JOURNAL OF COMBINATORIAL THEORY, Series A **79**, 223–254 (1997)
ARTICLE NO. TA962779

Generalized Quadrangles of Order (s, s^2), II

J. A. Thas*

Department of Pure Mathematics and Computer Algebra, University of Ghent, Krijgslaan 281, B-9000 Ghent, Belgium

Communicated by Francis Buekenhout

Received November 26, 1996

1. INTRODUCTION

View metadata, citation and similar papers at core.ac.uk

For terminology, notation, results, etc., concerning finite generalized quadrangles and not explicitly given here, see the monograph of Payne and Thas [1984] denoted FGQ.

Let $\mathcal{S} = (P, B, \mathrm{I})$ be a (finite) generalized quadrangle (GQ) of order (s, t), $s \geqslant 1$, $t \geqslant 1$. So $\mathcal{S}$ has $v = |P| = (1+s)(1+st)$ points and $b = |B| = (1+t)(1+st)$ lines. If $s \neq 1 \neq t$, then $t \leqslant s^2$ and, dually, $s \leqslant t^2$; also $s + t$ divides $st(1+s)(1+t)$.

There is a point-line duality for GQ (of order (s, t)) for which in any definition or theorem the words "point" and "line" are interchanged and the parameters s and t are interchanged. Normally, we assume without further notice that the dual of a given theorem or definition has also been given.

Given two (not necessarily distinct) points x, x' of $\mathcal{S}$, we write $x \sim x'$ and say that x and x' are *collinear*, provided that there is some line L for which $x \mathrm{I} L \mathrm{I} x'$; hence $x \not\sim x'$ means that x and x' are not collinear. Dually, for $L, L' \in B$, we write $L \sim L'$ or $L \not\sim L'$ according as L and L' are *concurrent* or nonconcurrent. When $x \sim x'$ we also say that x is *orthogonal* or *perpendicular* to x'; similarly for $L \sim L'$. The line incident with distinct collinear points x and x' is denoted xx', and the point incident with distinct concurrent lines L and L' is denoted either LL' or $L \cap L'$.

For $x \in P$ put $x^\perp = \{x' \in P \parallel x \sim x'\}$, and note that $x \in x^\perp$. The *trace* of a pair $\{x, x'\}$ of distinct points is defined to be the set $x^\perp \cap x'^\perp$ and is denoted either $\mathrm{tr}(x, x')$ or $\{x, x'\}^\perp$; then $|\{x, x'\}^\perp| = s+1$ or $t+1$ according as $x \sim x'$ or $x \not\sim x'$. More generally, if $A \subset P$, A "*perp*" is defined by

* E-mail: jat@cage.rug.ac.be.

0097-3165/97 $25.00
Copyright © 1997 by Academic Press
All rights of reproduction in any form reserved.

$A^{\perp} = \bigcap \{x^{\perp} \parallel x \in A\}$. For $x \neq x'$, the *span* of the pair $\{x, x'\}$ is $\mathrm{sp}(x, x') = \{x, x'\}^{\perp\perp} = \{u \in P \parallel u \in z^{\perp}$ for all $z \in x^{\perp} \cap x'^{\perp}\}$. When $x \not\sim x'$, then $\{x, x'\}^{\perp\perp}$ is also called the *hyperbolic line* defined by x and x', and $|\{x, x'\}^{\perp\perp}| = s+1$ or $|\{x, x'\}^{\perp\perp}| \leqslant t+1$ according as $x \sim x'$ or $x \not\sim x'$.

2. REGULARITY, DUAL NETS, AND THE AXIOM OF VEBLEN

Let $\mathcal{S} = (P, B, \mathrm{I})$ be a finite GQ of order (s, t). If $x \sim x'$, $x \neq x'$, or if $x \not\sim x'$ and $|\{x, x'\}^{\perp\perp}| = t+1$, where $x, x' \in P$, we say the pair $\{x, x'\}$ is *regular*. The point x is *regular* provided $\{x, x'\}$ is regular for all $x' \in P$, $x' \neq x$. Regularity for lines is defined dually.

A (finite) *net* of *order* $k(\geqslant 2)$ and *degree* $r(\geqslant 2)$ is an incidence structure $\mathcal{N} = (P, B, \mathrm{I})$ satisfying

(i) each point is incident with r lines and two distinct points are incident with at most one line;

(ii) each line is incident with k points and two distinct lines are incident with at most one point;

(iii) if x is a point and L is a line not incident with x, then there is a unique line M incident with x and not concurrent with L.

For a net of order k and degree r we have $|P| = k^2$ and $|B| = kr$.

THEOREM 2.1 (1.3.1 of Payne and Thas [1984]). *Let x be a regular point of the* GQ $\mathcal{S} = (P, B, \mathrm{I})$ *of order* (s, t), $s > 1$. *Then the incidence structure with pointset $x^{\perp} - \{x\}$, with lineset the set of spans $\{y, z\}^{\perp\perp}$, where $y, z \in x^{\perp} - \{x\}$, $y \not\sim z$, and with the natural incidence, is the dual of a net of order s and degree $t+1$. If in particular $s = t > 1$, there arises a dual affine plane of order s. Also, in the case $s = t > 1$ the incidence structure π_x with pointset $x^{\perp}$, with lineset the set of spans $\{y, z\}^{\perp\perp}$, where $y, z \in x^{\perp}$, $y \neq z$, and with the natural incidence, is a projective plane of order s.*

Now we introduce the *Axiom of Veblen* for duals nets $\mathcal{N}^* = (P, B, \mathrm{I})$.

Axiom of Veblen. If $L_1 \,\mathrm{I}\, x \,\mathrm{I}\, L_2$, $L_1 \neq L_2$, $M_1 \,\not\mathrm{I}\, x \,\not\mathrm{I}\, M_2$, and if L_i is concurrent with M_j for all $i, j \in \{1, 2\}$, then M_1 is concurrent with M_2.

The only known dual net $\mathcal{N}^*$ which is not a dual affine plane and which satisfies the Axiom of Veblen is the dual net H_q^n, $n > 2$, which is constructed as follows: the points of H_q^n are the points of $\mathrm{PG}(n, q)$ not in a given subspace $\mathrm{PG}(n-2, q) \subset \mathrm{PG}(n, q)$, the lines of H_q^n are the lines of $\mathrm{PG}(n, q)$ which have no point in common with $\mathrm{PG}(n-2, q)$, the incidence in H_q^n is the natural one. By the following theorem these dual nets H_q^n are characterized by the Axiom of Veblen.

THEOREM 2.2 (Thas and De Clerck [1977]). *Let $\mathcal{N}^*$ be a dual net with $s+1$ points on any line and $t+1$ lines through any point, where $t+1>s$. If $\mathcal{N}^*$ satisfies the Axiom of Veblen, then $\mathcal{N}^* \cong H_q^n$ with $n>2$ (hence $s=q$ and $t+1=q^{n-1}$).*

3. TRANSLATION GENERALIZED QUADRANGLES, PROPERTY (G), AND THE AXIOM OF VEBLEN

Let $\mathcal{S}=(P, B, \mathrm{I})$ be a GQ of order (s, t), $s \neq 1$, $t \neq 1$. A collineation θ of $\mathcal{S}$ is an *elation* about the point p if $\theta=\mathrm{id}$ or if θ fixes all lines incident with p and fixes no point of $P-p^{\perp}$. If there is a group H of elations about p acting regularly on $P-p^{\perp}$, we say $\mathcal{S}$ is an *elation generalized quadrangle* (EGQ) with *elation group* H and *base point* p. Briefly, we say that $(\mathcal{S}^{(p)}, H)$ or $\mathcal{S}^{(p)}$ is an EGQ. If the group H is abelian, then we say that the EGQ $(\mathcal{S}^{(p)}, H)$ is a *translation generalized quadrangle*. For any TGQ $\mathcal{S}^{(p)}$ the point p is coregular, that is, each line incident with p is regular. Hence the parameters s and t of a TGQ satisfy $s \leqslant t$; see 8.2 of FGQ. Also, by 8.5.2 of FGQ, for any TGQ with $s \neq t$ we have $s=q^a$ and $t=q^{a+1}$, with q a prime power and a an odd integer; if s (or t) is even then by 8.6.1(iv) of FGQ either $s=t$ or $s^2=t$.

In $\mathrm{PG}(2n+m-1, q)$ consider a set $O(n, m, q)$ of q^m+1 $(n-1)$-dimensional subspaces $\mathrm{PG}^{(0)}(n-1, q), \mathrm{PG}^{(1)}(n-1, q), \ldots, \mathrm{PG}^{(q^m)}(n-1, q)$, every three of which generate a $\mathrm{PG}(3n-1, q)$ and such that each element $\mathrm{PG}^{(i)}(n-1, q)$ of $O(n, m, q)$ is contained in a $\mathrm{PG}^{(i)}(n+m-1, q)$ having no point in common with any $\mathrm{PG}^{(j)}(n-1, q)$ for $j \neq i$. It is easy to check that $\mathrm{PG}^{(i)}(n+m-1, q)$ is uniquely determined, $i=0, 1, \ldots, q^m$. The space $\mathrm{PG}^{(i)}(n+m-1, q)$ is called the *tangent space* of $O(n, m, q)$ at $\mathrm{PG}^{(i)}(n-1, q)$. For $n=m$ such a set $O(n, n, q)$ is called a *generalized oval* or an $[n-1]$-*oval* of $\mathrm{PG}(3n-1, q)$; a generalized oval of $\mathrm{PG}(2, q)$ is just an oval of $\mathrm{PG}(2, q)$. For $n \neq m$ such a set $O(n, m, q)$ is called a *generalized ovoid* or an $[n-1]$-*ovoid* or an *egg* of $\mathrm{PG}(2n+m-1, q)$; a $[0]$-ovoid of $\mathrm{PG}(3, q)$ is just an ovoid of $\mathrm{PG}(3, q)$.

Now embed $\mathrm{PG}(2n+m-1, q)$ in a $\mathrm{PG}(2n+m, q)$, and construct a point-line geometry $T(n, m, q)$ as follows.

Points are of three types :

(i) the points of $\mathrm{PG}(2n+m, q)-\mathrm{PG}(2n+m-1, q)$;

(ii) the $(n+m)$-dimensional subspaces of $\mathrm{PG}(2n+m, q)$ which intersect $\mathrm{PG}(2n+m-1, q)$ in one of the $\mathrm{PG}^{(i)}(n+m-1, q)$;

(iii) the symbol (∞).

Lines are of two types:

(a) the n-dimensional subspaces of $\mathrm{PG}(2n+m,q)$ which intersect $\mathrm{PG}(2n+m-1,q)$ in a $\mathrm{PG}^{(i)}(n-1,q)$;

(b) the elements of $O(n,m,q)$.

Incidence in $T(n,m,q)$ is defined as follows. A point of type (i) is incident only with lines of type (a); here the incidence is that of $\mathrm{PG}(2n+m,q)$. A point of type (ii) is incident with all lines of type (a) contained in it and with the unique element of $O(n,m,q)$ contained in it. The point (∞) is incident with no line of type (a) and with all lines of type (b).

THEOREM 3.1 (8.7.1 of Payne and Thas [1984]). *$T(n,m,q)$ is a* TGQ *of order (q^n,q^m) with base point (∞). Conversely, every* TGQ *is isomorphic to a $T(n,m,q)$. It follows that the theory of* TGQ *is equivalent to the theory of the sets $O(n,m,q)$.*

COROLLARY 3.2. *The following hold for any $O(n,m,q)$:*

(i) *$n=m$ or $n(a+1)=ma$ with a odd;*

(ii) *if q is even, then $n=m$ or $m=2n$.*

Let $O(n,2n,q)$ be an egg of $\mathrm{PG}(4n-1,q)$. We say that $O(n,2n,q)$ is *good at the element* $\mathrm{PG}^{(i)}(n-1,q)$ of $O(n,2n,q)$ if any $\mathrm{PG}(3n-1,q)$ containing $\mathrm{PG}^{(i)}(n-1,q)$ and at least two other elements of $O(n,2n,q)$, contains exactly q^n+1 elements of $O(n,2n,q)$. In such a case the corresponding TGQ $T(n,2n,q)$ contains at least $q^{3n}+q^{2n}$ translation subquadrangles of order q^n (see Thas [1994]).

THEOREM 3.3 (Thas and Van Maldeghem [1995]). *Let $\mathscr{S}^{(p)}$ be a* TGQ *of order (s,s^2), $s\neq 1$, with base point p. Then the dual net $\mathscr{N}_L^*$ defined by the regular line L, with $p\,\mathrm{I}\,L$, satisfies the Axiom of Veblen if and only if the egg $O(n,2n,q)$ which corresponds to $\mathscr{S}^{(p)}$ is good at its element* $\mathrm{PG}^{(i)}(n-1,q)$ *which corresponds to L.*

Let $O=O(n,2n,q)$ be an egg in $\mathrm{PG}(4n-1,q)$. By 8.7.2 of FGQ the $q^{2n}+1$ tangent spaces of O form an $O^*=O^*(n,2n,q)$ in the dual space of $\mathrm{PG}(4n-1,q)$. So in addition to $T(n,2n,q)=T(O)$ there arises a TGQ $T(O^*)$ with the same parameters. The TGQ $T(O^*)$ is called the *translation dual* of the TGQ $T(O)$. Examples are known for which $T(O)\cong T(O^*)$, and examples are known for which $T(O)\not\cong T(O^*)$; see Thas [1994].

Let $\mathscr{S}=(P,B,\mathrm{I})$ be a GQ of order (s,s^2), $s\neq 1$. Let x_1, y_1 be distinct collinear points. We say that the pair $\{x_1,y_1\}$ has *Property* (G), or that $\mathscr{S}$

has *Property* (G) *at* $\{x_1, y_1\}$, if every triple $\{x_1, x_2, x_3\}$ of points, with x_1, x_2, x_3 pairwise noncollinear and $y_1 \in \{x_1, x_2, x_3\}^{\perp}$, is 3-regular; for the definition of 3-regularity see 1.3 of FGQ. The GQ $\mathcal{S}$ has *Property* (G) *at the line* L, or the line L has *Property* (G), if each pair of points $\{x, y\}, x \neq y$ and $x \mathrm{I} L \mathrm{I} y$, has Property (G). If (x, L) is a flag, that is, if $x \mathrm{I} L$, then we say that $\mathcal{S}$ has *Property* (G) *at* (x, L), or that (x, L) has *Property* (G), if every pair $\{x, y\}, x \neq y$ and $y \mathrm{I} L$, has Property (G). Property (G) was introduced in Payne [1989] in connection with generalized quadrangles of order (q^2, q) arising from flocks of quadratic cones in $\mathrm{PG}(3, q)$.

THEOREM 3.4 (Thas and Van Maldeghem [1995]). *Let* $\mathcal{S} = (P, B, \mathrm{I})$ *be a* GQ *of order* (s^2, s), *s even, satisfying Property* (G) *at the point x. Then x is regular in* $\mathcal{S}$ *and the dual net* $\mathcal{N}_x^*$ *defined by x satisfies the Axiom of Veblen. Consequently* $\mathcal{N}_x^* \cong H_s^3$.

THEOREM 3.5 (Thas [1994]). *A* TGQ $T(n, 2n, q) = T(O)$ *satisfies Property* (G) *at* $\{(\infty), \bar{\zeta}\}$, *with* $\bar{\zeta}$ *a point of type* (ii) *incident with the line* ζ *of type* (b) (*or, equivalently, at the flag* $((\infty), \zeta)$) *if and only if for any two elements* ζ_i, ζ_j $(i \neq j)$ *of* $O(n, 2n, q) - \{\zeta\}$ *the* $(n-1)$*-dimensional space* $\mathrm{PG}(n-1, q) = \tau \cap \tau_i \cap \tau_j$, *with* τ, τ_i, τ_j *the respective tangent spaces of* $O(n, 2n, q)$ *at* ζ, ζ_i, ζ_j, *is contained in* $q^n + 1$ *tangent spaces of* $O(n, 2n, q)$.

THEOREM 3.6 (Thas and Van Maldeghem [1995]). *Let* $\mathcal{S}^{(p)}$ *be a* TGQ *of order* (s, s^2), $s \neq 1$, *with base point p. Then the dual net* $\mathcal{N}_L^*$ *defined by the regular line L, with* $p \mathrm{I} L$, *satisfies the Axiom of Veblen if and only if the translation dual* $\mathcal{S}'^{(p')}$ *of* $\mathcal{S}^{(p)}$ *satisfies Property* (G) *at the flag* (p', L'), *where* L' *corresponds to L; in the even case* $\mathcal{N}_L^*$ *satisfies the Axiom of Veblen if and only if* $\mathcal{S}^{(p)}$ *satisfies Property* (G) *at the flag* (p, L).

THEOREM 3.7. *Let* $\mathcal{S}^{(p)}$ *be a* TGQ *of order* (s, s^2), *s odd and* $s \neq 1$, *with base point p. If the dual net* $\mathcal{N}_L^*$ *defined by the regular line L, with* $p \mathrm{I} L$, *satisfies the Axiom of Veblen, then* $\mathcal{S}^{(p)}$ *contains at least* $s^3 + s^2$ *classical subquadrangles* $Q(4, s)$.

Proof. This follows immediately from Theorem 3.6 and Theorem 4.3.4 of Thas [1994]. ▮

4. FLOCK GENERALIZED QUADRANGLES AND THE AXIOM OF VEBLEN

Let F be a flock of the quadratic cone K with vertex x of $\mathrm{PG}(3, q)$, that is, a partition of $K - \{x\}$ into q disjoint irreducible conics. Then, by Thas

[1987], with F there corresponds a GQ $\mathcal{S}(F)$ of order (q^2, q). In Payne [1989] it was shown that $\mathcal{S}(F)$ satisfies Property (G) at its point (∞).

THEOREM 4.1 (Thas and Van Maldeghem [1995]). *For any* GQ $\mathcal{S}(F)$ *of order* (q^2, q) *arising from a flock* F, *the point* (∞) *is regular. If* q *is even, then the dual net* $\mathcal{N}^*_{(\infty)}$ *always satisfies the Axiom of Veblen and so* $\mathcal{N}^*_{(\infty)} \cong H^3_q$. *If* q *is odd, then the dual net* $\mathcal{N}^*_{(\infty)}$ *satisfies the Axiom of Veblen if and only if* F *is a Kantor flock.*

COROLLARY 4.2. *Suppose that the* TGQ $T(O)$, *with* $O = O(n, 2n, q)$ *and* q *odd, is a flock* GQ $\mathcal{S}(F)$ *where the point* (∞) *of* $\mathcal{S}(F)$ *corresponds to the line* ζ *of type* (b) *of* $T(O)$. *Then* $T(O)$ *is good at the element* ζ *if and only if* F *is a Kantor flock.*

Proof. This follows immediately from Theorems 4.1 and 3.3. ∎

5. VERONESE VARIETIES

In Section 6 we shall see that there is a strong connection between TGQ satisfying Property (G) (or, equivalently, satisfying the Axiom of Veblen) and the Veronesean $\mathcal{V}_2^4$ of all conics of PG(2, q). So we include a short section on Veronese varieties; a good reference is Chapter 25 of Hirschfeld and Thas [1991].

The *Veronese variety* of all quadrics of PG(n, K), $n \geqslant 1$ and K any commutative field, is the variety

$$\mathcal{V} = \{(x_0^2, x_1^2, ..., x_n^2, x_0x_1, x_0x_2, ..., x_0x_n, x_1x_2, ..., x_1x_n, ..., x_{n-1}x_n) \parallel \\ (x_0, x_1, ..., x_n) \text{ is a point of } \mathrm{PG}(n, K)\}$$

of PG(N, K) with $N = n(n+3)/2$. The variety $\mathcal{V}$ has dimension n and order 2^n; for $\mathcal{V}$ we also write $\mathcal{V}_n$ or $\mathcal{V}_n^{2^n}$. It is also called the *Veronesean* of quadrics of PG(n, K), or simply the *quadric Veronesean* of PG(n, K). It can be shown that the quadric Veronesean is absolutely irreducible and non-singular.

Let PG(n, K) consist of all points

$$(y_{00}, y_{11}, ..., y_{nn}, y_{01}, y_{02}, ..., y_{0n}, y_{12}, ..., y_{1n}, ..., y_{n-1,n});$$

for y_{ij} we also write y_{ji}. Let ζ: PG(n, K) $\to$ PG(N, K), with $N = n(n+3)/2$ and $n \geqslant 1$, be defined by

$$(x_0, x_1, ..., x_n) \mapsto (y_{00}, y_{11}, ..., y_{n-1,n}),$$

with $y_{ij} = x_i x_j$. Then ζ is a bijection of $\mathrm{PG}(n, K)$ onto the quadric Veronesean $\mathcal{V}$ of $\mathrm{PG}(n, K)$. It then follows that the variety $\mathcal{V}$ is rational.

THEOREM 5.1. *The quadrics of* $\mathrm{PG}(n, K)$ *are mapped by* ζ *onto all hyperplane sections of* $\mathcal{V}$.

COROLLARY 5.2. *No hyperplane of* $\mathrm{PG}(N, K)$ *contains the quadric Veronesean* $\mathcal{V}$.

THEOREM 5.3. *Any two distinct points of* $\mathcal{V}$ *are contained in a unique irreducible conic of* $\mathcal{V}$.

If $K = \mathrm{GF}(q)$, then it is clear that $\mathcal{V}_n$ contains $\theta(n) = q^n + q^{n-1} + \cdots + q + 1$ points. As no three points of $\mathcal{V}$ are collinear we have the following theorem.

THEOREM 5.4. *The quadric Veronesean* $\mathcal{V}_n$ *is a* $\theta(n)$*-cap of* $\mathrm{PG}(N, q)$, $N = n(n+3)/2$.

Let $n = 2$. Then $\mathcal{V}$ is a surface of order 4 in $\mathrm{PG}(5, K)$. Apart from the conic, the variety $\mathcal{V}_2^4$ is the quadric Veronesean which is most studied and characterized. Assume also that $K = \mathrm{GF}(q)$. To the conics (irreducible or not) of $\mathrm{PG}(2, q)$ there correspond all hyperplane sections of $\mathcal{V}_2^4$. The hyperplane is uniquely determined by the conic if and only if the latter is not a single point. If the conic of $\mathrm{PG}(2, q)$ is one line, then the corresponding hyperplane of $\mathrm{PG}(5, q)$ meets $\mathcal{V}_2^4$ in an irreducible conic; the surface $\mathcal{V}_2^4$ contains no other irreducible conics. It follows that $\mathcal{V}_2^4$ contains exactly $q^2 + q + 1$ irreducible conics, that any two distinct points of $\mathcal{V}_2^4$ are contained in a unique irreducible conic, and that any two distinct irreducible conics on $\mathcal{V}_2^4$ meet in a unique point. If the conic $\mathcal{C}$ of $\mathrm{PG}(2, q)$ is two distinct lines, then the corresponding hyperplane $\mathrm{PG}(4, q)$ meets $\mathcal{V}_2^4$ in two irreducible conics with exactly one point in common; if $\mathcal{C}$ is irreducible, then $\mathrm{PG}(4, q)$ meets $\mathcal{V}_2^4$ in a rational quartic curve. The planes of $\mathrm{PG}(5, q)$ which meet $\mathcal{V}_2^4$ is an irreducible conic are called the *conic planes* of $\mathcal{V}_2^4$.

THEOREM 5.5. *Any two distinct conic planes* π *and* π' *of* $\mathcal{V}_2^4$ *have exactly one point in common, and this common point belongs to* $\mathcal{V}_2^4$.

The tangent lines of the irreducible conics of $\mathcal{V}_2^4$ are called the *tangents* or *tangent lines* of $\mathcal{V}_2^4$. Since no point of the surface $\mathcal{V}_2^4$ is singular, all tangent lines of $\mathcal{V}_2^4$ at the point p of $\mathcal{V}_2^4$ are contained in a plane $\pi(p)$. This plane $\pi(p)$ is called the *tangent plane* of $\mathcal{V}_2^4$ at p. Since p is contained in exactly $q+1$ irreducible conics of $\mathcal{V}_2^4$ and since no two conic planes through p have a line in common, the tangent plane $\pi(p)$ is the union of the $q+1$ tangent lines of $\mathcal{V}_2^4$ through p. Also $\mathcal{V}_2^4 \cap \pi(p) = \{p\}$.

THEOREM 5.6. *For any two distinct points p_1 and p_2 of $\mathcal{V}_2^4$, the tangent planes $\pi(p_1)$ and $\pi(p_2)$ have exactly one point in common.*

THEOREM 5.7. *Suppose that q is odd. Then* PG(5, q) *admits a polarity which maps the set of all conic planes of $\mathcal{V}_2^4$ onto the set of all tangent planes of $\mathcal{V}_2^4$.*

COROLLARY 5.8. *Suppose that q is odd. Then for any three distinct points p_1, p_2, p_3 of $\mathcal{V}_2^4$, the intersection $\pi(p_1) \cap \pi(p_2) \cap \pi(p_3)$ of the tangent planes is empty.*

COROLLARY 5.9. *Suppose that q is odd. Then each point of* PG(5, q) $-$ $\mathcal{V}_2^4$ *is on* 0 *or* 2 *tangent planes of $\mathcal{V}_2^4$.*

THEOREM 5.10. *Suppose that* PG(4, q) *is a hyperplane for which* PG(4, q) $\cap \mathcal{V}_2^4$ *is a non-singular conic $\mathscr{C}$. Then* PG(4, q) *contains exactly $q+1$ tangent planes $\pi(p_0), \pi(p_1), ..., \pi(p_q)$. Also $\mathscr{C} = \{p_0, p_1, ..., p_q\}$, and if π is the conic plane determined by $\mathscr{C}$, then the intersections $\pi \cap \pi(p_0)$, $\pi \cap \pi(p_1)$, ..., $\pi \cap \pi(p_q)$ are the tangent lines of the conic $\mathscr{C}$.*

Proof. Let $\mathscr{C} = \{p_0, p_1, ..., p_q\}$ and consider the tangent plane $\pi(p_i)$. Then the plane $\pi(p_i)$ contains the tangent line of $\mathscr{C}$ at p_i. Let π_i be the threedimensional space containing $\pi(p_i)$ and the plane π of $\mathscr{C}$. Considering the hyperplanes containing π_i we see that q of them intersect $\mathcal{V}_2^4$ in two non-singular conics through p_i (one of which is $\mathscr{C}$) while the remaining one intersects $\mathcal{V}_2^4$ in $\mathscr{C}$. So this last hyperplane is the hyperplane PG(4, q) of the statement of the theorem. Consequently PG(4, q) contains the tangent planes $\pi(p_0), \pi(p_1), ..., \pi(p_q)$. If $\pi(p)$ is any tangent plane in PG(4, q), then, as PG(4, q) $\cap \mathcal{V}_2^4 = \mathscr{C}$, it is clear that $p \in \mathscr{C}$. ∎

THEOREM 5.11. *Suppose that $\mathscr{C}$ is a non-singular conic on $\mathcal{V}_2^4$, that π is the plane of $\mathscr{C}$ and that $p' \in \mathcal{V}_2^4 - \mathscr{C}$. Then $\pi(p') \cap \pi = \emptyset$.*

Proof. Assume, by way of contradiction, that $r \in \pi(p') \cap \pi$. Then rp' is the tangent line at p' of some non-singular conic $\mathscr{C}'$ on $\mathcal{V}_2^4$. Hence the plane π' of $\mathscr{C}'$ and the conic plane π have a point $r \notin \mathcal{V}_2^4$ in common, contradicting Theorem 5.5. ∎

THEOREM 5.12. *Suppose that q is odd and that* PG(4, q) *is a hyperplane for which* PG(4, q) $\cap \mathcal{V}_2^4$ *is a non-singular conic $\mathscr{C}$. If $p' \in \mathcal{V}_2^4 - \mathscr{C}$, then the* *line* $\pi(p') \cap \mathrm{PG}(4, q)$ *intersects* $\pi(p_i)$, $p_i \in \mathscr{C}$, *in a point* r_i, *where* r_i *is not in the plane of* $\mathscr{C}$.

Proof. Let r be a point of $\pi(p_i) - \pi$, with $p_i \in \mathscr{C}$ and π the plane of $\mathscr{C}$. By Corollary 5.9 r is contained in 2 tangent planes $\pi(p_i)$ and $\pi(u)$. Then u is not on $\mathscr{C}$ as otherwise the unique common point r of $\pi(p_i)$ and $\pi(u)$ would be in π. So $u \in \mathscr{V}_2^4 - \mathscr{C}$. By Corollary 5.8 $r \mapsto u$ defines a bijection of $\pi(p_i) - \pi$ onto $\mathscr{V}_2^4 - \mathscr{C}$. Hence the line $\pi(p') \cap \mathrm{PG}(4, q)$ in the statement of the theorem intersects the plane $\pi(p_i)$ in a point $r_i \notin \pi$. ∎

6. TRANSLATION GENERALIZED QUADRANGLES OF ORDER (s, s^2), $s \neq 1$ AND s ODD, SATISFYING THE AXIOM OF VEBLEN AND VERONESE SURFACES

Consider a TGQ $T(n, 2n, q) = T(O)$, with $O = O(n, 2n, q)$. The egg $O(n, 2n, q)$ is good at its element $\mathrm{PG}(n-1, q) = L$ if and only if the dual net $\mathscr{N}_L^*$ defined by the regular line L satisfies the Axiom of Veblen if and only if the translation dual $T(O^*)$ of $T(O)$ satisfies Property (G) at the flag $((\infty), \tau)$, with τ the tangent space of O at $\mathrm{PG}(n-1, q)$; see Theorems 3.3 and 3.6.

Notation. If $\mathrm{GF}(q^h)$ is an extension of $\mathrm{GF}(q)$, then the corresponding extension of $\mathrm{PG}(m, q)$ will be denoted by $\mathrm{PG}(m, q^h)$ or $\overline{\mathrm{PG}(m, q)}$.

THEOREM 6.1. *Consider a non-classical* TGQ $T(n, 2n, q) = T(O)$, *q odd, with* $O = O(n, 2n, q) = \{\mathrm{PG}(n-1, q), \mathrm{PG}^{(1)}(n-1, q), \mathrm{PG}^{(2)}(n-1, q), ..., \mathrm{PG}^{(q^{2n})}(n-1, q)\}$ *an egg in* $\mathrm{PG}(4n-1, q)$. *If O is good at* $\mathrm{PG}(n-1, q)$, *then one of the following two cases occurs*

(a) *There exists a* $\mathrm{PG}(4, q^n)$ *in* $\mathrm{PG}(4n-1, q^n)$ *which intersects* $\mathrm{PG}(n-1, q^n)$ *in a line M and which has exactly one point r_i in common with any space* $\mathrm{PG}^{(i)}(n-1, q^n)$, $i = 1, 2, ..., q^{2n}$.

(b) *We are not in Case* (a) *and there exists a* $\mathrm{PG}(5, q^n)$ *in* $\mathrm{PG}(4n-1, q^n)$ *which intersects* $\mathrm{PG}(n-1, q^n)$ *in a plane μ and which has exactly one point r_i in common with any space* $\mathrm{PG}^{(i)}(n-1, q^n)$, $i = 1, 2, ..., q^n$.

Proof. Consider a non-classical TGQ $T(n, 2n, q) = T(O)$, q odd, with $O = O(n, 2n, q) = \{\mathrm{PG}(n-1, q), \mathrm{PG}^{(1)}(n-1, q), \mathrm{PG}^{(2)}(n-1, q), ..., \mathrm{PG}^{(q^{2n})}(n-1, q)\}$ an egg in $\mathrm{PG}(4n-1, q)$, and assume that O is good at $\mathrm{PG}(n-1, q)$. Let $\mathrm{PG}(3n-1, q)$ be a subspace of $\mathrm{PG}(4n-1, q)$ which is skew to $\mathrm{PG}(n-1, q)$, and let $\langle \mathrm{PG}(n-1, q), \mathrm{PG}^{(i)}(n-1, q) \rangle \cap \mathrm{PG}(3n-1, q) = \pi_i$, $i = 1, 2, ..., q^{2n}$. The tangent space τ of O at $\mathrm{PG}(n-1, q)$ intersects $\mathrm{PG}(3n-1, q)$ in a $(2n-1)$-dimensional space π. Clearly $\mathrm{PG}(3n-1, q) = \pi \cup \pi_1 \cup \pi_2 \cup \cdots \cup \pi_{q^{2n}}$.

By the proof of Theorem 4.3.4 in Thas [1994], in $PG(3n-1, q)$ there are n planes $\eta_1, \eta_2, ..., \eta_n$ over $GF(q^n)$, which are conjugate with respect to the nth extension $GF(q^n)$ of $GF(q)$ and which generate $PG(3n-1, q)$, such that each plane η_j has just one point in common with $\bar{\pi}_i$ and such that η_j has just one line in common with $\bar{\pi}, j=1, 2, ..., n$ and $i=1, 2, ..., q^{2n}$. Let $\langle PG(n-1, q^n), \eta_j \rangle = \rho_j, j=1, 2, ..., n$; then ρ_j is $(n+2)$-dimensional. The space $\bar{\pi}_i$ has just one point in common with η_1, and so $PG^{(i)}(n-1, q^n)$ has exactly one point r_i in common with ρ_1, with $i=1, 2, ..., q^{2n}$. In $PG(n-1, q)$ we choose a $PG(n-2, q)$, and in ρ_1 we choose a $PG(3, q^n) = \Phi_1$ skew to $PG(n-2, q^n)$. Then Φ_1 and $PG(n-1, q^n)$ have exactly one point s_1 in common. Now we project the point r_i from $PG(n-2, q^n)$ onto Φ_1, and we obtain the point $r'_i, i=1, 2, ..., q^{2n}$. Clearly $r'_i \neq s_1$ and if $r'_i = r'_j, i \neq j$, then $r_i r_j$ has a point in common with $PG(n-1, q^n)$, a contradiction as O is an egg. Hence the set $T_1 = \{s_1, r'_1, r'_2, ..., r'_{q^{2n}}\}$ contains $q^{2n}+1$ distinct points. If $r'_i r'_j, i \neq j$, contains s_1, then $\langle PG(n-1, q^n), r_i, r_j \rangle$ is n-dimensional, a contradiction. Now assume, by way of contradiction, that r'_i, r'_j, r'_k are distinct collinear points of $T_1 - \{s_1\}$. Then the plane $r_i r_j r_k = \xi$ has at least one point l in common with $PG(n-2, q^n)$. Without loss of generality we may assume that $l \notin r_i r_j$, so $\xi = l r_i r_j$. Any space $PG^{(u)}(n-1, q^n)$ is generated by r_u and its conjugates with respect to $GF(q^n)$, hence the space $\psi = \langle PG^{(i)}(n-1, q^n), PG^{(j)}(n-1, q^n), PG^{(k)}(n-1, q^n) \rangle$ is generated by the plane $\xi = r_i r_j r_k = l r_i r_j$ and its conjugates. As l belongs to $PG(n-2, q^n) = \overline{PG(n-2, q)}$, the space ψ is generated by a subspace of $PG(n-2, q^n)$ and $\langle PG^{(i)}(n-1, q^n), PG^{(j)}(n-1, q^n) \rangle$. Consequently Ψ is at most $(3n-2)$-dimensional, a contradiction as O is an egg. So we conclude that no three points of T_1 are collinear, that is, T_1 is an ovoid of the space Φ_1.

The extension $\bar{\tau}$ of the tangent space τ of O at $PG(n-1, q)$ intersects ρ_1 in the $(n+1)$-dimensional space $PG(n+1, q^n) = \langle \bar{\pi} \cap \eta_1, PG(n-1, q^n) \rangle$. Clearly $PG(n+1, q^n) \cap \Phi_1$ is a plane θ_1. This plane θ_1 contains s_1 but no one of the points $r'_1, r'_2, ..., r'_{q^{2n}}$. Hence θ_1 is the tangent plane of the ovoid T_1 at s_1.

Since q^n is odd, by Barlotti [1955] and Panella [1955], the ovoid T_1 is an elliptic quadric of Φ_1.

Each space $\langle PG(n-1, q^n), r_i, r_j \rangle, i \neq j$, contains exactly q^n points of $\eta_1 - \bar{\pi}$; so $\langle PG(n-1, q^n), r_i, r_j \rangle$ contains exactly q^n of the points $r_1, r_2, ..., r_{q^{2n}}$, say $r_{l_1}, r_{l_2}, ..., r_{l_{q^n}}$. Let $\langle PG(n-1, q), PG^{(i)}(n-1, q), PG^{(j)}(n-1, q) \rangle \cap PG(3n-1, q) = PG(2n-1, q)$. Then $\eta_j \cap PG(2n-1, q^n)$ is a line $L_j, j=1, 2, ..., n$. Clearly the space $\langle PG(n-1, q^n), r_i, r_j \rangle$ intersects $PG(2n-1, q^n)$ in the line L_1. By Theorem 4.3.4 of Thas [1994] the q^n+1 spaces $PG(n-1, q), PG^{(l_1)}(n-1, q), PG^{(l_2)}(n-1, q), ...$ define q^n TGQ $T(n, n, q)$ isomorphic to the classical GQ $Q(4, q^n)$; we remark that $PG(n-1, q), PG^{(l_1)}(n-1, q), ...$ are the elements of O in $\langle PG(n-1, q^n),$

$L_1, L_2, ..., L_n\rangle$. Now we show that $r_{l_1}, r_{l_2}, ...$ together with some point of $\mathrm{PG}(n-1, q^n)$, form a conic $\mathscr{C}$ over $\mathrm{GF}(q^n)$.

Let us consider one of the q^n TGQ $T(n, n, q)$. Let u_1, u_2 be non-collinear points of $T(n, n, q)$, with $u_1 \not\sim (\infty) \not\sim u_2$. Further consider all grids of $T(n, n, q)$ containing u_1, u_2 and two lines of type (b) of $T(n, n, q)$. As $T(n, n, q) \cong Q(4, q^n) \cong T(1, 1, q^n)$ all these grids have q^n common points of type (i). For the sake of convenience we now simplify our notation : $\{\mathrm{PG}(n-1, q), \ \mathrm{PG}^{(l_1)}(n-1, q), ...\} = \{\alpha_0, \alpha_1, ..., \alpha_{q^n}\}$. Let $\beta = \alpha_i\alpha_j \cap \alpha_u\alpha_k$, with all indices distinct. We now choose u_1, u_2 in such a way that the line u_1u_2 (in the projective space containing $T(n, n, q)$) contains a point g of β. Then all grids of $T(n, n, q)$ containing u_1, u_2 and two lines of type (b) of $T(n, n, q)$ contain all q^n points of $\langle \alpha_i, \alpha_j, u_1\rangle \cap \langle \alpha_u, \alpha_k, u_1\rangle$ not in $\mathrm{PG}(4n-1, q)$; clearly $\langle \alpha_i, \alpha_j, u_1\rangle \cap \langle \alpha_u, \alpha_k, u_1\rangle \cap \mathrm{PG}(4n-1, q) = \beta$. It easily follows that if $\alpha_a\alpha_b, a \neq b$, contains g, then it also contains β. If $|\{(\infty), u_1, u_2\}^{\perp}| = 2$ in $T(n, n, q)$, then β belongs to $(q^n-1)/2$ of the spaces $\alpha_a\alpha_b$ (in such a case g belongs to 2 tangent spaces of the $[n-1]$-oval $O(n, n, q) = \{\alpha_0, \alpha_1, ..., \alpha_{q^n}\}$); if $|\{(\infty), u_1, u_2\}^{\perp}| = 0$ in $T(n, n, q)$, then β belongs to $(q^n+1)/2$ of the spaces $\alpha_a\alpha_b$ (in such a case g belongs to 0 tangent spaces of the $[n-1]$-oval $O(n, n, q)$). Assume that g belongs to 2 tangent spaces α_{c_1} and α_{c_2} of $O(n, n, q)$, and let $g' \in \beta$. Interchanging roles of g and g', we see that $\langle g', \alpha_{c_1}\rangle$ and $\langle g', \alpha_{c_2}\rangle$ are contained in the tangent spaces of $O(n, n, q)$ at respectively α_{c_1} and α_{c_2}. Consequently the tangent spaces of $O(n, n, q)$ at α_{c_1} and α_{c_2} contain β. It now follows that all spaces $\alpha_i\alpha_j \cap \alpha_u\alpha_k$ with all indices distinct, together with the elements of $O(n, n, q)$, constitute a $(n-1)$-spread S in the space $\mathrm{PG}(3n-1, q)$ of $O(n, n, q)$. Let β_1, β_2 be distinct elements of S. Let y be a point of type (i) in $T(n, n, q)$. Then the spaces $\langle y, \beta_1\rangle$ and $\langle y, \beta_2\rangle$ correspond to lines M_1 and M_2 in the space $\mathrm{PG}(3, q^n)$ of the GQ $T(1, 1, q^n) \cong T(n, n, q)$. Let $\{y'\} = M_1 \cap M_2$ and consider all lines $M_1, M_2, ..., M_{q^n+1}$ through y' in the plane $\langle M_1, M_2\rangle$. In the space $\mathrm{PG}(3n, q)$ of $T(n, n, q)$, with these q^n+1 lines there correspond q^n+1 n-dimensional spaces $\langle y, \beta_1\rangle = \gamma_1, \langle y, \beta_2\rangle = \gamma_2, ..., \langle y, \beta_{q^n+1}\rangle = \gamma_{q^n+1}$, with $\beta_1, \beta_2, ..., \beta_{q^n+1} \in S$. Let $\gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_{q^n+1} = W$; then $|W| = (q^{2n+1}-1)/(q-1)$. Let $w_1, w_2 \in W$, $w_1 \neq w_2$, with $w_1, w_2 \notin \mathrm{PG}(3n-1, q)$. Considering again $T(1, 1, q^n)$, we see that there is a β_i, $i \in \{1, 2, ..., q^n+1\}$, such that w_1w_2 has a point in common with β_i and $\langle w_1, \beta_i\rangle \subset W$ (two lines in the plane $\langle M_1, M_2\rangle$, but not in the plane $\mathrm{PG}(2, q^n)$ of the oval $O(1, 1, q^n)$, which contain a common point of $\mathrm{PG}(2, q^n)$ define the same element of S). So $w_1w_2 \subset W$. Now it easily follows that W is a $2n$-dimensional space, hence $\beta_1 \cup \beta_2 \cup \cdots \cup \beta_{q^n+1}$ is a $(2n-1)$-dimensional space. We conclude that in all spaces $\beta_1\beta_2$, with β_1 and β_2 distinct elements of S, a $(n-1)$-spread is induced by S. Consequently the spread S is geometric; see 8.2 of Thas [1995]. Hence by a theorem of Segre, see Theorem 2 in 8.2 of Thas [1995], in $\mathrm{PG}(3n-1, q)$

there are n planes $\delta_1, \delta_2, ..., \delta_n$ over $GF(q^n)$, which are conjugate with respect to the nth extension $GF(q^n)$ of $GF(q)$ and which generate $PG(3n-1, q)$, such that each plane δ_i intersects each element $\bar{\beta}$, with $\beta \in S$. In particular each such plane δ_i intersects $\bar{\alpha}_0, \bar{\alpha}_1, ..., \bar{\alpha}_{q^n}$. Projecting from $PG(n-1, q^n)$ onto $PG(2n-1, q^n)$, the planes $\delta_1, \delta_2, ..., \delta_n$ are projected onto the lines $L_1, L_2, ..., L_n$; notations can be chosen in such a way that δ_i is projected onto L_i, $i = 1, 2, ..., n$. The space $\langle PG(n-1, q^n), L_1 \rangle$ intersects $PG^{(l_1)}(n-1, q^n)$, $PG^{(l_2)}(n-1, q^n), ...$ in the points common to $PG^{(l_1)}(n-1, q^n)$, $PG^{(l_2)}(n-1, q^n), ...$ and the plane δ_1. It follows that $r_{l_1}, r_{l_2}, ...$ belong to δ_1. Now it is clear that the point $\delta_1 \cap PG(n-1, q^n)$ together with the q^n points $r_{l_1}, r_{l_2}, ...$ form a conic $\mathscr{C}$ over $GF(q^n)$.

Now we choose $\Phi_1 = PG(3, q^n)$ in such a way that it contains δ_1, that is, in such a way that it contains $\mathscr{C}$. The point of $\mathscr{C}$ in $PG(n-1, q^n)$ will be denoted by s_1. In $PG(n-1, q)$ we now choose spaces $PG(n-2, q) = \varepsilon$, $PG'(n-2, q) = \varepsilon'$, ... the extensions of which to $GF(q^n)$ do not contain s_1. Projecting all points $r_{l_1}, r_{l_2}, ...$ from the extensions $\bar{\varepsilon}, \overline{\varepsilon'}, ...$ of the resp. spaces $\varepsilon, \varepsilon', ...$ onto Φ_1, we obtain elliptic quadrics $T_1, T'_1, ...$ containing the conic $\mathscr{C}$. The tangent plane of $T_1, T'_1, ...$ at s_1 is the intersection of Φ_1 with the extension $\bar{\tau}$ of the tangent space τ of O at $PG(n+1, q)$. Hence the quadrics $T_1, T'_1, ...$ have in common the tangent plane at s_1.

Consider a point $r_k \notin \mathscr{C}$. If the 3-dimensional space defined by $\mathscr{C}$ and r_k contains a line of $PG(n-1, q^n)$, then by projecting $\mathscr{C} \cup \{r_k\}$ from $PG(n-2, q^n)$ onto Φ_1 there arises a plane (q^n+2)-arc, a contradiction as q is odd. Hence we may choose Φ_1 in such a way that it contains $\mathscr{C}$ and r_k. With the notations of the preceding paragraph, there arise quadrics $T_1, T'_1, ...$ containing the conic $\mathscr{C}$, the point r_k and having at s_1 a common tangent plane.

First, assume that $T_1, T'_1, ...$ all coincide. By way of contradiction, let $r_l \notin T_1$, so $r_l \notin \Phi_1$. By projecting r_l from $\bar{\varepsilon}, \overline{\varepsilon'}, ...$ onto Φ_1, we obtain points of T_1 which all belong to the line $\langle PG(n-1, q^n), r_l \rangle \cap \Phi_1$ through s_1. Hence T_1 contains at least three collinear points, a contradiction. Consequently $r_l \in T_1$, and so $T_1 - \{s_1\}$ is the set of all points r_i, $i = 1, 2, ..., q^{2n}$. It follows that any space $PG^{(i)}(n-1, q^n)$ has a point in common with Φ_1. But then $T(n, 2n, q) \cong T(1, 2, q^n)$, and so $T(n, 2n, q)$ is isomorphic to the classical GQ of order (q^n, q^{2n}), a contradiction. We conclude that the quadrics $T_1, T'_1, ...$ do not all coincide.

Next, assume that Φ_1 contains a point r_l, with $r_l \neq r_k$ and $r_l \notin \mathscr{C}$. Then the quadrics $T_1, T'_1, ...$ all contain $\mathscr{C}, r_l, r_k$ and have at s_1 a common tangent plane. Hence they all belong to a uniquely defined pencil of quadrics in Φ_1. The base $\mathscr{B}$ of the pencil $\mathscr{P}$, that is, the quartic curve common to all elements of $\mathscr{P}$, contains $\mathscr{C}$ as a component, contains r_k, and has s_1 as multiple point. As $\mathscr{P}$ contains elliptic quadrics, $\mathscr{B}$ does not contain a line over $GF(q^n)$ as component; if $\mathscr{B}$ contains a line over $GF(q^{2n})$ through s_1

as component, then also $s_1 r_k$ is a component, a contradiction. Consequently, either $\mathscr{B}$ consists of two non-singular conics $\mathscr{C}, \mathscr{C}'$ over $GF(q^n)$ intersecting in distinct points s_1 and s'_1, or $\mathscr{B}$ consists of two non-singular conics $\mathscr{C}, \mathscr{C}'$ over $\mathrm{GF}(q^n)$ which are mutually tangent at s_1.

Assume that $\mathscr{B}$ consists of two non-singular conics $\mathscr{C}, \mathscr{C}'$ over $\mathrm{GF}(q^n)$ which intersect in distinct points s_1 and s'_1. Let $\langle \bar{\varepsilon}, r_u \rangle$ and $\langle \overline{\varepsilon'}, r_u \rangle$, with $r_u \notin \Phi_1$, intersect Φ_1 in r'_u and r''_u. We choose ε and ε' in such a way that $r'_u \neq r''_u$. Then put $r'_u r_u \cap \mathrm{PG}(n-1, q^n) = \{l'_u\}$ and $r''_u r_u \cap \mathrm{PG}(n-1, q^n) = \{l''_u\}$. The lines $r'_u r''_u$ and $l'_u l''_u$ have a point in common which belongs to Φ_1 and $\mathrm{PG}(n-1, q^n)$. Hence $r'_u r''_u$ contains s_1. Consider a point $c \in \mathscr{C}' - \{s_1, s'_1\}$. Assume c is the projection from $\bar{\varepsilon}$ of a point r_u with $r_u \notin \Phi_1$; so $c = r'_u$. Choose ε' in such a way that $r'_u \neq r''_u$. Then s_1, r'_u, r''_u are collinear. As they all belong to T'_1 we have a contradiction. Hence $\mathscr{C}' - \{s_1\}$ consists of points r_d. It follows that $(\mathscr{C} \cup \mathscr{C}') - \{s_1\}$ is the set of all points r_d in Φ_1.

Next assume that $\mathscr{B}$ consists of two non-singular conics $\mathscr{C}, \mathscr{C}'$ over $\mathrm{GF}(q^n)$ which are mutually tangent at s_1. As in the previous section one shows that $(\mathscr{C} \cup \mathscr{C}') - \{s_1\}$ is the set of all points r_d in Φ_1.

Any two points r_i and r_j, $i \neq j$, define exactly one non-singular conic $\mathscr{C}$ over $\mathrm{GF}(q^n)$. The tangent line U of $\mathscr{C}$ at s_1 is contained in the extension of the tangent space of $O(n, n, q) \subset O(n, 2n, q)$, where $O(n, n, q)$ contains $\mathrm{PG}(n-1, q)$, $\mathrm{PG}^{(i)}(n-1, q)$, $\mathrm{PG}^{(j)}(n-1, q)$, at $\mathrm{PG}(n-1, q)$, so is contained in $\bar{\tau}$. Hence $\langle \mathrm{PG}(n-1, q^n), U \rangle$ intersects $\mathrm{PG}(2n-1, q^n)$ in the point $L_1 \cap \bar{\tau}$. Let V be the set of all conics $\mathscr{C}$ in $\mathrm{PG}(n+2, q^n) = \rho_1$. Any two distinct conics $\mathscr{C}, \mathscr{C}^* \in V$ either have a point of $\rho_1 - \mathrm{PG}(n-1, q^n)$ in common, or do not have a point of $\rho_1 - \mathrm{PG}(n-1, q^n)$ in common in which case $\langle \mathrm{PG}(n-1, q^n), U \rangle = \langle \mathrm{PG}(n-1, q^n), U^* \rangle$, where U is the tangent line of $\mathscr{C}$ at s_1 ($\{s_1\} = \mathscr{C} \cap \mathrm{PG}(n-1, q^n)$) and U^* is the tangent line of $\mathscr{C}^*$ at s_1^* ($\{s_1^*\} = \mathscr{C}^* \cap \mathrm{PG}(n-1, q^n)$).

Consider again the conic $\mathscr{C}'$, where $\mathscr{C}$ and $\mathscr{C}'$ are the components of the base $\mathscr{B}$ of the pencil $\mathscr{P}$ of quadrics. By projecting $\mathscr{C}' - \{s_1\}$ from $\mathrm{PG}(n-1, q^n)$ onto η_1 there arise q^n points of a line, so it is clear that also the conic $\mathscr{C}'$ belongs to V.

Let us now consider a point $r_u \in \mathscr{C} - \mathscr{C}'$ and a point $r'_u \in \mathscr{C}' - \mathscr{C}$. The conic $\mathscr{C}''$ of V through r_u and $r_{u'}$ intersects Φ_1 exactly in the points $r_u, r_{u'}$. Let Φ'_1 be the 4-dimensional space containing $\mathscr{C}, \mathscr{C}', \mathscr{C}''$. Further, let $r_v \notin \mathscr{C} \cup \mathscr{C}' \cup \mathscr{C}''$. In the plane η_1 there correspond with $\mathscr{C}, \mathscr{C}', \mathscr{C}''$ distinct lines K, K', K'', and with r_v the point $\tilde{r}_v$. Now let K''' be a line of η_1 through $\tilde{r}_v$ which intersects K, K', K'' in distinct points of $\eta_1 - \bar{\tau}$. With K''' there corresponds a conic $\mathscr{C}'''$ of V through r_v which intersects $\mathscr{C} \cup \mathscr{C}' \cup \mathscr{C}''$ in at least three distinct points. Hence $\mathscr{C}'''$ belongs to Φ'_1, hence r_v belongs to Φ'_1. We conclude that the q^{2n} points r_d belong to Φ'_1.

Still assuming that Φ_1 contains a point r_l, with $r_l \neq r_k$ and $r_l \notin \mathscr{C}$, we now assume, by way of contradiction, that there exists a 3-dimensional space

$PG(3, q^n) = \Delta$ which contains $\mathscr{C}_1 \in V$ and exactly one point $r_d \notin \mathscr{C}_1$. Suppose there is a second 3-dimensional space Δ' containing $\mathscr{C}_1$ and exactly one point $r_{d'} \notin \mathscr{C}_1$. Let $r_t \in \mathscr{C}_1$ and let $\mathscr{C}'_1 \neq \mathscr{C}_1$ be a conic of V with $r_t \in \mathscr{C}'_1$ and $r_d, r_{d'} \notin \mathscr{C}'_1$. As $\mathscr{C}'_1 \subset \Phi'_1$, $\mathscr{C}'_1 \cap \Delta \subset \mathscr{C}_1$ and $\mathscr{C}'_1 \cap \Delta' \subset \mathscr{C}_1$, the conics $\mathscr{C}_1$ and $\mathscr{C}'_1$ necessarily have a common tangent line at r_t. Hence $\mathscr{C}_1$ and $\mathscr{C}'_1$ belong to a common 3-dimensional space Δ_1. Now by a preceding section either $|\mathscr{C}_1 \cap \mathscr{C}'_1| = 2$ or $\mathscr{C}_1$ and $\mathscr{C}'_1$ are mutually tangent at some point in $PG(n-1, q^n)$, a contradiction. Next we consider all 3-dimensional subspaces of Φ'_1 which contain $\mathscr{C}_1$. Any such subspace distinct from Δ either contains $2q^n$ or $2q^n - 1$ points r_d. Let α be the number of such subspaces which contain $2q^n - 1$ points r_d, and let β be the number of such subspaces which contain $2q^n$ points r_d. Then we have

$$\alpha(q^n - 1) + \beta q^n + 1 + q^n = q^{2n}, \qquad \text{and} \quad \alpha + \beta + 1 \leqslant q^n + 1.$$

If $\alpha + \beta + 1 = q^n + 1$, then $\alpha = q^n + 1$, so $\alpha + \beta + 1 > q^n + 1$, a contradiction; if $\alpha + \beta + 1 \leqslant q^n - 1$, then $\alpha \leqslant -q^n + 1$, also a contradiction. Hence $\alpha + \beta + 1 = q^n$, so $\alpha = 1$ and $\beta = q^n - 2$. So there arise $q^n - 2$ conics $\mathscr{C}_2, ..., \mathscr{C}_{q^n - 1}$, distinct from $\mathscr{C}_1$, which are tangent to $\mathscr{C}_1$ at the common point s_1 of $\mathscr{C}_1$ and $PG(n-1, q^n)$, and one conic $\mathscr{C}_{q^n}$ which intersects $\mathscr{C}_1$ in distinct points s_1 and r_u. With $\mathscr{C}_1, \mathscr{C}_2, ..., \mathscr{C}_{q^n - 1}$ there correspond in η_1 $q^n - 1$ lines $K_1, K_2, ..., K_{q^n - 1}$ through a common point on $\eta_1 \cap \bar{\tau}$; with $\mathscr{C}_{q^n}$ there corresponds in η_1 a line K_{q^n} which intersects K_1 in a point not on $\eta_1 \cap \bar{\tau}$. Hence K_{q^n} intersects $\mathscr{C}_i$ in a point not on $\eta_1 \cap \bar{\tau}$, $i = 1, 2, ..., q^n - 1$. Consequently the 3-dimensional space containing $\mathscr{C}_1$ and $\mathscr{C}_{q^n}$, coincides with the 3-dimensional space containing $\mathscr{C}_1$ and $\mathscr{C}_i$, $i = 1, 2, 3, ..., q^n - 1$, clearly a contradiction. We conclude that there exists no 3-dimensional space containing a conic $\mathscr{C}_1 \in V$ and just one point $r_d \notin \mathscr{C}_1$.

From the preceding section it follows that either all 3-dimensional spaces containing any conic $\mathscr{C}_1 \in V$ and a point $r_d \notin \mathscr{C}_1$ also contain a second point $r_{d'} \notin \mathscr{C}_1$ (and then there arises a uniquely defined pencil of quadrics in any such 3-dimensional space), or no 3-dimensional space containing any conic $\mathscr{C}_1 \in V$ contains more than one point $r_d \notin \mathscr{C}_1$.

(a) First assume that all 3-dimensional spaces containing any conic $\mathscr{C}_1 \in V$ and a point $r_d \notin \mathscr{C}_1$ also contain a second point $r_{d'} \notin \mathscr{C}_1$. Hence in each such 3-dimensional space a pencil $\mathscr{P}$ of quadrics is defined. Let s_1 be the common point of $\mathscr{C}_1$ and $PG(n-1, q^n)$ and let $r_u \notin \mathscr{C}_1$. Further let $\mathscr{B}$ be the base of the pencil $\mathscr{P}$ of quadrics defined in the 3-dimensional space containing $\mathscr{C}_1$ and r_u. Then $\mathscr{B}$ consists of two conics $\mathscr{C}_1, \mathscr{C}'_1$ through s_1, with $r_u \in \mathscr{C}'_1$ and $\mathscr{C}'_1 \in V$. Consequently any of the q^{2n} points r_d belongs to a conic of V through s_1. Now we consider all conics $\mathscr{C}_1, \mathscr{C}_2, ...$ of V containing the point s_1 of $PG(n-1, q^n)$. Let $PG(n+1, q^n)$ be a hyperplane of ρ_1 which does not contain s_1 and project all points of $\mathscr{C}_i - \{s_1\}$ from s_1 onto

$PG(n+1, q^n)$; then there arise q^n points of some line R_i. The set of all these lines R_i will be denoted by $\mathscr{R}$. Now we project the q^{2n} points r_i from s_1 onto $PG(n+1, q^n)$; the projection of r_i is denoted by r_i^*. The set of all points r_i^* is denoted by B; we have $|B| = q^{2n}$. Further, let A be the set of all intersections of $PG(n+1, q^n)$ with the tangent lines of the conics $\mathscr{C}_i$ at s_1. The tangent line U_i of $\mathscr{C}_i$ at s_1 belongs to $\bar{\tau}$, hence $\mathscr{C}_j \cap U_i = \{s_1\}$. It follows that $A \cap B = \emptyset$. Each line $R_i \in \mathscr{R}$ contains q^n points of B and just one point of A. Let π be the plane containing R_i and r_j^*, $r_j^* \notin R_i$. If we now take for Φ_1 the space $\langle R_i, s_1, r_j^* \rangle$, then the base $\mathscr{B}$ of the pencil $\mathscr{P}$ consists of two conics $\mathscr{C}_i$ and $\mathscr{C}_j$ and $(\mathscr{C}_i \cup \mathscr{C}_j) - \{s_1\}$ is the set of all points r_d in Φ_1. It immediately follows that π contains the line R_j, and that π contains no line $R_l \in \mathscr{R}$ with $i \neq l \neq j$; also, $\pi \cap B$ consists of all points of $R_i \cup R_j$ in B.

Suppose that r_i^* and r_j^* are points of B not on a common line of $\mathscr{R}$. Considering all planes of $PG(n+1, q^n)$, the previous section shows that r_i^* and r_j^* are on the same number of lines of $\mathscr{R}$.

Now let r_i^* and r_j^*, $i \neq j$, be points of B on a common line of $\mathscr{R}$. Let $t_u + 1$ be the number of lines of $\mathscr{R}$ containing r_u^*. By way of contradiction we assume that $t_i \neq t_j$. Let r_l^* be a point of B not on $r_i^* r_j^*$. The plane $r_i^* r_j^* r_l^*$ contains exactly one line R_k of $\mathscr{R}$ through r_l^*. If $r_i^* \notin R_k$ and $r_j^* \notin R_k$, then $r_i^* r_l^* \notin \mathscr{R}$ and $r_j^* r_l^* \notin \mathscr{R}$, so $t_i = t_l = t_j$, a contradiction. Hence either $r_i^* \in R_k$ or $r_j^* \in R_k$, say $r_i^* \in R_k$. Then r_l^* is on $t_j + 1$ lines of $\mathscr{R}$. Each point of $r_i^* r_l^* - \{r_i^*\}$ not in A, is on $t_j + 1$ lines of $\mathscr{R}$. Also, each point of $r_i^* r_j^* - \{r_i^*\}$ not in A, is on $t_j + 1$ lines of $\mathscr{R}$. Let $r_u^* \in B$ be not on $r_i^* r_j^*$ and not on $r_i^* r_l^*$. Assume, by way of contradiction that $r_u^* r_j^* \in \mathscr{R}$. As $r_i^* r_u^* \notin \mathscr{R}$, the point r_u^* is on $t_i + 1$ lines of $\mathscr{R}$. For any point $r_c^* \in r_i^* r_j^* - \{r_j^*, r_i^*\}$ the line $r_u^* r_c^*$ does not belong to $\mathscr{R}$, and so r_u^* is on $t_j + 1$ lines of $\mathscr{R}$. Hence $t_i = t_j$, a contradiction. Consequently $r_u^* r_j^* \notin \mathscr{R}$. Interchanging roles of r_u^* and r_l^*, we see that necessarily $r_u^* r_i^* \in \mathscr{R}$. It follows that all lines of $\mathscr{R}$ contain r_i^*. Hence $t_j = 0$. So $(t_1 + 1)(q^n - 1) + 1 = |B| = q^{2n}$, that is, $t_1 = q^n$. Consequently there are $q^n + 1$ conics in V containing s_1, and they all share a point r_i. In $PG(n-1, q^n)$ we now consider a point $s_2 \neq s_1$ on an element of V. Let $\mathscr{C}$ be a conic of V containing r_i and s_2. Choose on $\mathscr{C}$ a point r_w different from r_i. Then there is a conic $\mathscr{C}'$ in V containing r_w and s_1, and so $\mathscr{C}'$ contains r_i. Consequently V contains two distinct conics $\mathscr{C}$ and $\mathscr{C}'$ through r_i and r_w, a contradiction. We conclude that the number of lines of $\mathscr{R}$ through any point $r_i^* \in B$ is independent of the index i.

The constant number of lines of $\mathscr{R}$ which contains $r_i^* \in B$ will be denoted by $t + 1$.

Fix a line $R \in \mathscr{R}$ and count on two ways the number of ordered pairs (r_i^*, R'), with $r_i^* \in B$, $r_i^* \notin R$, $R' \cap R \neq \emptyset$ and $r_i^* \in R'$. We obtain

$$q^n t(q^n - 1) + \beta q^n = q^{2n} - q^n,$$

where $\beta+1$ is the number of lines of $\mathscr{R}$ through $R\cap A$. Hence

$$(q^n-1)(t-1)+\beta=0.$$

(1) $t=0$ and $\beta=q^n-1$.

The q^n lines of $\mathscr{R}$ through $R\cap A$ form a partition of B, and as $t=0$ the set $\mathscr{R}$ only contains these q^n lines. It follows that s_1 belongs to exactly q^n conics of V; also these conics are mutually tangent at s_1.

(2) $t=1$ and $\beta=0$.

Then $\mathscr{R}$ contains $q^{2n}\cdot 2/q^n=2q^n$ lines and $|A|=2q^n$. So s_1 belongs to exactly q^{2n} conics of V. Let $R_{d_1}, R_{d_2}, ..., R_{d_{qn}}$ be the q^n lines of $\mathscr{R}$ which intersect R in B. The remaining q^n lines of $\mathscr{R}$ necessarily intersect $R_{d_1}, R_{d_2}, ..., R_{d_{q^n}}$. So the intersections $R_i\cap B$, $R_i\in\mathscr{R}$, form a grid $\mathscr{G}$ of order $(q^n-1, 1)$. Clearly $\mathscr{G}$ is contained in a 3-dimenional space Ψ, A consists of two distinct intersecting lines N_1, N_2 minus their common point, and $\mathscr{R}\cup\{N_1, N_2\}$ is the set of all lines of a hyperbolic quadric in Ψ.

In Case (2) $A\cup B$ is contained in a 3-dimensional space Ψ, hence all points r_i are contained in a PG$(4, q^n)$.

Now we consider Case (1). Let $s_2\neq s_1$ be a point of PG$(n-1, q^n)$ on a conic $\mathscr{C}\in V$. Suppose that $\mathscr{C}', \mathscr{C}''$ are distinct conics of V containing s_1. The q^n elements of V through s_1 correspond with the q^n lines of η_1, distinct from $\eta_1-\bar{\pi}=\eta_1-\bar{\tau}$ and containing a common point e on $\eta_1\cap\bar{\tau}$. Hence with $\mathscr{C}$ there corresponds a line of η_1 not containing e. So $|\mathscr{C}\cap\mathscr{C}'|=|\mathscr{C}\cap\mathscr{C}''|=1$. So $\mathscr{C}$ belongs to the space PG$(4, q^n)$ defined by $\mathscr{C}', \mathscr{C}''$ and s_2. It follows that all conics of V containing s_2 belong to PG$(4, q^n)$. Consequently all points r_i are contained in PG$(4, q^n)$.

Since PG$(4, q^n)\subset\langle$PG$(n-1, q^n), \eta_1\rangle=\rho_1$ and PG$^{(i)}(n-1, q^n)$ has exactly one point r_i in common with ρ_1, it has also one point r_i in common with PG$(4, q^n)$.

As $\langle$PG$(4, q^n)$,PG$(n-1, q^n)\rangle=\rho_1$, the space PG$(4, q^n)$ intersects PG$(n-1, q^n)$ in a line M. Consequently all the common points of PG$(n-1, q^n)$ and the conics of V belong to M.

(b) Next assume that no 3-dimensional space containing any conic $\mathscr{C}\in V$ contains more than one point $r_d\notin\mathscr{C}$. Let $\mathscr{W}$ be the set of all points r_i; then $|\mathscr{W}|=q^{2n}$.

Let $\mathscr{C}\in V$ and $r_d\in\mathscr{W}-\mathscr{C}$. Further, let Φ_1 be the 3-dimensional space containing $\mathscr{C}$ and r_d. The common point of $\mathscr{C}$ and PG$(n-1, q^n)$ will be denoted by s. If $\mathscr{C}=\{s, r_1, r_2, ..., r_{q^n}\}$, then r_j and r_d are on exactly one conic $\mathscr{C}_j$ of V, with $j=1, 2, ..., q^n$. Let $\mathscr{C}_j\cap$PG$(n-1, q^n)=\{s_j\}$, $j=1, 2, ..., q^n$. If $s_j=s_{j'}$, with $j\neq j'$ and $j, j'\in\{1, 2, ..., q^n\}$, then the 3-dimensional space defined by $\mathscr{C}_j$ and $r_{j'}$, contains $\mathscr{C}_{j'}$, so contains more than one point $r_d \in \mathscr{W} - \mathscr{C}_j$, a contradiction. So the points $s_1, s_2, ..., s_{q^n}$ are distinct. A same argument shows that $s \neq s_j, j \neq 1, 2, ..., q^n$. Let $\tilde{\mathscr{C}} = \{s, s_1, s_2, ..., s_{q^n}\}$.

Let Φ'_1 be the space generated by $\mathscr{C}, \mathscr{C}_1, \mathscr{C}_2$. Further, let $r_d \in \mathscr{W} - (\mathscr{C} \cup \mathscr{C}_1 \cup \mathscr{C}_2)$. In the plane η_1 there correspond with $\mathscr{C}, \mathscr{C}_1, \mathscr{C}_2$ distinct lines K, K_1, K_2, and with r_d the point $\tilde{r}_d$. Now let K' be a line of η_1 through $\tilde{r}_d$ which intersects K, K_1, K_2 in distinct points of $\eta_1 - \bar{\tau}$. With K' there corresponds a conic $\mathscr{C}'$ of V through r_d which intersects $\mathscr{C} \cup \mathscr{C}_1 \cup \mathscr{C}_2$ in at least three distinct points. Hence $\mathscr{C}'$ belongs to Φ'_1, hence r_d belongs to Φ'_1. We conclude that $\mathscr{W}$ is a subset of Φ'_1.

Clearly Φ'_1 is either 4-dimensional or 5-dimensional. If Φ'_1 is 4-dimensional, then considering all hyperplanes of Φ'_1 on $\mathscr{C}$ we have that $\mathscr{W}$ contains at most $2q^n + 1$ points, a contradiction. Consequently Φ'_1 is 5-dimensional.

Assume, by way of contradiction, that $\tilde{\mathscr{C}}$ is a line. Then $\mathscr{W}$ is contained in the 4-dimensional space generated by $\mathscr{C}, s_1, r_d$, a contradiction. So $\tilde{\mathscr{C}}$ is not a line, hence it also follows that $\Phi'_1 \cap \mathrm{PG}(n-1, q^n)$ is at least 2-dimensional. If $\mathrm{PG}(5, q^n) \cap \mathrm{PG}(n-1, q^n) = \mu$ is at least 3-dimensional, then $\mu \cap \Phi_1$ is at least a line, which was shown to be impossible in the first half of the proof. It follows that μ is a plane.

Finally, since $\Phi'_1 \subset \langle \mathrm{PG}(n-1, q^n), \eta_1 \rangle = \rho_1$ and $\mathrm{PG}^{(i)}(n-1, q^n)$ has exactly one point r_i in common with ρ_1, it has also one point r_i in common with Φ'_1. ∎

Let $\mathscr{W}$ be the set of all points r_i; then $|\mathscr{W}| = q^{2n}$. Assume that we are in Case (a) of Theorem 6.1. Then the set $\mathscr{W}$ belongs to $\mathrm{PG}(4, q^n)$ and $\mathrm{PG}(4, q^n) \cap \mathrm{PG}(n-1, q^n)$ is a line M. The following observations are taken from the proof of Theorem 6.1. The set $\mathscr{W}$ provided with the conics $\mathscr{C}$ of V forms a $2-(q^{2n}, q^n, 1)$ design, that is an affine plane of order q^n. If $r_i \in \mathscr{C}$, with $\mathscr{C} \in V$, and if $\{s_1\} = \mathscr{C} \cap \mathrm{PG}(n-1, q^n)$, then any point $r_j \in \mathscr{W}$ is on a conic of V through s_1. If the 3-dimensional space Φ_1 contains $\mathscr{C} \in V$ and a point of $\mathscr{W}$ not on $\mathscr{C}$, then it contains two conics $\mathscr{C}, \mathscr{C}'$ of V. Also $|\mathscr{C} \cap \mathscr{C}'| \leqslant 2$ and the conics $\mathscr{C}, \mathscr{C}'$ have a common point on M. Further, $(\mathscr{C}' \cup \mathscr{C}) - \{s_1\} = \Phi_1 \cap \mathscr{W}$.

Let s_1 be the common point of $\mathscr{C} \in V$ and $\mathrm{PG}(n-1, q^n)$. We distinguish two cases :

(1) The point s_1 belongs to q^n elements of V and all these conics have a common tangent line U at s_1; this line U does not belong to $\mathrm{PG}(n-1, q^n)$.

(2) The point s_1 belongs to $2q^n$ elements of V. The set of these $2q^n$ conics can be partitioned into sets $\mathscr{A}_1$ and $\mathscr{A}_2$, where each conic of $\mathscr{A}_1$ has two distinct points in common with each conic of $\mathscr{A}_2$ and where any two distinct conics of $\mathscr{A}_i$ have just s_1 in common, $i = 1, 2$. The tangents at s_1 of

all conics of $\mathscr{A}_i$ are all distinct and belong to a plane θ_i, $i=1,2$; also, $\theta_1 \cap \theta_2$ is a line, distinct from these $2q^n$ tangent lines. With the q^n conics of $\mathscr{A}_i$ there correspond q^n lines of η_1 containing a common point e_i of $\bar{\tau} \cap \eta_1$, $i=1,2$. It follows that the tangent lines of the elements of $\mathscr{A}_i$ at s_1 all belong to a n-dimensional space through $\mathrm{PG}(n-1, q^n)$, and so $\theta_i \cap \mathrm{PG}(n-1, q^n)$ is a line M_i, $i=1,2$. As $\theta_i \subset \mathrm{PG}(4, q^n)$ and $\mathrm{PG}(4, q^n) \cap \mathrm{PG}(n-1, q^n) = M$ we necessarily have $M_1 = M_2 = M$, and so $\theta_1 \cap \theta_2 = M$.

Let s_1 be the common point of $\mathscr{C} \in V$ and $\mathrm{PG}(n-1, q^n)$, and let s_2 be the common point of $\mathscr{C}' \in V$ and $\mathrm{PG}(n-1, q^n)$, where $s_1 \neq s_2$. With $\mathscr{C}$, resp. $\mathscr{C}'$, there corresponds a line N, resp. N', of η_1. From the preceding section follows that N and N' intersect the line $\eta_1 \cap \bar{\tau}$ in distinct points. As $N \cap N'$ does not belong to $\eta_1 \cap \bar{\tau}$, the conics $\mathscr{C}$ and $\mathscr{C}'$ have exactly one point (not in $\mathrm{PG}(n-1, q^n)$) in common.

The set of all common points of M with elements of V will be denoted by $\mathscr{M}$.

Theorem 6.2. *Consider a non-classical* TGQ $T(n, 2n, q) = T(O)$, *q odd, with* $O = O(n, 2n, q) = \{\mathrm{PG}(n-1, q), \mathrm{PG}^{(1)}(n-1, q), \mathrm{PG}^{(2)}(n-1, q), \ldots, \mathrm{PG}^{(q^{2n})}(n-1, q)\}$. *If O is good at* $\mathrm{PG}(n-1, q)$ *and if the set $\mathscr{W}$ is contained in a* $\mathrm{PG}(4, q^n)$, *then the set $\mathscr{W} \cup \mathscr{M}$ of* $\mathrm{PG}(4, q^n)$ *is the projection of a quadric Veronesean $\mathscr{V}_2^4$ from a point p in a conic plane onto a hyperplane* $\mathrm{PG}(4, q^n)$.

Proof. We use the notations of Theorem 6.1 and its proof, and these of the sections preceding Theorem 6.2.

Let α_1 be the number of points in $\mathscr{M}$ which belong to q^n elements of V, and let α_2 be the number of points in $\mathscr{M}$ which belong to $2q^n$ elements of V. Now we count on two ways the number of ordered pairs $(s, \mathscr{C})$, with $s \in \mathscr{M}$ and $\mathscr{C} \in V$ containing s. We obtain

$$\alpha_1 q^n + \alpha_2 (2q^n) = q^{2n} + q^n,$$

that is,

$$\alpha_1 + 2\alpha_2 = q^n + 1.$$

It follows that α_1 is even. We distinguish two cases.

(i) $\underline{\alpha_1 \geqslant 2}$

Let $s_1, s_2, \ldots, s_{\alpha_1}$ be the points of $\mathscr{M}$ which are contained in q^n conics of V. All conics of V through s_i have a common tangent line U_i at s_i, $i=1, 2, \ldots, \alpha_1$. As $\langle \mathrm{PG}(n-1, q^n), U_1 \rangle$ and $\langle \mathrm{PG}(n-1, q^n), U_2 \rangle$ intersect η_1 in different points, the lines U_1 and U_2 do not intersect. Let $\mathscr{C}$ be a conic in V not containing s_1 or s_2, and let s be the point of $\mathscr{C}$ in $PG(n-1, q^n)$. Further, let $r_i \in \mathscr{W}$. There is a conic $\mathscr{C}' \in V$ containing r_i and s_1. Then $\mathscr{C}$ and $\mathscr{C}'$ have a point r_j in common. As $\mathscr{C}'$ is tangent to U_1 at s_1, it is contained in the plane $\langle U_1, r_j \rangle$. Hence $\mathscr{C}'$ belongs to the quadratic cone $\mathscr{Q}_1$ with vertex U_1 and base $\mathscr{C}$ (as $\langle PG(n-1, q^n), U_1 \rangle \cap \eta_1$ is not on the line of η_1 defined by $\mathscr{C}$, the line U_1 is skew to the plane σ of $\mathscr{C}$). Consequently $\mathscr{W}$ belongs to $\mathscr{Q}_1$. Clearly also M belongs to $\mathscr{Q}_1$. Analogously $\mathscr{W}$, and also M, belong to the quadratic cone $\mathscr{Q}_2$ with vertex U_2 and base $\mathscr{C}$.

Let ξ be the 3-dimensional space $\langle U_1, U_2 \rangle$. As $\xi \subset \bar{\tau}$, $\xi \subset PG(4, q^n)$, and as $PG(4, q^n) \not\subset \bar{\tau}$, we have $\bar{\tau} \cap PG(4, q^n) = \xi$. Hence the tangent line U of $\mathscr{C}$ at s is contained in the subspace ξ.

Now let us choose coordinates in $PG(4, q^n)$. Let $s(0, 0, 1, 0, 0)$, $(0, 1, 0, 0, 0) \in \mathscr{C}$, and let $(1, 0, 0, 0, 0) \in U$ be the pole of the line $\langle (0, 0, 1, 0, 0), (0, 1, 0, 0, 0) \rangle$ with respect to $\mathscr{C}$. So the plane σ of $\mathscr{C}$ has equation $X_3 = X_4 = 0$. Further assume that $s_1(0, 0, 0, 1, 0)$, so M has equations $X_0 = X_1 = X_4 = 0$. On U_1 we choose the point $(0, 0, 0, 0, 1)$ in such a way that the line $\langle (1, 0, 0, 0, 0), (0, 0, 0, 0, 1) \rangle$ intersects U_2 (this is possible as U, U_1, U_2 are contained in the 3-dimensional space ξ). Then the hyperplane ξ of $PG(4, q^n)$ has equation $X_1 = 0$. Let e' be any point of $\mathscr{C} - \{s, (0, 1, 0, 0, 0)\}$ and let e be the intersection of the planes $U_1 e'$ and $\langle U_2, (0, 1, 0, 0, 0) \rangle$. Choose e as point $(1, 1, 1, 1, 1)$. Then $\langle e, (0, 1, 0, 0, 0) \rangle \cap \xi$ is the point $(1, 0, 1, 1, 1)$, and so $(1, 0, 1, 1, 1)$ is on U_2. It follows that $U_2 \cap M = \{(0, 0, 1, 1, 0)\}$ and $U_2 \cap \langle (0, 0, 0, 0, 1), (1, 0, 0, 0, 0) \rangle = \{(1, 0, 0, 0, 1)\}$. Further, $U_1 e \cap \sigma = \{e'(1, 1, 1, 0, 0)\}$, with σ the plane of $\mathscr{C}$. It follows that $\mathscr{C}$ has equations $X_3 = X_4 = X_0^2 - X_1 X_2 = 0$.

Consequently

$$\mathscr{Q}_1 \colon X_0^2 = X_1 X_2,$$

$$\mathscr{Q}_2 \colon (X_0 - X_4)^2 = X_1 (X_2 - X_3).$$

So we have

$$\mathscr{W} \subseteq \mathscr{Q}_1 \cap \mathscr{Q}_2 = \mathbf{V}(X_0^2 - X_1 X_2, (X_0 - X_4)^2 - X_1 (X_2 - X_3)).$$

Now we calculate the number of points of $\mathscr{Q}_1 \cap \mathscr{Q}_2$ not in ξ. So we put $x_1 = 1$. As x_0 and x_4 can be chosen and define uniquely x_2 and x_3, we have $|(\mathscr{Q}_1 \cap \mathscr{Q}_2) - \xi| = q^{2n} = |\mathscr{W}|$. So $\mathscr{W} = (\mathscr{Q}_1 \cap \mathscr{Q}_2) - \xi$. As $\mathscr{Q}_1 \cap \mathscr{Q}_2 \cap \xi = M$, we also have $\mathscr{W} = (\mathscr{Q}_1 \cap \mathscr{Q}_2) - M$. Clearly $\mathscr{W} = \{(m, 1, m^2, 2mn - n^2, n) \parallel m, n \in GF(q^n)\}$.

Now we consider the quadric Veronesean $\mathscr{V}_2^4$ in $PG(5, q^n)$, where

$$\mathscr{V}_2^4 = \{(ml, l^2, m^2, 2mn - n^2, nl, n^2) \parallel (l, m, n) \text{ is a point of } PG(2, q^n)\}.$$

Assume that PG$(4, q^n)$ is the hyperplane $X_5=0$ of PG$(5, q^n)$. Project $\mathscr{V}_2^4$ from $(0, 0, 0, 0, 0, 1) \notin \mathscr{V}_2^4$ onto PG$(4, q^n)$. Then we obtain the surface

$$\Gamma = \{(ml, l^2, m^2, 2mn - n^2, nl, 0) \parallel (l, m, n) \neq (0, 0, 0)\}.$$

For $l=1$ we obtain the points $(m, 1, m^2, 2mn-n^2, n, 0)$, with $m, n \in GF(q^n)$. Hence $\Gamma - \xi = \mathscr{W} = (\mathscr{Q}_1 \cap \mathscr{Q}_2) - M$. For $l=0$ we obtain the points $(0, 0, m^2, 2mn-n^2, 0, 0)$, with $(m, n) \neq (0, 0)$. So $\Gamma \cap \xi \subseteq M$ and $|\Gamma \cap \xi| = (q^n+3)/2$. The point $(0, 0, 0, 0, 0, 1)$ from which we project is in the conic plane φ: $X_0 = X_1 = X_4 = 0$ of $\mathscr{V}_2^4$, and is an exterior point of the conic $\varphi \cap \mathscr{V}_2^4$.

Let $p_1, p_2 \in \mathscr{W}$, $p_1 \neq p_2$, let $\mathscr{C}$ be the conic of V containing p_1 and p_2, and let $\tilde{\mathscr{C}}$ be any non-singular conic of Γ containing p_1 and p_2. Assume, by way of contradiction that $\mathscr{C} \neq \tilde{\mathscr{C}}$. If σ is the plane of $\mathscr{C}$ and if p is a point of $\tilde{\mathscr{C}}$ not on $\mathscr{C}$ and not in ξ, then $\Phi_1 = \langle \sigma, p \rangle$ contains two conics $\mathscr{C}$ and $\mathscr{C}'$ of V. Also $\tilde{\mathscr{C}} - \xi \subseteq (\mathscr{C} \cup \mathscr{C}') - \xi$. Consequently $\mathscr{C}' = \tilde{\mathscr{C}}$, a contradiction as $|(\mathscr{C}' \cap \mathscr{C}) - \xi| \leqslant 1$. Hence $\mathscr{C} = \tilde{\mathscr{C}}$. Now it follows easily that V is the set of all non-singular conics on Γ, and that V is the set of the projections of all non-singular conics on $\mathscr{V}_2^4$ but not in the plane φ.

From the preceding section it also follows that the set $\mathscr{M}$ of all common points of M with elements of V, is the projection from $(0, 0, 0, 0, 0, 1)$ onto PG$(4, q^n)$ of the conic $\varphi \cap \mathscr{V}_2^4$. Hence $|\mathscr{M}| = (q^n+3)/2$. If T_1, T_2 are the tangent lines of $\varphi \cap \mathscr{V}_2^4$ through $(0, 0, 0, 0, 0, 1)$, then the points $T_1 \cap M$ and $T_2 \cap M$ are contained in q^n conics of V; if Z_i is any line of φ through $(0, 0, 0, 0, 0, 1)$ which intersects $\varphi \cap \mathscr{V}_2^4$ in two distinct points, then $Z_i \cap M$ is contained in $2q^n$ conics of V. Hence $\alpha_1 = 2$.

We conclude that $\mathscr{W} \cup \mathscr{M}$ is the projection of the quadric Veronesean $\mathscr{V}_2^4$ from $(0, 0, 0, 0, 0, 1)$ onto the space PG$(4, q^n)$.

(ii) $\underline{\alpha_1 = 0}$

In this case each point of $\mathscr{M}$ is contained in exactly $2q^n$ conics of V. Further $\alpha_2 = |\mathscr{M}| = (q^n+1)/2$.

Let $\mathscr{C}_1, \mathscr{C}_2$ be conics of V intersecting M in distinct points s_1, s_2. The tangent line of $\mathscr{C}_i$ at s_i will be denoted by U_i, $i=1, 2$. As $\langle$PG$(n-1, q^n), U_1\rangle$ and $\langle$PG$(n-1, q^n), U_2\rangle$ intersect η_1 in different points, the lines U_1 and U_2 do not intersect. Let ξ be the 3-dimensional space $\langle U_1, U_2\rangle$. As $\xi \subset \bar{\tau}$, $\xi \subset$ PG$(4, q^n)$, and as PG$(4, q^n) \not\subset \bar{\tau}$, we have $\bar{\tau} \cap$ PG$(4, q^n) = \xi$. Hence the tangent line U of any conic $\mathscr{C} \in V$ at $\mathscr{C} \cap M$ is contained in the subspace ξ.

Let $\mathscr{C}, \mathscr{C}' \in V$, $\mathscr{C} \neq \mathscr{C}'$, with $M \cap \mathscr{C} = M \cap \mathscr{C}' = \{s\}$ and assume that $\mathscr{C} \cap \mathscr{C}' = \{s, r\}$.

Now let us choose coordinates in PG$(4, q^n)$. Let the hyperplane ξ of PG$(4, q^n)$ have equation $X_1 = 0$, and let the hyperplane π of PG$(4, q^n)$

containing $\mathscr{C} \cup \mathscr{C}'$ have equation $X_3 = 0$. Further, let $X_0 = X_3 = 0$ be the plane σ of $\mathscr{C}$, and let $X_4 = X_3 = 0$ be the plane σ' of $\mathscr{C}'$. The conics $\mathscr{C}$ and $\mathscr{C}'$ define distinct lines of η_1, and so M is not contained in π. As M is contained in ξ and intersects $\sigma \cap \sigma'$ we may choose M to be the line $X_0 = X_1 = X_4 = 0$. Then $s(0, 0, 1, 0, 0)$. Further, let $r(0, 1, 0, 0, 0)$ and let $s_1(0, 0, 0, 1, 0) \in \mathscr{M}$. Also, let the pole of sr with respect to $\mathscr{C}$ be the point $(0, 0, 0, 0, 1)$ and let the pole of sr with respect to $\mathscr{C}'$ be the point $(1, 0, 0, 0, 0)$. Call t a point of $\mathscr{C} - \{s, r\}$. Now let $s_2 \in \mathscr{M} - \{s_1, s\}$ and let w be the intersection of st and $\langle r, (0, 0, 0, 0, 1)\rangle$. Further, let t' be the common point of $s_2 w$ and $s_1 t$. Finally let e be a point of $\langle (1, 0, 0, 0, 0), t'\rangle - \{(1, 0, 0, 0, 0), t'\}$, and choose e as point $(1, 1, 1, 1, 1)$. Then $t'(0, 1, 1, 1, 1)$, $t(0, 1, 1, 0, 1)$, $w(0, 1, 0, 0, 1)$, $s_2(0, 0, 1, 1, 0)$.

Consequently,

$$\mathscr{C}\colon X_0 = X_3 = X_1 X_2 - X_4{}^2 = 0,$$

$$\mathscr{C}'\colon X_3 = X_4 = X_1 X_2 - v X_0{}^2 = 0,\ v \neq 0.$$

Let ψ be the pencil of quadrics in $X_3 = 0$ with base curve $\mathscr{C} \cup \mathscr{C}'$. Then ψ is the set of quadrics Φ_h with

$$\Phi_h\colon X_3 = 2hX_0X_4 + (X_1X_2 - vX_0^2 - X_4^2) = 0, \qquad h \in \mathrm{GF}(q^n) \cup \{\infty\}.$$

Clearly $\sigma \cup \sigma'$ is an element of ψ. If γ_1 is the number of cones (with a point vertex) in ψ, γ_2 is the number of non-singular elliptic quadrics in ψ, and γ_3 is the number of non-singular hyperbolic quadrics in ψ, then, counting in two ways the number of pairs (p, Φ_h), with p a point of π not on $\mathscr{C} \cup \mathscr{C}'$ and $p \in \Phi_h \in \psi$, we obtain

$$\begin{aligned}(2q^{2n} + q^n + 1 - 2q^n) &+ \gamma_1(q^{2n} + q^n + 1 - 2q^n) \\ &+ \gamma_2(q^{2n} + 1 - 2q^n) + \gamma_3((q^n + 1)^2 - 2q^n) \\ &= q^{3n} + q^{2n} + q^n + 1 - 2q^n.\end{aligned}$$

Also $\gamma_1 + \gamma_2 + \gamma_3 = q^n$. The singular points of Φ_h, $h \neq \infty$, are determined by the equations $X_3 = hX_4 - vX_0 = X_2 = X_1 = hX_0 - X_4 = 0$. Hence if v is a non-square then Φ_h, $h \neq \infty$, is always non-singular, and if v is a square $(\neq 0)$ then for two distinct values of h, $h \neq \infty$, the quadric Φ_h is a quadratic cone.

So, if v is a square, then $\gamma_1 = 2$, $\gamma_2 = (q^n - 1)/2$, $\gamma_3 = (q^n - 3)/2$, and if v is a non-square, then $\gamma_1 = 0$, $\gamma_2 = (q^n + 1)/2$, $\gamma_3 = (q^n - 1)/2$.

Let s' be any point of $\mathscr{M}$. Then the union of all the lines through s' and a point of $\mathscr{W}$, all the tangent lines at s' of the conics of V through s', and the line M, is a singular hyperbolic quadric $\mathscr{H}_{s'}$ with vertex s'. Clearly $\mathscr{H}_{s'}$

contains $\mathscr{C} \cup \mathscr{C}'$. If $s \neq s'$, then $\mathscr{H}_{s'} \cap \pi$ is a non-singular hyperbolic quadric of the pencil ψ; if $s = s'$, then $\mathscr{H}_{s'} \cap \pi = \sigma \cup \sigma'$. Assume, by way of contradiction, that for distinct points s', s'' of $\mathscr{M}$, with $s' \neq s \neq s''$, we have $\mathscr{H}_{s'} \cap \pi = \mathscr{H}_{s''} \cap \pi$. Let $d \in (\mathscr{H}_{s'} \cap \pi) - (\mathscr{C} \cup \mathscr{C}')$. Then $d \notin \mathscr{W}$. On $s'd$ there is a point r_i of $\mathscr{W}$ and on $s''d$ there is a point r_j of $\mathscr{W}$. So $r_i r_j$ intersects M, a contradiction. So $\mathscr{H}_{s'} \cap \pi \neq \mathscr{H}_{s''} \cap \pi$. It follows that $\gamma_3 \geqslant |\mathscr{M}| - 1 = (q^n - 1)/2$. Consequently v is a non-square, $\gamma_1 = 0$, $\gamma_2 = (q^n + 1)/2$ and $\gamma_3 = (q^n - 1)/2$. Also, the $(q^n - 1)/2$ non-singular hyperbolic quadrics of ψ are the intersections of π with the $(q^n - 1)/2$ singular hyperbolic quadrics $\mathscr{H}_{s'}$, with s' any point of $\mathscr{M} - \{s\}$.

Let $\mathscr{Q}_1$ be the singular hyperbolic quadric $\mathscr{H}_{s'}$, with $s'(0, 0, 0, 1, 0)$

$$\mathscr{Q}_1 \colon 2h_1 X_0 X_4 + (X_1 X_2 - vX_0^2 - X_4^2) = 0, \qquad \text{with} \quad h_1^2 - v (\neq 0) \text{ a square.}$$

Let $\mathscr{Q}_2$ be the singular hyperbolic quadric $\mathscr{H}_{s''}$, with $s''(0, 0, 1, 1, 0)$

$$\mathscr{Q}_2 \colon 2h_2 X_0 X_4 + (X_1(X_2 - X_3) - vX_0^2 - X_4^2) = 0, \quad \text{with } h_2^2 - v (\neq 0) \text{ a square.}$$

As $\mathscr{H}_{s'} \cap \pi \neq \mathscr{H}_{s''} \cap \pi$, we have $h_1 \neq h_2$. So

$$\begin{aligned} \mathscr{W} \subseteq \mathscr{Q}_1 \cap \mathscr{Q}_2 &= \mathbf{V}(2h_1 X_0 X_4 + (X_1 X_2 - vX_0^2 - X_4^2), \\ &\qquad 2h_2 X_0 X_4 + (X_1(X_2 - X_3) - vX_0^2 - X_4^2)) \\ &= \mathbf{V}(2h_1 X_0 X_4 + (X_1 X_2 - vX_0^2 - X_4^2), \\ &\qquad 2(h_2 - h_1)\, X_0 X_4 - X_1 X_3). \end{aligned}$$

Now we calculate the number of points of $\mathscr{Q}_1 \cap \mathscr{Q}_2$ not in ξ. So we put $x_1 = 1$. As x_0 and x_4 can be chosen and define uniquely x_2 and x_3, we have $|(\mathscr{Q}_1 \cap \mathscr{Q}_2) - \xi| = q^{2n} = |\mathscr{W}|$. So $\mathscr{W} = (\mathscr{Q}_1 \cap \mathscr{Q}_2) - \xi$. As $\mathscr{Q}_1 \cap \mathscr{Q}_2 \cap \xi = M$, we also have $\mathscr{W} = (\mathscr{Q}_1 \cap \mathscr{Q}_2) - M$. Clearly

$$\mathscr{W} = \{(m, 1, vm^2 + n^2 - 2h_1 mn, 2(h_2 - h_1) mn, n) \parallel m, n \in \mathrm{GF}(q^n)\}.$$

Now we consider the quadric Veronesean $\mathscr{V}_2^4$ in $\mathrm{PG}(5, q^n)$, where

$$\begin{aligned} \mathscr{V}_2^4 = \{&(ml, l^2, vm^2 + n^2 - 2h_1 mn, 2(h_2 - h_1) mn, nl, n^2) \parallel (l, m, n) \\ &\text{is a point of } \mathrm{PG}(2, q^n)\}. \end{aligned}$$

Assume that $\mathrm{PG}(4, q^n)$ is the hyperplane $X_5 = 0$ of $\mathrm{PG}(5, q^n)$. Project $\mathscr{V}_2^4$ from $(0, 0, 0, 0, 0, 1) \notin \mathscr{V}_2^4$ onto $\mathrm{PG}(4, q^n)$. Then we obtain the surface

$$\Gamma = \{(ml, l^2, vm^2 + n^2 - 2h_1 mn, 2(h_2 - h_1) mn, nl, 0) \parallel (l, m, n) \neq (0, 0, 0)\}.$$

For $l = 1$ we obtain the points $(m, 1, vm^2 + n^2 - 2h_1 mn, 2(h_2 - h_1) mn, n, 0)$, with $m, n \in \mathrm{GF}(q^n)$. Hence $\Gamma - \xi = \mathscr{W} = (\mathscr{Q}_1 \cap \mathscr{Q}_2) - M$. For $l = 0$ we obtain the points $(0, 0, vm^2+n^2-2h_1mn, 2(h_2-h_1)mn, 0, 0)$, with $(m, n) \neq (0, 0)$. So $\Gamma \cap \xi \subseteq M$ and $|\Gamma \cap \xi| = (q^n+1)/2$. The point $(0, 0, 0, 0, 0, 1)$ from which we project is in the conic plane φ: $X_0 = X_1 = X_4 = 0$ of $\mathscr{V}_2^4$, and is an interior point of the conic $\varphi \cap \mathscr{V}_2^4$.

Let $p_1, p_2 \in \mathscr{W}$, $p_1 \neq p_2$, let $\mathscr{C}$ be the conic of V containing p_1 and p_2, and let $\tilde{\mathscr{C}}$ be any non-singular conic of Γ containing p_1 and p_2. Assume, by way of contradiction that $\mathscr{C} \neq \tilde{\mathscr{C}}$. If σ is the plane of $\mathscr{C}$ and if p is a point of $\tilde{\mathscr{C}}$ not on $\mathscr{C}$ and not in ξ, then $\Phi_1 = \langle \sigma, p \rangle$ contains two conics $\mathscr{C}$ and $\mathscr{C}'$ of V. Also $\tilde{\mathscr{C}} - \xi \subseteq (\mathscr{C} \cup \mathscr{C}') - \xi$. Consequently $\mathscr{C}' = \tilde{\mathscr{C}}$, a contradiction as $|(\mathscr{C}' \cap \mathscr{C}) - \xi| \leqslant 1$. Hence $\mathscr{C} = \tilde{\mathscr{C}}$. Now it follows easily that V is the set of all non-singular conics on Γ, and that V is the set of the projections of all non-singular conics on $\mathscr{V}_2^4$ but not in the plane φ.

From the preceding section it also follows that the set $\mathscr{M}$ of all common points of M with elements of V, is the projection from $(0, 0, 0, 0, 0, 1)$ onto $\mathrm{PG}(4, q^n)$ of the conic $\varphi \cap \mathscr{V}_2^4$.

We conclude that $\mathscr{W} \cup \mathscr{M}$ is the projection of the quadric Veronesean $\mathscr{V}_2^4$ from $(0, 0, 0, 0, 0, 1)$ onto the space $\mathrm{PG}(4, q^n)$. ∎

COROLLARY 6.3. *In $\mathscr{M}$ there are either* 0 *or* 2 *points which belong to exactly q^n elements of V.*

Proof. This follows immediately from the proof of Theorem 6.2. ∎

THEOREM 6.4. *Consider a non-classical* TGQ $T(n, 2n, q) = T(O)$, *q odd, with O an egg which is good at* $\mathrm{PG}(n-1, q)$. *Assume also that the set $\mathscr{W}$ is contained in a* $\mathrm{PG}(4, q^n)$. *Then the set $\mathscr{M}$ contains exactly* 2 *points which belong to exactly q^n elements of V, that is, the set $\mathscr{M}$ is the projection of an irreducible conic $\mathscr{K}$ from an exterior point p of $\mathscr{K}$ onto a line M of the plane of $\mathscr{K}$.*

Proof. We use the notations of the preceding theorems.

Let $\mathrm{PG}(n-2, q)$ be any hyperplane of $\mathrm{PG}(n-1, q)$. Assume by way of contradiction that the extension $\mathrm{PG}(n-2, q^n)$ of $\mathrm{PG}(n-2, q)$ contains a point s of $\mathscr{M}$. As s belongs to a conic $\mathscr{C}$ of V, the point s together with its conjugates with respect to the extension $\mathrm{GF}(q^n)$ of $\mathrm{GF}(q)$ generate $\mathrm{PG}(n-1, q^n)$. But s and all its conjugates belong to $\mathrm{PG}(n-2, q^n)$, a contradiction. Hence $\mathrm{PG}(n-2, q^n) \cap \mathscr{M} = \emptyset$.

Now we dualize in $\mathrm{PG}(n-1, q^n)$, in such a way that the points of $\mathrm{PG}(n-1, q)$ become hyperplanes of $\mathrm{PG}(n-1, q)$. The dual of the line M containing $\mathscr{M}$ is a $\mathrm{PG}(n-3, q^n)$. The dual of a hyperplane $\mathrm{PG}(n-2, q)$ of $\mathrm{PG}(n-1, q)$ is a point z of $\mathrm{PG}(n-1, q)$, and the dual of the common point of M and the hyperplane $\mathrm{PG}(n-2, q^n)$ which extends $\mathrm{PG}(n-2, q)$ is the hyperplane $\langle z, \mathrm{PG}(n-3, q^n) \rangle$. Let $\mathrm{PG}(1, q^n) = U$ be a line of

$PG(n-1, q^n)$ which is skew to $PG(n-3, q^n)$. Then there is a linear isomorphism θ of M onto U which maps the set of the intersections $M \cap PG(n-2, q^n)$, with $PG(n-2, q^n) = \overline{PG(n-2, q)}$, onto the set of the intersections $U \cap \langle z, PG(n-3, q^n)\rangle$, with z a point of $PG(n-1, q)$.

We embed $PG(n-1, q)$ in a $PG(n, q)$, and in the extension $PG(n, q^n)$ of $PG(n, q)$ we choose a $PG(2, q^n)$ with $PG(2, q^n) \cap PG(n-1, q^n) = U$. Now we project $PG(n, q)$ from $PG(n-3, q^n)$ onto the plane $PG(2, q^n)$. Assume, by way of contradiction, that the points z_1, z_2, with $z_1 \neq z_2$, of $PG(n, q) - PG(n-1, q)$ are projected onto the same point z of $PG(2, q^n)$. Then $z_1 z_2$ has a unique point in common with $PG(n-1, q)$ which also belongs to $PG(n-3, q^n)$. Hence the line M is contained in the extension $PG(n-2, q^n)$ of a hyperplane $PG(n-2, q)$ of $PG(n-1, q)$, a contradiction. It follows that the q^n points z of $PG(n, q) - PG(n-1, q)$ are projected onto q^n distinct points z' of $PG(2, q^n) - U$. If $z'_1, z'_2 \in PG(2, q^n) - U$, with $z'_1 \neq z'_2$, are such projections, then the set of all intersections $U \cap z'_1 z'_2$ is exactly the set of all intersections $U \cap <z, PG(n-3, q^n)>$, with z a point of $PG(n-1, q)$.

We now embed $PG(2, q^n)$ in a $PG(3, q^n)$, and in $PG(3, q^n)$ we choose a plane π in such a way that $\pi \cap PG(2, q^n) = U$. Assume, by way of contradiction, that the set $\mathcal{M}$ in the statement of the theorem does not contain a point which belongs to exactly q^n elements of V. If $\mathcal{M}^\theta = \tilde{\mathcal{M}}$, then there exists a non-singular conic $\mathcal{C}$ in π, with $\mathcal{C} \cap U = \emptyset$, such that $\tilde{\mathcal{M}}$ is the set of all exterior points of $\mathcal{C}$ on U; see Thas [1981]. Dualizing in $PG(3, q^n)$, we see that with the q^n projections $z' \in PG(2, q^n) - U$ there correspond q^n planes which intersect some quadratic cone K in q^n mutually disjoint non-singular conics (the points of K different from its vertex v correspond with the planes of $PG(3, q^n)$ which intersect π in a tangent line of $\mathcal{C}$; the point v corresponds with the plane π). Hence there arises a flock F of K. As all points z' are coplanar, the corresponding planes all contain a common point w. As the set of all intersections $M \cap PG(n-2, q^n)$ is not a singleton all points z' are not collinear, and so by Thas [1987] the point w necessarily is an exterior point of K. It follows that $PG(2, q^n)$ contains 2 points of $\mathcal{C}$, that is $|U \cap \mathcal{C}| = 2$, a contradiction.

We conclude that the set $\mathcal{M}$ contains exactly 2 points which belong to exactly q^n elements of V. ∎

LEMMA 6.5. *Consider a non-classical* TGQ $T(n, 2n, q) = T(O)$, q *odd, with* O *an egg which is good at* $PG(n-1, q)$. *Assume also that the set* $\mathcal{W}$ *is contained in a* $PG(4, q^n)$. *If* s_1, s_2 *are the points of* $\mathcal{M}$ *which belong to exactly* q^n *elements of* V, *then* s_1, s_2 *are conjugate with respect to the* n*th extension* $GF(q^n)$ *of* $GF(q)$, *that is,* s_1, s_2 *are in the same orbit of the group* $\langle\theta\rangle$ *with* θ *the collineation of order* n *of* $PG(n-1, q^n)$ *which fixes* $PG(n-1, q)$ *pointwise.*

Proof. Let $s_1(x_0, x_1, ..., x_{n-1})$ and $s_2(y_0, y_1, ..., y_{n-1})$ in $\mathrm{PG}(n-1, q^n)$. As each point $s \in \mathscr{M}$ belongs to a conic $\mathscr{C} \in V$, the space $\langle s, s^\theta, s^{\theta^2}, ..., s^{\theta^{n-1}} \rangle$ is $(n-1)$-dimensional. So the extension $\overline{\mathrm{PG}(n-2, q)}$ of any hyperplane $\mathrm{PG}(n-2, q)$ of $\mathrm{PG}(n-1, q)$ intersects $M = s_1 s_2$ in a point not in $\mathscr{M}$. As $\mathscr{M}$ is the projection onto M of a non-singular conic $\mathscr{K}$ from an exterior point of $\mathscr{K}$, we have that the cross-ratio $\{s_1, s_2; v, w\} = \square$ for any two distinct points v, w not in $\mathscr{M}$.

The hyperplane $X_i = 0$ of $\mathrm{PG}(n-1, q^n)$ intersects M in the point

$$v_i(x_0 y_i - x_i y_0, ..., x_{i-1} y_i - x_i y_{i-1}, 0, x_{i+1} y_i - x_i y_{i+1}, x_n y_i - x_i y_n),$$

$i = 0, 1, ..., n$. As $\{s_1, s_2; v_i, v_j\} = \{(1, 0), (0, 1); (y_i, -x_i), (y_j, -x_j)\}$, we have $\{s_1, s_2; v_i, v_j\} = (y_j x_i)/(y_i x_j) = \square$ for all $i, j = 0, 1, ..., n$ with $i \neq j$. Hence $x_i y_i x_j y_j = \square$ for all $i, j = 0, 1, ..., n$ with $i \neq j$. Next, let

$$\delta\colon a_0 X_0 + a_1 X_1 + \cdots + a_{n-1} X_{n-1} = 0, \qquad a_i \in \mathrm{GF}(q),$$

be any hyperplane of $\mathrm{PG}(n-1, q)$. If v is the common point of $\bar{\delta}$ and M, then

$$\begin{aligned} v = (a_0 y_0 + a_1 y_1 + \cdots + a_{n-1} y_{n-1}) s_1 \\ + (-a_0 x_0 - a_1 x_1 - \cdots - a_{n-1} x_{n-1}) s_2. \end{aligned}$$

If $\delta' \neq \delta$ is a second hyperplane of $\mathrm{PG}(n-1, q)$, with

$$\delta'\colon b_0 X_0 + b_1 X_1 + \cdots + b_{n-1} X_{n-1} = 0, \qquad b_i \in \mathrm{GF}(q),$$

and if w is the common point of $\overline{\delta'}$ and M, then

$$\begin{aligned} w = (b_0 y_0 + b_1 y_1 + \cdots + b_{n-1} y_{n-1}) s_1 \\ + (-b_0 x_0 - b_1 x_1 - \cdots - b_{n-1} x_{n-1}) s_2. \end{aligned}$$

Hence

$$\{s_1, s_2; v, w\} = \left\{(1, 0), (0, 1); \left(\sum_i a_i y_i, -\sum_i a_i x_i\right), \left(\sum_i b_i y_i, -\sum_i b_i x_i\right)\right\}.$$

Consequently

$$\{s_1, s_2; v, w\} = \left(\sum_i b_i y_i \sum_i a_i x_i\right) \Big/ \left(\sum_i b_i x_i \sum_i a_i y_i\right) = \square,$$

and so

$$\sum_i a_i x_i \sum_i a_i y_i \sum_i b_i x_i \sum_i b_i y_i = \square.$$

As the points s_1, s_2 do not belong to the hyperplane $X_0 = 0$ of $\mathrm{PG}(n-1, q^n)$, we may assume that $x_0 = 1 = y_0$. Then

$$x_i y_i = \square, \qquad i = 1, 2, ..., n-1,$$

and

$$\left(a_0 + \sum_{i=1}^{n-1} a_i x_i\right)\left(a_0 + \sum_{i=1}^{n-1} a_i y_i\right) = \square, \tag{1}$$

with $a_0, a_1, ..., a_{n-1} \in \mathrm{GF}(q)$ and $(a_1, a_2, ..., a_{n-1}) \neq (0, 0, ..., 0)$.

If the elements $1, x_1, x_2, ..., x_{n-1}$ are linearly dependent over $\mathrm{GF}(q)$, then $l_0 + l_1 x_1 + l_2 x_2 + \cdots + l_{n-1} x_{n-1} = 0$ for some $l_0, l_1, ..., l_{n-1} \in \mathrm{GF}(q)$ with $(l_0, l_1, ..., l_{n-1}) \neq (0, 0, ..., 0)$. Hence the point s_1 belongs to the extension of the hyperplane $l_0 X_0 + l_1 X_1 + \cdots + l_{n-1} X_{n-1} = 0$ of $\mathrm{PG}(n-1, q)$, a contradiction. Consequently the elements $1, x_1, x_2, ..., x_{n-1}$ are linearly independent over $\mathrm{GF}(q)$. It folllows that

$$\mathrm{GF}(q^n) = \{c_0 + c_1 x_1 + c_2 x_2 + \cdots + c_{n-1} x_{n-1} \parallel c_i \in \mathrm{GF}(q)\}.$$

Analogously,

$$\mathrm{GF}(q^n) = \{c_0 + c_1 y_1 + c_2 y_2 + \cdots + c_{n-1} y_{n-1} \parallel c_i \in \mathrm{GF}(q)\}.$$

Now we consider the mapping φ defined by

$$\begin{aligned}\varphi\colon \mathrm{GF}(q^n) \to \mathrm{GF}(q^n),\ & c_0 + c_1 x_1 + \cdots + c_{n-1} x_{n-1} \\ & \to c_0 + c_1 y_1 + \cdots + c_{n-1} y_{n-1}.\end{aligned}$$

Clearly φ is a permutation of $\mathrm{GF}(q^n)$.

Let h, k be distinct elements of $\mathrm{GF}(q^n)$, with

$$h = a_0 + a_1 x_1 + \cdots + a_{n-1} x_{n-1},$$

$$k = b_0 + b_1 x_1 + \cdots + b_{n-1} x_{n-1}.$$

Then

$$(h^\varphi - k^\varphi)(h - k) = \left(a_0 - b_0 + \sum_{i=1}^{n-1} (a_i - b_i)\, y_i\right)\left(a_0 - b_0 + \sum_{i=1}^{n-1} (a_i - b_i) x_i\right).$$

By (1) we have

$$(h^{\varphi} - k^{\varphi})(h - k) = \square.$$

Also $0^{\varphi} = 0$ and $1^{\varphi} = 1$. Now by a theorem of Carlitz [1960]

$$\varphi\colon \mathrm{GF}(q^n) \to \mathrm{GF}(q^n), \qquad h \mapsto h^{p^r},$$

with $q = p^h$, p prime, and $1 \leqslant r \leqslant hn - 1$ ($r \neq 0$ as $s_1 \neq s_2$). For any $a \in \mathrm{GF}(q)$ we have

$$ay_1 = (ax_1)^{\varphi} = (ax_1)^{p^r} = a^{p^r} x_1^{p^r} = a^{p^r} x_1^{\varphi} = a^{p^r} y_1.$$

As $y_1 \neq 0$, we have

$$a^{p^r} = a, \qquad \text{for all} \quad a \in \mathrm{GF}(q).$$

It follows that $r = dh$ for some d, with $1 \leqslant d \leqslant n - 1$, and so

$$\varphi\colon \mathrm{GF}(q^n) \to \mathrm{GF}(q^n), \qquad h \mapsto h^{q^d}.$$

As

$$\theta\colon \mathrm{PG}(n-1, q^n) \to \mathrm{PG}(n-1, q^n),\ (z_0, z_1, ..., z_{n-1}) \mapsto (z_0^q, z_1^q, ..., z_{n-1}^q),$$

we have

$$s_2 = (1, y_1, ..., y_{n-1}) = (1, x_1^{\varphi}, ..., x_{n-1}^{\varphi}) = (1, x_1^{q^d}, ..., x_{n-1}^{q^d}) = s_1^{\theta d}.$$

We conclude that s_1 and s_2 are in the same orbit of the group $\langle \theta \rangle$. ∎

THEOREM 6.6. *Consider a non-classical* TGQ $T(n, 2n, q) = T(O)$, *q odd, with O an egg which is good at* $\mathrm{PG}(n-1, q)$. *If the set $\mathcal{W}$ is contained in a* $\mathrm{PG}(4, q^n)$, *then the translation dual $T(O^*)$ of $T(O)$ is good at τ with τ the tangent space of O at* $\mathrm{PG}(n-1, q)$.

Proof. We use the notations of Lemma 6.5.

The point s_i belongs to q^n elements of V and all these conics have a common tangent line U_i at s_i, $i = 1, 2$; this line U_i does not belong to $\mathrm{PG}(n-1, q^n)$, $i = 1, 2$. The collineation of order n of $\mathrm{PG}(4n-1, q^n)$ which fixes $\mathrm{PG}(4n-1, q)$ pointwise (and so is an extension of θ), will also be denoted by θ. If $\mathscr{C} \in V$ contains s_i, then $\langle U_i, U_i^{\theta}, U_i^{\theta^2}, ..., U_i^{\theta^{n-1}} \rangle$ is the tangent space at $\mathrm{PG}(n-1, q)$ of the $[n-1]$-oval $O(n, n, q)$ defined by $\mathscr{C}$, $i \in \{1, 2\}$. If $\theta^d = \theta'$ then the line $U_1^{\theta'}$ contains the point $s_1^{\theta'} = s_2$, and the line $U_2^{\theta'^{-1}}$ contains the point s_1. If $U_2 = U_1^{\theta'}$, then the q^n $[n-1]$-ovals defined by the conics of V through s_1 and the q^n $[n-1]$-ovals defined by the conics of V through s_2 all have the same tangent space at $\mathrm{PG}(n-1, q)$, a contradiction. So $U_2 \neq U_1^{\theta'}$ and $U_2^{\theta'^{-1}} \neq U_1$. Let $\kappa = \langle U_1, U_2^{\theta'^{-1}} \rangle$; then $\kappa^{\theta'} = \langle U_1^{\theta'}, U_2 \rangle$.

Consider an element $\mathrm{PG}^{(i)}(n-1, q)$ of $O(n, 2n, q) = O$, and the tangent space τ_i of O at $\mathrm{PG}^{(i)}(n-1, q)$. Let r_i be the point of $\mathscr{W}$ which corresponds with $\mathrm{PG}^{(i)}(n-1, q)$, and let $\mathscr{C}_j$ be the conic of V which contains r_i and s_j, $j = 1, 2$. Let Z_j be the tangent line of $\mathscr{C}_j$ at $r_i, j = 1, 2$. Then $\bar{\tau}_i = \langle Z_1, Z_2, Z_1^{\theta}, Z_2^{\theta}, ..., Z_1^{\theta^{n-1}}, Z_2^{\theta^{n-1}} \rangle$. If $U_1 \cap Z_1 = \{z_1\}$ and $U_2 \cap Z_2 = \{z_2\}$, then $\bar{\tau}_i \cap \kappa = z_1 z_2^{\theta'^{-1}}$ and $\bar{\tau}_i \cap \kappa^{\theta'} = z_2 z_1^{\theta'}$. Let $\bar{\tau}_i \cap \kappa = Y_i$; then $\bar{\tau}_i \cap \kappa^{\theta'} = Y_i^{\theta'}$. We have $\bar{\tau} \cap \bar{\tau}_i = \langle Y_i, Y_i^{\theta}, ..., Y_i^{\theta^{n-1}} \rangle$. As $\bar{\tau} \cap \bar{\tau}_i \neq \bar{\tau} \cap \bar{\tau}_j$ for $i \neq j$, there holds $Y_i \neq Y_j$ for $i \neq j$. So with the q^{2n} elements of $O - \{\mathrm{PG}(n-1, q)\}$ there correspond the q^{2n} lines of κ not through s_1. Let t be a point of κ, with $t \neq s_1$, and let $Y_{i_1}, Y_{i_2}, ..., Y_{i_{q^n}}$ be the q^n lines of κ not containing s_1. The extensions $\mathrm{PG}^{(i_1)}(n-1, q^n), \mathrm{PG}^{(i_2)}(n-1, q^n), ..., \mathrm{PG}^{(i_{q^n})}(n-1, q^n)$ of the elements $\mathrm{PG}^{(i_1)}(n-1, q), \mathrm{PG}^{(i_2)}(n-1, q), ..., \mathrm{PG}^{(i_{q^n})}(n-1, q) \in O - \{\mathrm{PG}(n-1, q)\}$ all contain the point t, hence they all contain the $(n-1)$-dimensional space $\bar{\psi} = \langle t, t^{\theta}, t^{\theta^2}, ..., t^{\theta^{n-1}} \rangle \subset \bar{\tau}$. So ψ is contained in $q^n + 1$ tangent spaces of O. Let $\mathrm{PG}^{(i)}(n-1, q), \mathrm{PG}^{(j)}(n-1, q) \in O - \{\mathrm{PG}(n-1, q)\}$, $i \neq j$, let τ_i be the tangent space of O at $\mathrm{PG}^{(i)}(n-1, q)$, and let τ_j be the tangent space of O at $\mathrm{PG}^{(j)}(n-1, q)$. If $\bar{\tau} \cap \bar{\tau}_i = Y_1$, $\bar{\tau} \cap \bar{\tau}_j = Y_j$ and $Y_i \cap Y_j = \{t\}$, then $\psi = \tau \cap \tau_i \cap \tau_j$ with $\bar{\psi} = \langle t, t^{\theta}, t^{\theta^2}, ..., t^{\theta^{n-1}} \rangle$. So $\tau \cap \tau_i \cap \tau_j$ is contained in exactly $q^n + 1$ tangent spaces of O. Hence the translation dual $T(O^*)$ of $T(O)$ is good at τ. ∎

THEOREM 6.7. *Consider a non-classical* GQ $\mathscr{S}(F)$ *of order* (q^2, q), *q odd, arising from the flock F and assume that the point-line dual* $\widehat{\mathscr{S}(F)}$ *of* $\mathscr{S}(F)$ *is a* TGQ $T(n, 2n, q) = T(O)$. *Then the translation dual $T(O^*)$ is good at the tangent space τ of O at the element ζ of O, where ζ is the point (∞) of $\mathscr{S}(F)$. If the corresponding set $\mathscr{W}$ is contained in a* $\mathrm{PG}(4, q^n)$, *then the flock F is a Kantor flock.*

Proof. As $\mathscr{S}(F)$ is not classical, by 3.3 of Payne and Thas [1991] the point (∞) of $\mathscr{S}(F)$ is a line of type (b) of $T(O)$, that is, a line which is an element $\mathrm{PG}(n-1, q) = \zeta$ of O. By Section 4 $T(O)$ satisfies Property (G) at $\mathrm{PG}(n-1, q)$ and then by Theorem 3.5 the translation dual $T(O^*)$ of $T(O)$ is good at the tangent space τ of O at $\mathrm{PG}(n-1, q)$. As for $T(O^*)$ the corresponding set $\mathscr{W}$ is contained in a $\mathrm{PG}(4, q^n)$, by Theorem 6.6 the TGQ $T(O)$ is good at $\mathrm{PG}(n-1, q)$. Now by Corollary 4.2 the flock F is a Kantor flock. ∎

THEOREM 6.8. *Consider a non-classical* TGQ $T(n, 2n, q) = T(O)$, *q odd, with* $O = O(n, 2n, q) = \{\mathrm{PG}(n-1, q), \mathrm{PG}^{(1)}(n-1, q), ..., \mathrm{PG}^{(q^{2n})}(n-1, q)\}$. *Assume that O is good at* $\mathrm{PG}(n-1, q)$ *and that the set $\mathscr{W}$ is contained in* *a* PG$(5, q^n)$, *but not in a* PG$(4, q^n)$. *If* $\mathscr{P}$ *is the set of all common points of the plane* $\mu = \mathrm{PG}(n-1, q^n) \cap \mathrm{PG}(5, q^n)$ *with conics of* V, *then* $\mathscr{W} \cup \mathscr{P}$ *is a quadric Veronesean* $\mathscr{V}_2^4$ *in* PG$(5, q^n)$.

Proof. Let $\mathscr{C}, \mathscr{C}' \in V$, with $\mathscr{C} \neq \mathscr{C}'$, and assume that $\mathscr{C} \cap \mu = \mathscr{C}' \cap \mu = \{s\}$. If $|\mathscr{C} \cap \mathscr{C}'| = 2$, then $\mathscr{C}$ and $\mathscr{C}'$ are contained in a 3-dimensional space PG$(3, q^n)$ which contains $\mathscr{C}$ and more than one point $r_d \in \mathscr{W} - \mathscr{C}$, a contradiction (see section preceding (a) in the proof of Theorem 6.1). Hence $\mathscr{C} \cap \mathscr{C}' = \{s\}$.

Consider a conic $\mathscr{C} \in V$ and all conics $\mathscr{C}' \in V - \{\mathscr{C}\}$ containing a point of $\mathscr{C} - \mu$. As no 3-dimensional space containing $\mathscr{C}$ contains more than one point $r_d \in \mathscr{W} - \mathscr{C}$, the conics $\mathscr{C}$ and $\mathscr{C}'$ generate a 4-dimensional space. Assume, by way of contradiction, that the 4-dimensional spaces π'_1 and π'_2, respectively defined by $\mathscr{C}, \mathscr{C}'_1$ and $\mathscr{C}, \mathscr{C}'_2$ with $\mathscr{C}'_1, \mathscr{C}'_2 \in V - \{\mathscr{C}\}$, $|\mathscr{C} \cap \mathscr{C}'_1 \cap \mathscr{W}| \geqslant 1$, $|\mathscr{C} \cap \mathscr{C}'_2 \cap \mathscr{W}| \geqslant 1$ and $\mathscr{C}'_1 \neq \mathscr{C}'_2$, coincide. Let $r_v \notin \mathscr{C} \cup \mathscr{C}'_1 \cup \mathscr{C}'_2$, $r_v \in \mathscr{W}$. In the plane η_1 (we still use the notations introduced at the beginning of the proof of Theorem 6.1) there correspond with $\mathscr{C}, \mathscr{C}'_1, \mathscr{C}'_2$ distinct lines K, K'_1, K'_2, and with r_v the point $\tilde{r}_v$. Now let K'' be a line of η_1 through $\tilde{r}_v$ which intersects K, K'_1, K'_2 in distinct points of $\eta_1 - \bar{\tau}$. With K'' there corresponds a conic $\mathscr{C}''$ of V through r_v which intersects $\mathscr{C} \cup \mathscr{C}'_1 \cup \mathscr{C}'_2$ in at least three distinct points. Hence $\mathscr{C}''$ belongs to π'_1, and so r_v belongs to π'_1. We conclude that $\mathscr{W} \subset \pi'_1$, a contradiction. Consequently $\pi'_1 \neq \pi'_2$, that is, the q^{2n} conics $\mathscr{C}'$ define q^{2n} 4-dimensional spaces. Let π' be the 4-dimensional space generated by $\mathscr{C}$ and $\mathscr{C}'$, with $\mathscr{C}' \in V - \{\mathscr{C}\}$ and $|\mathscr{C} \cap \mathscr{C}' \cap \mathscr{W}| \geqslant 1$. Assume, by way of contradiction, that $\pi' \cap \mathscr{W}$ contains a point r_d not in $\mathscr{C} \cup \mathscr{C}'$. In the plane η_1 there correspond with $\mathscr{C}, \mathscr{C}'$ distinct lines K, K', and with r_d the point $\tilde{r}_d$. Now let K'' be a line of η_1 through $\tilde{r}_d$ which intersects K, K' in distinct points of $\eta_1 - \bar{\tau}$. With K'' there corresponds a conic $\mathscr{C}''$ of V through r_d which intersects each of $\mathscr{C}, \mathscr{C}'$ in a point not in μ. So $\mathscr{C}''$ belongs to π', and consequently the 4-dimensional spaces generated by $\mathscr{C}, \mathscr{C}'$, respectively $\mathscr{C}, \mathscr{C}''$, coincide, a contradiction. It follows that $\pi' \cap \mathscr{W} = (\mathscr{C} \cup \mathscr{C}') \cap \mathscr{W}$.

Next, assume that the conics $\mathscr{C}_1, \mathscr{C}_2 \in V$, with $\mathscr{C}_1 \neq \mathscr{C}_2$, define lines of the plane η_1 which intersect $\bar{\tau}$ in a common point. Suppose, by way of contradiction, that $\mathscr{C}_1 \cap \mathscr{C}_2 = \varnothing$. Let γ_i be the plane of $\mathscr{C}_i$, $i = 1, 2$. Assume first that $\langle \gamma_1, \gamma_2 \rangle$ is 5-dimensional. Then the q^{2n} 4-dimensional spaces of the preceding section defined by $\mathscr{C}_1$, intersect γ_2 in q^{2n} distinct lines. Hence one of these spaces, say δ, intersects $\mathscr{C}_2$ in two distinct points of $\mathscr{W}$. If $\mathscr{C}_1, \mathscr{C}'$ are the conics of V in δ, then $\mathscr{C}'$ contains two distinct points of $\mathscr{C}_2 \cap \mathscr{W}$, clearly a contradiction. Consequently $\langle \gamma_1, \gamma_2 \rangle$ is 4-dimensional. If $r_i \in \mathscr{C}_2 \cap \mathscr{W}$, then the 3-dimensional space generated by $\mathscr{C}_1$ and r_i contains just one point of $\mathscr{C}_2 \cap \mathscr{W}$, hence intersects γ_2 in the tangent line of $\mathscr{C}_2$ at r_i. This tangent line contains the common point l of γ_1 and γ_2. Consequently

q^n tangent lines of $\mathscr{C}_2$ are concurrent, a contradiction as q is odd. So we conclude that $|\mathscr{C}_1 \cap \mathscr{C}_2| = 1$, that is, $\mathscr{C}_1 \cap \mu = \mathscr{C}_2 \cap \mu$.

From the foregoing follows that the $q^{2n} + q^n$ conics of V intersect μ in exactly $q^n + 1$ distinct points; the set of these $q^n + 1$ points will be denoted by $\mathscr{P}$.

Now we will use Theorem 25.2.13 of Hirschfeld and Thas [1991] to prove that $\mathscr{P} \cup \mathscr{W}$ is a quadric Veronesean in PG$(5, q^n)$. Call $\mathscr{L}$ the set with as elements the plane μ and the $q^{2n} + q^n$ planes of the conics of V. Any point of $\mathscr{P} \cup \mathscr{W}$ belongs to $q^n + 1$ elements of $\mathscr{L}$, hence no point of $\mathscr{P} \cup \mathscr{W}$ belongs to all elements of $\mathscr{L}$. Relying also on the section preceding (a) in the proof of Theorem 6.1 we have : (i) if $\mathscr{C} \cap \mu = \mathscr{C}' \cap \mu = \{s\}$, with $\mathscr{C}, \mathscr{C}' \in V$ and $\mathscr{C} \neq \mathscr{C}'$, then the planes of $\mathscr{C}$ and $\mathscr{C}'$ have just s in common, (ii) if $\mathscr{C} \cap \mathscr{C}' \cap \mu = \emptyset$, with $\mathscr{C}, \mathscr{C}' \in V$ and $\mathscr{C} \neq \mathscr{C}'$, then the common point of $\mathscr{C}$ and $\mathscr{C}'$ is the unique common point of the planes of $\mathscr{C}$ and $\mathscr{C}'$, and (iii) for any $\mathscr{C} \in V$, the plane of $\mathscr{C}$ contains exactly one point of the plane μ. It follows that any two distinct elements of $\mathscr{L}$ have exactly one point in common. Now we consider any three distinct elements π_1, π_2, π_3 of $\mathscr{L}$. First, assume that $\pi_i \neq \mu$, with $i = 1, 2, 3$. Let $\mathscr{C}_i$ be the element of V in π_i, with $i = 1, 2, 3$. Assume, by way of contradiction, that π_1, π_2, π_3 generate a PG$(4, q^n)$. Let $r_v \notin \mathscr{C}_1 \cup \mathscr{C}_2 \cup \mathscr{C}_3$, $r_v \in \mathscr{W}$. In the plane η_1 there correspond with $\mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_3$ distinct lines K_1, K_2, K_3 and with r_v the point $\tilde{r}_v$. Now let K' be a line of η_1 through $\tilde{r}_v$ which intersects K_1, K_2, K_3 in distinct points of $\eta_1 - \bar{\tau}$. With K' there corresponds a conic $\mathscr{C}'$ of V through r_v which intersects $\mathscr{C}_1 \cup \mathscr{C}_2 \cup \mathscr{C}_3$ in three distinct points. Hence $\mathscr{C}'$ belongs to PG$(4, q^n)$, hence r_v belongs to PG$(4, q^n)$, so $\mathscr{W} \subset$ PG$(4, q^n)$, a contradiction. It follows that $\langle \pi_1, \pi_2, \pi_3 \rangle =$ PG$(5, q^n)$. Next, suppose that $\pi_3 = \mu$. Let $\mathscr{C}_i$ be the element of V in π_i, with $i = 1, 2$. Assume, by way of contradiction, that PG$(4, q^n) = \langle \pi_1, \pi_2 \rangle$ also contains μ. Let $r_v \notin \mathscr{C}_1 \cup \mathscr{C}_2$, $r_v \in \mathscr{W}$. In the plane η_1 there correspond with $\mathscr{C}_1, \mathscr{C}_2$ distinct lines K_1, K_2 and with r_v the point $\tilde{r}_v$. Now let K' be a line of η_1 through $\tilde{r}_v$ which intersects K_1, K_2 in distinct points of $\eta_1 - \bar{\tau}$. With K' there corresponds a conic $\mathscr{C}'$ of V through r_v which intersects $\mathscr{C}_1 \cup \mathscr{C}_2$ in two distinct points of $\mathscr{W}$ and which contains a point of μ not on $\mathscr{C}_1 \cup \mathscr{C}_2$. Hence $\mathscr{C}'$ belongs to PG$(4, q^n)$, hence r_v belongs to PG$(4, q^n)$, so $\mathscr{W} \subset$ PG$(4, q^n)$, a contradiction. Now by Theorem 25.2.13 of Hirschfeld and Thas [1991], which is a corollary of a theorem by Tallini [1958], $\mathscr{L}$ is the set of all conic planes of a quadric Veronesean $\mathscr{V}_2^4$ in PG$(5, q^n)$. Also, $\mathscr{V}_2^4$ is the set of all intersections of pairs of distinct elements of $\mathscr{L}$, that is, $\mathscr{V}_2^4 = \mathscr{W} \cup \mathscr{P}$. ∎

MAIN THEOREM 6.9. *Consider a* TGQ $T(n, 2n, q) = T(O)$, q *odd, with* $O = O(n, 2n, q) = \{\mathrm{PG}(n-1, q), \mathrm{PG}^{(1)}(n-1, q), ..., \mathrm{PG}^{(q^{2n})}(n-1, q)\}$. *If* O *is good at* PG$(n-1, q)$, *then we have one of the following.*

(a) *There exists a* $\mathrm{PG}(3, q^n)$ *in the extension* $\mathrm{PG}(4n-1, q^n)$ *of the space* $\mathrm{PG}(4n-1, q)$ *of* $O(n, 2n, q)$ *which has exactly one point in common with each of the spaces* $\mathrm{PG}(n-1, q^n)$, $\mathrm{PG}^{(1)}(n-1, q^n)$, ..., $\mathrm{PG}^{(q^{2n})}(n-1, q^n)$. *The set of these* $q^{2n}+1$ *points is an elliptic quadric of* $\mathrm{PG}(3, q^n)$ *and* $T(O)$ *is isomorphic to the classical* GQ $Q(5, q^n)$.

(b) *We are not in Case* (a) *and there exists a* $\mathrm{PG}(4, q^n)$ *in* $\mathrm{PG}(4n-1, q^n)$ *which intersects* $\mathrm{PG}(n-1, q^n)$ *in a line* M *and which has exactly one point* r_i *in common with any space* $\mathrm{PG}^{(i)}(n-1, q^n)$, $i = 1, 2, ..., q^{2n}$. *Let* $\mathscr{W} = \{r_i \parallel i = 1, 2, ..., q^{2n}\}$ *and let* $\mathscr{M}$ *be the set of all common points of* M *and the conics which contain exactly* q^n *points of* $\mathscr{W}$. *Then the set* $\mathscr{W} \cup \mathscr{M}$ *is the projection of a quadric Veronesean* $\mathscr{V}_2^4$ *from a point* p *in a conic plane of* $\mathscr{V}_2^4$ *onto a hyperplane* $\mathrm{PG}(4, q^n)$; *the point* p *is an exterior point of the conic of* $\mathscr{V}_2^4$ *in the conic plane. Also, if the point-line dual of the translation dual of* $T(O)$ *is a flock* GQ $\mathscr{S}(F)$, *then* F *is a Kantor flock.*

(c) *We are not in Cases* (a) *and* (b) *and there exists a* $\mathrm{PG}(5, q^n)$ *in* $\mathrm{PG}(4n-1, q^n)$ *which intersects* $\mathrm{PG}(n-1, q^n)$ *in a plane* μ *and which has exactly one point* r_i *in common with any space* $\mathrm{PG}^{(i)}(n-1, q^n)$, $i = 1, 2, ..., q^{2n}$. *Let* $\mathscr{W} = \{r_i \parallel i = 1, 2, ..., q^{2n}\}$ *and let* $\mathscr{P}$ *be the set of all common points of* μ *and the conics which contain exactly* q^n *points of* $\mathscr{W}$. *Then the set* $\mathscr{W} \cup \mathscr{P}$ *is a quadric Veronesean in* $\mathrm{PG}(5, q^n)$.

Proof. In Case (a), $T(n, 2n, q)$ is the interpretation over $\mathrm{GF}(q)$ of $T(1, 2, q^n) \cong Q(5, q^n)$. If we are not in Case (a), then the theorem directly follows from Theorems 6.1, 6.2, 6.4, 6.7 and 6.8. ▮

Remark. If we project the Veronesean $\mathscr{V}_2^4$ in $\mathrm{PG}(5, q^n)$ onto some $\mathrm{PG}(3, q^n) \subset \mathrm{PG}(5, q^n)$ from a line N which intersects $\mathscr{V}_2^4$ in two points of $\mathrm{PG}(5, q^{2n}) - \mathrm{PG}(5, q^n)$ which are conjugate with respect to the extension $\mathrm{GF}(q^{2n})$ of $\mathrm{GF}(q^n)$, then we obtain an elliptic quadric of $\mathrm{PG}(3, q^n)$.

Conjectures. (a) In Case (c) of the Main Theorem and if the point-line dual of the translation dual of $T(O)$ is a flock GQ, then the flock GQ is a Roman GQ of Payne (see Payne [1988], [1989]). As by the Main Theorem, and also by some other results not mentioned here, we have strong information on this class of GQ, we hope to prove shortly this conjecture. This would yield the complete classification of all TGQ of order (s, s^2), s odd and $s \neq 1$, for which the point-line dual is a flock TGQ.

(b) Any TGQ $T(O)$ of order (s, s^2) and $s \neq 1$, with O good at some element, is the point-line dual of the translation dual of a translation flock GQ.

Final remark. For s even Johnson [1987] proves that any TGQ for which the point-line dual is a flock GQ, is the classical GQ $Q(5, s)$. We conjecture that any TGQ $T(O)$ of order (s, s^2), s even and $s \neq 1$, with O good at some element, is the classical GQ $Q(5, s)$.

REFERENCES

A. Barlotti, Un' estensione del teorema di Segre-Kustaanheimo, *Boll. Un. Mat. Ital.* **10** (1955), 96–98.

L. Carlitz, A theorem on permutations in a finite field, *Proc. Amer. Math. Soc.* **11** (1960), 456–459.

J. W. P. Hirschfeld and J. A. Thas, "General Galois Geometries," Oxford Univ. Press, 1991.

N. L. Johnson, Semifield flocks of quadratic cones, *Simon Stevin* **61** (1987), 313–326.

G. Panella, Caratterizzazione delle quadriche di uno spazio (tridimensionale) lineare sopra un corpo finito, *Boll. Un. Mat. Ital.* **10** (1955), 507–513.

S. E. Payne, *q*-clans e quadrangoli generalizzati, *Sem. Geom. Combin.* **79** (1988), 1–16.

S. E. Payne, An essay on skew translation generalized quadrangles, *Geom. Dedicata* **32** (1989), 93–118.

S. E. Payne and J. A. Thas, "Finite Generalized Quadrangles," Pitman, Boston, 1984.

S. E. Payne and J. A. Thas, Conical flocks, partial flocks, derivation, and generalized quadrangles, *Geom. Dedicata* **38** (1991), 229–243.

G. Tallini, Una proprietà grafica caratteristica delle superficie di Veronese negli spazi finiti (Note I; II), *Atti Accad. Naz. Lincei Rend.* **24** (1958), 19–23; 135–138.

J. A. Thas, Some results on quadrics and a new class of partial geometries, *Simon Stevin* **55** (1981), 129–139.

J. A. Thas, Generalized quadrangles and flocks of cones, *European J. Combin.* **8** (1987), 441–452.

J. A. Thas, Generalized quadrangles of order (s, s^2), I, *J. Combin. Theory Ser. A* **67** (1994), 140–160.

J. A. Thas, Projective geometry over a finite field, *in* "Handbook of Incidence Geometry, Buildings and Foundations" (F. Buekenhout, Ed.), Chapt. 7, pp. 295–348, Amsterdam, North-Holland, 1995.

J. A. Thas and F. De Clerck, Partial geometries satisfying the axiom of Pasch, *Simon Stevin* **51** (1977), 123–137.

J. A. Thas and H. Van Maldeghem, Generalized quadrangles and the axiom of Veblen, *in* "Geometry, Combinatorial Designs, and Related Structures," Proc. of the First Pythagorean Conference, Cambridge Univ. Press, Cambridge, 1995, to appear.