hX_0X_4 - vX_0^2 = X_1 = hX_0 - X_4 = 0$. Hence if $v$ is a non-square then $\Phi_\psi$, $h \neq \infty$, is always non-singular, and if $v$ is a square ($\neq 0$) then for two distinct values of $h$, $h \neq \infty$, the quadric $\Phi_\psi$ is a quadratic cone.

So, if $v$ is a square, then $\gamma_1 = 2, \gamma_2 = (q^n - 1)/2, \gamma_3 = (q^n - 3)/2$, and if $v$ is a non-square, then $\gamma_1 = 0, \gamma_2 = (q^n + 1)/2, \gamma_3 = (q^n - 1)/2$.

Let $s'$ be any point of $\mathcal{M}$. Then the union of all the lines through $s'$ and a point of $\mathcal{M}$, all the tangent lines at $s'$ of the conics of $V$ through $s'$, and the line $M$, is a singular hyperbolic quadric $\mathcal{M}'$, with vertex $s'$. Clearly $\mathcal{M}'$,
contains \( C \cup C' \). If \( s \neq s' \), then \( H_s \cap \pi \) is a non-singular hyperbolic quadric of the pencil \( \psi \); if \( s = s' \), then \( H_s \cap \pi = \sigma \cup \sigma' \). Assume, by way of contradiction, that for distinct points \( s', s'' \) of \( \mathcal{H} \), with \( s' \neq s \neq s'' \), we have \( H_s \cap \pi = H_{s''} \cap \pi \). Let \( d \in (H_s \cap \pi) - (C \cup C') \). Then \( d \notin \mathcal{H}' \). On \( s'd \) there is a point \( r \) of \( \mathcal{H} \) and on \( s'd \) there is a point \( r \) of \( \mathcal{H}' \). So \( r, r_1 \) intersects \( \mathcal{H} \), a contradiction. So \( H_s \cap \pi \neq H_{s''} \cap \pi \). It follows that \( \gamma \geq |\mathcal{H}| - 1 = (q^n - 1)/2 \). Consequently \( v \) is a non-square, \( \gamma_1 = 0, \gamma_2 = (q^n - 1)/2 \) and \( \gamma_3 = (q^n - 1)/2 \). Also, the \((q^n - 1)/2 \) non-singular hyperbolic quadrics of \( \psi \) are the intersections of \( \pi \) with the \((q^n - 1)/2 \) singular hyperbolic quadrics \( H_s \), with \( s' \) any point of \( \mathcal{H} \) \(- \{s\} \).

Let \( \mathcal{H}_s \) be the singular hyperbolic quadric \( H_s \), with \( s'(0, 0, 1, 0, 0) \)

\[ \mathcal{H}_s : \begin{align*}
2h_1X_0X_4 + (X_1X_2 - vX_0^2 - X_4^2) & = 0, \quad \text{with } h_1^2 - v \neq 0 \text{ a square.}
\end{align*} \]

Let \( \mathcal{H}_s \) be the singular hyperbolic quadric \( H_s \), with \( s''(0, 0, 1, 0, 0) \)

\[ \mathcal{H}_s : \begin{align*}
2h_2X_0X_4 + (X_1X_2 - vX_0^2 - X_4^2) & = 0, \quad \text{with } h_2^2 - v \neq 0 \text{ a square.}
\end{align*} \]

As \( H_s \cap \pi \neq H_{s''} \cap \pi \), we have \( h_1 \neq h_2 \). So

\[ \mathcal{H}' \subseteq \mathcal{H}_s \cap \mathcal{H}_s = \mathcal{V}(2h_1X_0X_4 + (X_1X_2 - vX_0^2 - X_4^2), \]

\[ 2h_2X_0X_4 + (X_1X_2 - vX_0^2 - X_4^2)) = \mathcal{V}(2h_1X_0X_4 + (X_1X_2 - vX_0^2 - X_4^2), \]

\[ 2(h_2 - h_1)X_0X_4 - X_1X_3) \]

Now we calculate the number of points of \( \mathcal{H}_s \cap \mathcal{H}_s \) not in \( \zeta \). So we put \( x_1 = 1 \), and \( x_0 \) and \( x_4 \) can be chosen and define uniquely \( x_4 \) and \( x_3 \), we have \( \{(h_1 \cap h_2) - \zeta \} = q^{20} = |\mathcal{H}'| \). So \( \mathcal{H}' = (\mathcal{H}_s \cap \mathcal{H}_s) - \zeta \). As \( \mathcal{H}_s \cap \mathcal{H}_s \cap \zeta = \mathcal{V} \), we also have \( \mathcal{H}' = (\mathcal{H}_s \cap \mathcal{H}_s) - \mathcal{V} \). Clearly

\[ \mathcal{H}' = \{(m, v, m^2 + n^2 - 2h_1 mn, 2(h_2 - h_1) mn, n) \mid m, n \in GF(q^n) \} \]

Now we consider the quadric Veronesean \( \mathcal{Y}_{2}^{n} \) in \( PG(5, q^n) \), where

\[ \mathcal{Y}_{2}^{n} = \{(m, l^2, v, m^2 + n^2 - 2h_1 mn, 2(h_2 - h_1) mn, n^2) \mid (l, m, n) \text{ is a point of } PG(2, q^n) \} \]

Assume that \( PG(4, q^n) \) is the hyperplane \( X_5 = 0 \) of \( PG(5, q^n) \). Project \( \mathcal{Y}_{2}^{n} \) from \( (0, 0, 0, 1) \) \( \mathcal{Y}_{2}^{n} \) onto \( PG(4, q^n) \). Then we obtain the surface

\[ \Gamma = \{(m, l^2, v, m^2 + n^2 - 2h_1 mn, 2(h_2 - h_1) mn, 0) \mid (l, m, n) \neq (0, 0, 0) \} \]

For \( l = 1 \) we obtain the points \( (m, 1, v, m^2 + n^2 - 2h_1 mn, 2(h_2 - h_1) mn, n, 0) \), with \( m, n \in GF(q^n) \). Hence \( \Gamma - \zeta = \mathcal{Y}' = (\mathcal{H}_s \cap \mathcal{H}_s) - \mathcal{V} \). For \( l = 0 \) we obtain
the points $(0, 0, m^2 + n^2 - 2h_1mn, 2(h_2 - h_1)mn, 0, 0)$, with $(m, n) \neq (0, 0)$. So $\Gamma \cap \xi \subseteq M$ and $|\Gamma \cap \xi| = (q^n + 1)/2$. The point $(0, 0, 0, 0, 0, 1)$ from which we project is in the conic plane $\varphi$: $X_0 = X_1 = X_3 = 0$ of $\mathbb{V}^4_2$, and is an interior point of the conic $\varphi \cap \mathbb{V}^4_2$. Let $p_1, p_2 \in \mathbb{V}^4_2$, $p_1 \neq p_2$, let $C$ be the conic of $V$ containing $p_1$ and $p_2$, and let $\bar{C}$ be any non-singular conic of $\Gamma$ containing $p_1$ and $p_2$. Assume, by way of contradiction that $\bar{C} \neq \bar{C}$. If $\sigma$ is the plane of $C$ and if $p$ is a point of $\bar{C}$ not on $C$ and not in $\xi$, then $\Phi_1 = \langle \sigma, p \rangle$ contains two conics $\bar{C}$ and $\bar{C}'$ of $V$. Also $\bar{C} - \xi \subseteq (\bar{C} \cup \bar{C}') - \xi$. Consequently $\bar{C}' = \bar{C}$, a contradiction as $|\bar{C}' \cap \bar{C}^{-1}| = 1$. Hence $\bar{C} = \bar{C}$. Now it follows easily that $V$ is the set of all non-singular conics on $\Gamma$, and that $V$ is the set of the projections of all non-singular conics on $\mathbb{V}^4_2$ but not in the plane $\varphi$.

From the preceding section it also follows that the set $\mathcal{M}$ of all common points of $M$ with elements of $V$, is the projection from $(0, 0, 0, 0, 1)$ onto $\text{PG}(4, q^n)$ of the conic $\varphi \cap \mathbb{V}^4_2$.

We conclude that $\mathbb{V}^4_2 \cup \mathcal{M}$ is the projection of the quadric Veronesean $\mathbb{V}^4_2$ from $(0, 0, 0, 0, 1)$ onto the space $\text{PG}(4, q^n)$. 

**Corollary 6.3.** In $\mathcal{M}$ there are either 0 or 2 points which belong to exactly $q^n$ elements of $V$.

**Proof.** This follows immediately from the proof of Theorem 6.2.

**Theorem 6.4.** Consider a non-classical TGQ $T(n, 2n, q) = T(O)$, $q$ odd, with $O$ an egg which is good at $\text{PG}(n-1, q)$. Assume also that the set $\mathbb{V}^4_2$ is contained in a $\text{PG}(4, q^n)$. Then the set $\mathcal{M}$ contains exactly 2 points which belong to exactly $q^n$ elements of $V$, that is, the set $\mathcal{M}$ is the projection of an irreducible conic $\mathcal{K}$ from an exterior point $p$ of $\mathcal{K}$ onto a line $M$ of the plane of $\mathcal{K}$.

**Proof.** We use the notations of the preceding theorems.

Let $\text{PG}(n-2, q)$ be any hyperplane of $\text{PG}(n-1, q)$. Assume by way of contradiction that the extension $\text{PG}(n-2, q^n)$ of $\text{PG}(n-2, q)$ contains a point $s$ of $\mathcal{M}$. As $s$ belongs to a conic $C$ of $V$, the point $s$ together with its conjugates with respect to the extension $\text{GF}(q^n)$ of $\text{GF}(q)$ generate $\text{PG}(n-1, q^n)$. But $s$ and all its conjugates belong to $\text{PG}(n-2, q^n)$, a contradiction. Hence $\text{PG}(n-2, q^n) \cap \mathcal{M} = \emptyset$.

Now we dualize in $\text{PG}(n-1, q^n)$, in such a way that the points of $\text{PG}(n-1, q)$ become hyperplanes of $\text{PG}(n-1, q)$. The dual of the line $M$ containing $\mathcal{M}$ is a $\text{PG}(n-3, q^n)$. The dual of a hyperplane $\text{PG}(n-2, q)$ of $\text{PG}(n-1, q)$ is a point $z$ of $\text{PG}(n-1, q)$, and the dual of the common point of $M$ and the hyperplane $\text{PG}(n-2, q^n)$ which extends $\text{PG}(n-2, q)$ is the hyperplane $\langle z, \text{PG}(n-3, q^n) \rangle$. Let $\text{PG}(1, q^n) = U$ be a line of
PG(n−1, qn) which is skew to PG(n−3, qn). Then there is a linear isomorphism θ of M onto U which maps the set of the intersections $M \cap PG(n−2, qn)$, with $PG(n−2, qn) = PG(n−2, q)$, onto the set of the intersections $U \cap \langle z, PG(n−3, qn) \rangle$, with z a point of PG(n−1, q).

We embed PG(n−1, q) in a PG(n, q), and in the extension PG(n, qn) of PG(n, q) we choose a PG(2, qn) with $PG(2, qn) \cap PG(n−1, qn) = U$. Now we project PG(n, q) from PG(n−3, qn) onto the plane PG(2, qn). Assume, by way of contradiction, that the points $z_1, z_2$, with $z_1 \neq z_2$, of PG(n, q) − PG(n−1, q) are projected onto the same point z of PG(2, qn).

Then $z_1 z_2$ has a unique point in common with PG(n−1, q) which also belongs to PG(n−3, qn). Hence the line M is contained in the extension PG(n−2, qn) of a hyperplane PG(n−2, q) of PG(n−1, q), a contradiction. It follows that the qn points $z$ of PG(n, q) − PG(n−1, q) are projected onto $qn$ distinct points $z'$ of PG(2, qn) − U. If $z'_1, z'_2 \in PG(2, qn) − U$, with $z'_1 \neq z'_2$, are such projections, then the set of all intersections $U \cap \langle z \rangle$, with z a point of PG(n−1, q).

We now embed PG(2, qn) in a PG(3, qn), and in PG(3, qn) we choose a plane π in such a way that $π \cap PG(2, qn) = U$. Assume, by way of contradiction, that the set $\mathcal{H}$ in the statement of the theorem does not contain a point which belongs to exactly qn elements of V. If $\mathcal{H}' = \mathcal{H}$, then there exists a non-singular conic $C$ in π, with $C \cap U = \emptyset$, such that $\mathcal{H}$ is the set of all exterior points of C on U; see Thas [1981]. Dualizing in PG(3, qn), we see that with the qn projections $z' \in PG(2, qn) − U$ there correspond qn planes which intersect some quadratic cone K in qn mutually disjoint non-singular conics (the points of K different from its vertex v correspond with the planes of PG(3, qn) which intersect π in a tangent line of C; the point v corresponds with the plane π). Hence there arises a flock F of K. As all points $z'$ are coplanar, the corresponding planes all contain a common point w. As the set of all intersections $M \cap PG(n−2, qn)$ is not a singleton all points $z'$ are not collinear, and so by Thas [1987] the point w necessarily is an exterior point of K. It follows that PG(2, qn) contains 2 points of $\mathcal{H}$, that is $|U \cap \mathcal{H}| = 2$, a contradiction.

We conclude that the set $\mathcal{H}$ contains exactly 2 points which belong to exactly qn elements of V.

**Lemma 6.5.** Consider a non-classical TGQ $T(n, 2n, q) = T(O), q$ odd, with O an egg which is good at PG(n−1, q). Assume also that the set $\mathcal{H}'$ is contained in a PG(4, qn). If $s_1, s_2$ are the points of $\mathcal{H}'$ which belong to exactly qn elements of V, then $s_1, s_2$ are conjugate with respect to the $n$th extension $GF(qn)$ of $GF(q)$, that is, $s_1, s_2$ are in the same orbit of the group $\langle \theta \rangle$ with $\theta$ the collineation of order $n$ of PG(n−1, qn) which fixes PG(n−1, q) pointwise.
Let \( s_1(x_0, x_1, \ldots, x_{n-1}) \) and \( s_2(y_0, y_1, \ldots, y_{n-1}) \) in \( \text{PG}(n-1, q^n) \). As each point \( s \in \mathcal{A} \) belongs to a conic \( \mathcal{C} \in V \), the space \( \langle s, s', s'', \ldots, s^{(n-1)} \rangle \) is \((n-1)\)-dimensional. So the extension \( \text{PG}(n-2, q) \) of any hyperplane \( \text{PG}(n-2, q) \) of \( \text{PG}(n-1, q) \) intersects \( M = s_1, s_2 \) in a point not in \( \mathcal{A} \). As \( \mathcal{M} \) is the projection onto \( M \) of a non-singular conic \( \mathcal{K} \) from an exterior point of \( \mathcal{K} \), we have that the cross-ratio \( \{s_1, s_2; v, w\} = \square \) for any two distinct points \( v, w \) not in \( \mathcal{A} \).

The hyperplane \( X_i = 0 \) of \( \text{PG}(n-1, q^n) \) intersects \( M \) in the point

\[
v_i(x_0 y_i - x_i y_0, \ldots, x_{i-1} y_i - x_i y_{i-1}, 0, x_{i+1} y_i - x_i y_{i+1}, x_n y_i - x_i y_n),
\]

\( i = 0, 1, \ldots, n \). As \( \{x_j, x_k; v, w\} = \{(1, 0), (0, 1); (y_j, y_k), (y_j, y_k)\} \), we have \( \{s_1, s_2; v, w\} = (y_i, x_i)/(y_j, x_j) = \square \) for all \( i, j = 0, 1, \ldots, n \) with \( i \neq j \). Hence \( x_i y_i x_j y_j = \square \) for all \( i, j = 0, 1, \ldots, n \) with \( i \neq j \). Next, let

\[
\delta: a_0 X_0 + a_1 X_1 + \cdots + a_{n-1} X_{n-1} = 0, \quad a_i \in \text{GF}(q),
\]

be any hyperplane of \( \text{PG}(n-1, q) \). If \( v \) is the common point of \( \delta \) and \( M \), then

\[
v = (a_0 y_0 + a_1 y_1 + \cdots + a_{n-1} y_{n-1}) s_1 + (-a_0 x_0 - a_1 x_1 - \cdots - a_{n-1} x_{n-1}) s_2.
\]

If \( \delta' \neq \delta \) is a second hyperplane of \( \text{PG}(n-1, q) \), with

\[
\delta': b_0 X_0 + b_1 X_1 + \cdots + b_{n-1} X_{n-1} = 0, \quad b_i \in \text{GF}(q),
\]

and if \( w \) is the common point of \( \delta' \) and \( M \), then

\[
w = (b_0 y_0 + b_1 y_1 + \cdots + b_{n-1} y_{n-1}) s_1 + (-b_0 x_0 - b_1 x_1 - \cdots - b_{n-1} x_{n-1}) s_2.
\]

Hence

\[
\{s_1, s_2; v, w\} = \left\{(1, 0), (0, 1); \left(\sum_i a_i y_i - \sum_i a_i x_i\right), \left(\sum_i b_i y_i - \sum_i b_i x_i\right)\right\}.
\]

Consequently

\[
\{s_1, s_2; v, w\} = \left(\sum_i b_i y_i \sum_i a_i x_i\right)/\left(\sum_i b_i y_i \sum_i a_i y_i\right) = \square,
\]
and so
\[ \sum a_i x_i \sum a_i y_i \sum h_i x_i \sum h_i y_i = \square. \]

As the points \( s_1, s_2 \) do not belong to the hyperplane \( X_0 = 0 \) of \( \text{PG}(n-1, q^n) \), we may assume that \( x_0 = 1 = y_0 \). Then
\[ x_i y_i = \square, \quad i = 1, 2, ..., n-1, \]
and
\[ \left( a_0 + \sum_{i=1}^{n-1} a_i x_i \right) \left( a_0 + \sum_{i=1}^{n-1} a_i y_i \right) = \square, \quad (1) \]
with \( a_0, a_1, ..., a_{n-1} \in \text{GF}(q) \) and \( (a_1, a_2, ..., a_{n-1}) \neq (0, 0, ..., 0) \).

If the elements \( 1, x_1, x_2, ..., x_{n-1} \) are linearly dependent over \( \text{GF}(q) \), then
\[ l_0 + l_1 x_1 + l_2 x_2 + \cdots + l_{n-1} x_{n-1} = 0 \]
for some \( l_0, l_1, ..., l_{n-1} \in \text{GF}(q) \) with \( (l_0, l_1, ..., l_{n-1}) \neq (0, 0, ..., 0) \). Hence the point \( s_1 \) belongs to the extension of the hyperplane \( l_0 X_0 + l_1 X_1 + \cdots + l_{n-1} X_{n-1} = 0 \) of \( \text{PG}(n-1, q^n) \), a contradiction. Consequently the elements \( 1, x_1, x_2, ..., x_{n-1} \) are linearly independent over \( \text{GF}(q) \). It follows that
\[ \text{GF}(q^n) = \{ c_0 + c_1 x_1 + c_2 x_2 + \cdots + c_{n-1} x_{n-1} \mid c_i \in \text{GF}(q) \}. \]

Analogously,
\[ \text{GF}(q^n) = \{ c_0 + c_1 y_1 + c_2 y_2 + \cdots + c_{n-1} y_{n-1} \mid c_i \in \text{GF}(q) \}. \]

Now we consider the mapping \( \varphi \) defined by
\[ \varphi: \text{GF}(q^n) \rightarrow \text{GF}(q^n), \quad c_0 + c_1 x_1 + \cdots + c_{n-1} x_{n-1} \rightarrow c_0 + c_1 y_1 + \cdots + c_{n-1} y_{n-1}. \]

Clearly \( \varphi \) is a permutation of \( \text{GF}(q^n) \).

Let \( h, k \) be distinct elements of \( \text{GF}(q^n) \), with
\[ h = a_0 + a_1 x_1 + \cdots + a_{n-1} x_{n-1}, \]
\[ k = b_0 + b_1 x_1 + \cdots + b_{n-1} x_{n-1}. \]

Then
\[ (h^n - k^n)(h - k) = \left( a_0 - b_0 + \sum_{i=1}^{n-1} (a_i - b_i) y_i \right) \left( a_0 - b_0 + \sum_{i=1}^{n-1} (a_i - b_i) x_i \right). \]
By (1) we have
\[(h^n - k^n)(h - k) = \square.\]
Also \(0^r = 0\) and \(1^r = 1.\) Now by a theorem of Carlitz [1960]
\[
\varphi: GF(q^r) \to GF(q^n), \quad h \mapsto h^r,
\]
with \(q = p^h, p\) prime, and \(1 \leq r \leq hn - 1\) \((r \neq 0\) as \(s_1 \neq s_2).\) For any \(a \in GF(q)\) we have
\[
ay_1 = (ax_1)^r = a^r x_1^r = a^r y_1.
\]
As \(y_1 \neq 0,\) we have
\[
a^r = a, \quad \text{for all } a \in GF(q).
\]
It follows that \(r = dh\) for some \(d,\) with \(1 \leq d \leq n - 1,\) and so
\[
\varphi: GF(q^r) \to GF(q^n), \quad h \mapsto h^r.
\]
As \(\vartheta: PG(n - 1, q^r) \to PG(n - 1, q^n),\) \((z_0, z_1, ..., z_{n-1}) \mapsto (z_0^r, z_1^r, ..., z_{n-1}^r),\)
we have
\[
s_2 = (1, y_1, ..., y_{n-1}) = (1, x_1^r, ..., x_{n-1}^r) = s_1^d.
\]
We conclude that \(s_1\) and \(s_2\) are in the same orbit of the group \(\langle \vartheta \rangle.\)

**Theorem 6.6.** Consider a non-classical TGQ \(T(n, 2n, q) = T(O),\) \(q\) odd, with \(O\) an egg which is good at \(PG(n - 1, q).\) If the set \(\mathcal{W}\) is contained in a \(PG(4, q^n),\) then the translation dual \(T(O^*)\) of \(T(O)\) is good at \(\tau\) with \(\tau\) the tangent space of \(O\) at \(PG(n - 1, q).\)

**Proof.** We use the notations of Lemma 6.5.

The point \(s_i\) belongs to \(q^n\) elements of \(V\) and all these conics have a common tangent line \(U_i\) at \(s_i, i = 1, 2;\) this line \(U_i\) does not belong to \(PG(n - 1, q^r), i = 1, 2.\) The collineation of order \(n\) of \(PG(4n - 1, q^n)\) which fixes \(PG(4n - 1, q^n)\) pointwise (and so is an extension of \(\vartheta\)), will also be denoted by \(\vartheta.\) If \(\mathcal{W} \in V\) contains \(s_i,\) then \(\langle U_i, U_i^?, U_i^??, ..., U_i^{??}\rangle\) is the tangent space at \(PG(n - 1, q)\) of the \([n - 1]\)-oval \(O(n, n, q)\) defined by \(\mathcal{W},\) \(i \in \{1, 2\}.\) If \(\vartheta^d = \vartheta\) then the line \(U_i^?\) contains the point \(s_i^d = s_2,\) and the line \(U_i^{??}\) contains the point \(s_1.\) If \(U_2 = U_1^?\), then the \(q^n\) \([n - 1]\)-ovals defined by the conics of \(V\) through \(s_1\) and the \(q^n\) \([n - 1]\)-ovals defined by the
Corresponding set \( \mathcal{H} \) of \( O(n, 2n, q) = O \), and the tangent space \( \tau_i \) of \( O \) at \( \mathcal{H} \). Let \( r_j \) be the point of \( \mathcal{H} \) which corresponds with \( \mathcal{H} \) of order \( n \), and let \( \mathcal{H}_j \) be the conic of \( V \) which contains \( r_j \) and \( s_j \), \( j = 1, 2 \). Let \( Z_j \) be the tangent line of \( \mathcal{H}_j \) at \( r_j, j = 1, 2 \). Then \( \tau_i = \langle Z_1, Z_2, \ldots, Z_{n-1} \rangle \). If \( U_1 \cap Z_1 = \{ z_1 \} \) and \( U_2 \cap Z_2 = \{ z_2 \} \), then \( \tau_i \cap \mathcal{H} = z_2 z_1 \). Let \( \tau_j \cap \mathcal{H} = Y_j \); then \( \tau_i \cap \mathcal{H} = Y_j \).

Consider an element \( \mathcal{H}(n-1, q) \) of \( O(n, 2n, q) = O \), and the tangent space \( \tau_i \) of \( O \) at \( \mathcal{H} \). Let \( r_j \) be the point of \( \mathcal{H} \) which corresponds with \( \mathcal{H} \) of order \( n \), and let \( \mathcal{H}_j \) be the conic of \( V \) which contains \( r_j \) and \( s_j \), \( j = 1, 2 \). Let \( Z_j \) be the tangent line of \( \mathcal{H}_j \) at \( r_j, j = 1, 2 \). Then \( \tau_i = \langle Z_1, Z_2, \ldots, Z_{n-1} \rangle \). If \( U_1 \cap Z_1 = \{ z_1 \} \) and \( U_2 \cap Z_2 = \{ z_2 \} \), then \( \tau_i \cap \mathcal{H} = z_2 z_1 \). Let \( \tau_j \cap \mathcal{H} = Y_j \); then \( \tau_i \cap \mathcal{H} = Y_j \).

Consider an element \( \mathcal{H}(n-1, q) \) of \( O(n, 2n, q) = O \), and the tangent space \( \tau_i \) of \( O \) at \( \mathcal{H} \). Let \( r_j \) be the point of \( \mathcal{H} \) which corresponds with \( \mathcal{H} \) of order \( n \), and let \( \mathcal{H}_j \) be the conic of \( V \) which contains \( r_j \) and \( s_j \), \( j = 1, 2 \). Let \( Z_j \) be the tangent line of \( \mathcal{H}_j \) at \( r_j, j = 1, 2 \). Then \( \tau_i = \langle Z_1, Z_2, \ldots, Z_{n-1} \rangle \). If \( U_1 \cap Z_1 = \{ z_1 \} \) and \( U_2 \cap Z_2 = \{ z_2 \} \), then \( \tau_i \cap \mathcal{H} = z_2 z_1 \). Let \( \tau_j \cap \mathcal{H} = Y_j \); then \( \tau_i \cap \mathcal{H} = Y_j \).

Consider an element \( \mathcal{H}(n-1, q) \) of \( O(n, 2n, q) = O \), and the tangent space \( \tau_i \) of \( O \) at \( \mathcal{H} \). Let \( r_j \) be the point of \( \mathcal{H} \) which corresponds with \( \mathcal{H} \) of order \( n \), and let \( \mathcal{H}_j \) be the conic of \( V \) which contains \( r_j \) and \( s_j \), \( j = 1, 2 \). Let \( Z_j \) be the tangent line of \( \mathcal{H}_j \) at \( r_j, j = 1, 2 \). Then \( \tau_i = \langle Z_1, Z_2, \ldots, Z_{n-1} \rangle \). If \( U_1 \cap Z_1 = \{ z_1 \} \) and \( U_2 \cap Z_2 = \{ z_2 \} \), then \( \tau_i \cap \mathcal{H} = z_2 z_1 \). Let \( \tau_j \cap \mathcal{H} = Y_j \); then \( \tau_i \cap \mathcal{H} = Y_j \).

Consider an element \( \mathcal{H}(n-1, q) \) of \( O(n, 2n, q) = O \), and the tangent space \( \tau_i \) of \( O \) at \( \mathcal{H} \). Let \( r_j \) be the point of \( \mathcal{H} \) which corresponds with \( \mathcal{H} \) of order \( n \), and let \( \mathcal{H}_j \) be the conic of \( V \) which contains \( r_j \) and \( s_j \), \( j = 1, 2 \). Let \( Z_j \) be the tangent line of \( \mathcal{H}_j \) at \( r_j, j = 1, 2 \). Then \( \tau_i = \langle Z_1, Z_2, \ldots, Z_{n-1} \rangle \). If \( U_1 \cap Z_1 = \{ z_1 \} \) and \( U_2 \cap Z_2 = \{ z_2 \} \), then \( \tau_i \cap \mathcal{H} = z_2 z_1 \). Let \( \tau_j \cap \mathcal{H} = Y_j \); then \( \tau_i \cap \mathcal{H} = Y_j \).

Consider an element \( \mathcal{H}(n-1, q) \) of \( O(n, 2n, q) = O \), and the tangent space \( \tau_i \) of \( O \) at \( \mathcal{H} \). Let \( r_j \) be the point of \( \mathcal{H} \) which corresponds with \( \mathcal{H} \) of order \( n \), and let \( \mathcal{H}_j \) be the conic of \( V \) which contains \( r_j \) and \( s_j \), \( j = 1, 2 \). Let \( Z_j \) be the tangent line of \( \mathcal{H}_j \) at \( r_j, j = 1, 2 \). Then \( \tau_i = \langle Z_1, Z_2, \ldots, Z_{n-1} \rangle \). If \( U_1 \cap Z_1 = \{ z_1 \} \) and \( U_2 \cap Z_2 = \{ z_2 \} \), then \( \tau_i \cap \mathcal{H} = z_2 z_1 \). Let \( \tau_j \cap \mathcal{H} = Y_j \); then \( \tau_i \cap \mathcal{H} = Y_j \).
a PG(5, q^n), but not in a PG(4, q^n). If Ω is the set of all common points of the plane μ = PG(3, q^n) ∩ PG(5, q^n) with conics of V, then Ω ∪ Ω is a quadric Veronesean \( r^+ \) in PG(5, q^n).

Proof. Let \( C, C' \in V \), with \( C \neq C' \), and assume that \( C \cap Ω = C' \cap Ω = \{s\} \). If \( |C \cap Ω| = 2 \), then \( C \) and \( C' \) are contained in a 3-dimensional space PG(3, q^n) which contains \( C \) and more than one point \( r_d \in Ω - C \), a contradiction (see section preceding (a) in the proof of Theorem 6.1). Hence \( C \cap C' = \{s\} \).

Consider a conic \( C \in V \) and all conics \( C' \in V - \{C\} \) containing a point of \( C - μ \). As no 3-dimensional space containing \( C \) contains more than one point \( r_d \in Ω - C \), the conics \( C \) and \( C' \) generate a 4-dimensional space. Assume, by way of contradiction, that the 4-dimensional spaces \( \pi'_1 \) and \( \pi'_2 \), respectively defined by \( C, C' \) and \( C', C'' \) with \( C', C'' \in V - \{C\} \), \( |C \cap C' \cap Ω| \geq 1 \), \( |C \cap C'' \cap Ω| \geq 1 \) and \( C' \neq C'' \), coincide. Let \( r \neq C \cup C' \cup C'' \), \( r \in Ω \). In the plane \( η_1 \) (we still use the notations introduced at the beginning of the proof of Theorem 6.1) there correspond with \( C, C'_1, C'_2 \) distinct lines \( K, K'_1, K'_2 \), and with \( r \), the point \( \tilde{r} \). Now let \( K'' \) be a line of \( η_1 \) through \( \tilde{r} \), which intersects \( K, K'_1, K'_2 \) in distinct points of \( η_1 - \tilde{r} \). With \( K'' \) there corresponds a conic \( \eta'' \) of \( V \) through \( \tilde{r} \), which intersects \( C \cup C'_1 \cup C'_2 \) in at least three distinct points. Hence \( \eta'' \) belongs to \( \pi'_1 \), and so \( r \) belongs to \( \pi'_1 \). We conclude that \( Ω = \pi'_1 \), a contradiction. Consequently \( \pi'_1 \neq \pi'_2 \), that is, the \( q^{2n} \) conics \( C' \) define \( q^{2n} \) 4-dimensional spaces. Let \( \pi' \) be the 4-dimensional space generated by \( C \) and \( C' \), with \( C' \in V - \{C\} \) and \( |C \cap C' \cap Ω| \geq 1 \). Assume, by way of contradiction, that \( \pi' \cap Ω \) contains a point \( r_d \) not in \( C \cup C' \). In the plane \( η_1 \) there correspond with \( C \cup C'_1 \cup C'_2 \) distinct lines \( K, K' \), and with \( r_d \) the point \( \tilde{r} \). Now let \( K'' \) be a line of \( η_1 \) through \( \tilde{r} \), which intersects \( K, K' \) in distinct points of \( η_1 - \tilde{r} \). With \( K'' \) there corresponds a conic \( \eta'' \) of \( V \) through \( \tilde{r} \), which intersects each of \( C \), \( C' \) in a point not in \( μ \). So \( \eta'' \) belongs to \( \pi'' \), and consequently the 4-dimensional spaces generated by \( C, C' \), respectively \( C, C'' \), coincide, a contradiction. It follows that \( \pi' \cap Ω = (C \cup C') \cap Ω \).

Next, assume that the conics \( C_1, C_2 \in V \), with \( C_1 \neq C_2 \), define lines of the plane \( η_1 \) which intersect \( \tau \) in a common point. Suppose, by way of contradiction, that \( η_i \cap Ω \neq \emptyset \). Let \( γ_i, γ_{i+1}, i = 1, 2 \). Assume first that \( \langle γ_1, γ_2 \rangle \) is 3-dimensional. Then the \( q^{2n} \) 4-dimensional spaces of the preceding section defined by \( C_i \), intersect \( γ_2 \) in \( q^{2n} \) distinct lines. Hence one of these spaces, say \( δ \), intersects \( C_2 \) in two distinct points of \( Ω \). If \( C_1, C' \) are the conics of \( V \) in \( δ \), then \( C' \) contains two distinct points of \( C_2 \cap Ω \), clearly a contradiction. Consequently \( \langle γ_1, γ_2 \rangle \) is 3-dimensional. If \( r \in C_2 \cap Ω \), then the 3-dimensional space generated by \( C_1 \) and \( r \) contains just one point of \( C_2 \cap Ω \), hence intersects \( γ_2 \) in the tangent line of \( C_2 \) at \( r \). This tangent line contains the common point \( I \) of \( γ_1 \) and \( γ_2 \). Consequently
$q^o$ tangent lines of $C_2$ are concurrent, a contradiction as $q$ is odd. So we conclude that $|\{C_1 \cap C_2\}| = 1$, that is, $C_1 \cap C = C_2 \cap C$.

From the foregoing follows that the $q^{2n} + q^n$ conics of $V$ intersect $\mu$ in exactly $q^n + 1$ distinct points; the set of these $q^n + 1$ points will be denoted by $\mathcal{P}$.

Now we will use Theorem 25.2.13 of Hirschfeld and Thas [1991] to prove that $\mathcal{P} \cup \mathcal{W}$ is a quadratic Veronesean in $PG(5, q^n)$. Call $\mathcal{L}$ the set with as elements the plane $\mu$ and the $q^{2n} + q^n$ planes of the conics of $V$. Any point of $\mathcal{P} \cup \mathcal{W}$ belongs to $q^n + 1$ elements of $\mathcal{L}$, hence no point of $\mathcal{P} \cup \mathcal{W}$ belongs to all elements of $\mathcal{L}$. Relying also on the section preceding (a) in the proof of Theorem 6.1 we have: (i) if $C \cap \mu = C' \cap \mu = \{x\}$, with $C, C' \in \mathcal{V}$ and $C \neq C'$, then the planes of $C$ and $C'$ have just $s$ in common, (ii) if $C \cap C' \cap \mu = \{x\}$, with $C, C' \in \mathcal{V}$ and $C \neq C'$, then the common point of $C$ and $C'$ is the unique common point of the planes of $C$ and $C'$, and (iii) for any $C \in V$, the plane of $C$ contains exactly one point of the plane $\mu$. It follows that any two distinct elements of $\mathcal{L}$ have exactly one point in common. Now we consider any three distinct elements $\pi_1, \pi_2, \pi_3$ of $\mathcal{L}$. First, assume that $\pi_i \neq \mu$, with $i = 1, 2, 3$. Let $C_i$ be the element of $V$ in $\pi_i$, with $i = 1, 2, 3$. Assume, by way of contradiction, that $\pi_1, \pi_2, \pi_3$ generate a $PG(4, q^n)$. Let $r_s \not\in C_1 \cup C_2 \cup C_3, r_s \in \mathcal{W}$. In the plane $\eta_1$ there correspond with $C_1, C_2, C_3$ distinct lines $K_1, K_2, K_3$ and with $r_s$ the point $\tilde{r_s}$. Now let $K'$ be a line of $\eta_1$ through $\tilde{r_s}$ which intersects $K_1, K_2, K_3$ in distinct points of $\eta_1 - r_s$. With $K'$ there corresponds a conic $\mathcal{C}'$ of $V$ through $r_s$ which intersects $C_1 \cup C_2 \cup C_3$ in three distinct points. Hence $\mathcal{C}'$ belongs to $PG(4, q^n)$, hence $r_s$ belongs to $PG(4, q^n)$, so $\mathcal{W} \subset PG(4, q^n)$, a contradiction. It follows that $\langle \pi_1, \pi_2, \pi_3 \rangle = PG(5, q^n)$. Next, suppose that $\pi_1 = \pi_2$. Let $C_i$ be the element of $V$ in $\pi_i$, with $i = 1, 2$. Assume, by way of contradiction, that $PG(4, q^n) = \langle \pi_1, \pi_2 \rangle$ also contains $\mu$. Let $r_s \not\in C_1 \cup C_2, r_s \in \mathcal{W}$. In the plane $\eta_1$ there correspond with $C_1, C_2$ distinct lines $K_1, K_2$ and with $r_s$ the point $\tilde{r_s}$. Now let $K'$ be a line of $\eta_1$ through $\tilde{r_s}$ which intersects $K_1, K_2$ in distinct points of $\eta_1 - r_s$. With $K'$ there corresponds a conic $\mathcal{C}'$ of $V$ through $r_s$ which intersects $C_1 \cup C_2$ in two distinct points of $\mathcal{W}$ and which contains a point of $\mu$ not on $C_1 \cup C_2$. Hence $\mathcal{C}'$ belongs to $PG(4, q^n)$, hence $r_s$ belongs to $PG(4, q^n)$, so $\mathcal{W} \subset PG(4, q^n)$, a contradiction. Now by Theorem 25.2.13 of Hirschfeld and Thas [1991], which is a corollary of a theorem by Tallini [1958], $\mathcal{L}$ is the set of all conic planes of a quadratic Veronesean $\tau^2_2$ in $PG(5, q^n)$. Also, $\tau^2_2$ is the set of all intersections of pairs of distinct elements of $\mathcal{L}$, that is, $\tau^2_2 = \mathcal{W} \cup \mathcal{P}$.

Main Theorem 6.9. Consider a TGQ $T(n, 2n, q) = T(O), q$ odd, with $O = O(n, 2n, q) = \{PG(n - 1, q), PG^{1/n}(n - 1, q), \ldots, PG^{n_2}(n - 1, q)\}$. If $O$ is good at $PG(n - 1, q)$, then we have one of the following.
There exists a $\text{PG}(3, q^n)$ in the extension $\text{PG}(4n-1, q^n)$ of the space $\text{PG}(4n-1, q)$ which has exactly one point in common with each of the spaces $\text{PG}(n-1, q^n)$, $\text{PG}(n-1, q^n)$, ..., $\text{PG}(n-1, q^n)$. The set of these $q^{2n}+1$ points is an elliptic quadric of $\text{PG}(3, q^n)$ and $\mathcal{T}(O)$ is isomorphic to the classical $\text{GQ}(5, q^n)$.

(b) We are not in Case (a) and there exists a $\text{PG}(4, q^n)$ in $\text{PG}(4n-1, q^n)$ which intersects $\text{PG}(n-1, q^n)$ in a plane $M$ and which has exactly one point $r_i$ in common with any space $\text{PG}(n-1, q^n)$, $i=1, 2, ..., q^{2n}$. Let $\mathcal{W} = \{r_i \mid i=1, 2, ..., q^{2n}\}$ and let $\mathcal{A}$ be the set of all common points of $M$ and the conics which contain exactly $q^n$ points of $\mathcal{W}$. Then the set $\mathcal{W} \cup \mathcal{A}$ is the projection of a quadric Veronesean $\mathcal{V}^2$ from a point $p$ in a conic plane of $\mathcal{V}^2$ onto a hyperplane $\text{PG}(4, q^n)$; the point $p$ is an exterior point of the conic of $\mathcal{V}^2$ in the conic plane. Also, if the point-line dual of the translation dual of $\mathcal{T}(O)$ is a flock $\mathcal{GQ}$, then $\mathcal{F}$ is a Kantor flock.

(c) We are not in Cases (a) and (b) and there exists a $\text{PG}(5, q^n)$ in $\text{PG}(4n-1, q^n)$ which intersects $\text{PG}(n-1, q^n)$ in a plane $\mu$ and which has exactly one point $r_i$ in common with any space $\text{PG}(n-1, q^n)$, $i=1, 2, ..., q^{2n}$. Let $\mathcal{W} = \{r_i \mid i=1, 2, ..., q^{2n}\}$ and let $\mathcal{P}$ be the set of all common points of $\mu$ and the conics which contain exactly $q^n$ points of $\mathcal{W}$. Then the set $\mathcal{W} \cup \mathcal{P}$ is a quadric Veronesean in $\text{PG}(5, q^n)$.

Proof. In Case (a), $\mathcal{T}(n, 2n, q)$ is the interpretation over $\text{GF}(q)$ of $\mathcal{T}(1, 2, q^n)$ $\mathcal{O}(5, q^n)$. If we are not in Case (a), then the theorem directly follows from Theorems 6.1, 6.2, 6.4, 6.7 and 6.8.

Remark. If we project the Veronesean $\mathcal{V}^2$ in $\text{PG}(5, q^n)$ onto some $\text{PG}(3, q^n)$ in $\text{PG}(4n-1, q^n)$ which intersects $\text{PG}(n-1, q^n)$ in a line $N$ which intersects $\mathcal{V}^2$ in two points of $\text{PG}(5, q^n)$, $\text{PG}(5, q^n)$ are conjugate with respect to the extension $\text{GF}(q^n)$ of $\text{GF}(q^n)$, then we obtain an elliptic quadric of $\text{PG}(3, q^n)$.

Conjectures. (a) In Case (c) of the Main Theorem and if the point-line dual of the translation dual of $\mathcal{T}(O)$ is a flock $\mathcal{GQ}$, then the flock $\mathcal{GQ}$ is a Roman $\mathcal{GQ}$ of Payne (see Payne [1988], [1989]). As by the Main Theorem, and also by some other results not mentioned here, we have strong information on this class of $\mathcal{GQ}$, we hope to prove shortly this conjecture. This would yield the complete classification of all $\mathcal{TQG}$ of order $(s, s')$, $s$ odd and $s \neq 1$, for which the point-line dual is a flock $\mathcal{TQG}$.

(b) Any $\mathcal{TQG}$ $\mathcal{T}(O)$ of order $(s, s')$ and $s \neq 1$, with $O$ good at some element, is the point-line dual of the translation dual of a translation flock $\mathcal{GQ}$.

Final remark. For $s$ even Johnson [1987] proves that any $\mathcal{TQG}$ for which the point-line dual is a flock $\mathcal{GQ}$, is the classical $\mathcal{GQ}(5, s)$. We
conjecture that any TGQ $T(O)$ of order $(s, s^2)$, $s$ even and $s \neq 1$, with $O$ good at some element, is the classical GQ $Q(5, s)$.

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