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ORIGINAL ARTICLE

Solution for a problem of linear plane elasticity with mixed boundary conditions on an ellipse by the method of boundary integrals

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Abstract A numerical boundary integral scheme is proposed for the solution of the system of field equations of plane, linear elasticity in stresses for homogeneous, isotropic media in the domain bounded by an ellipse under mixed boundary conditions. The stresses are prescribed on one half of the ellipse, while the displacements are given on the other half. The method relies on previous analytical work within the Boundary Integral Method [1,2].

The considered problem with mixed boundary conditions is replaced by two subproblems with homogeneous boundary conditions, one of each type, having a common solution. The equations are reduced to a system of boundary integral equations, which is then discretized in the usual way and the problem at this stage is reduced to the solution of a rectangular linear system of algebraic equations. The unknowns in this system of equations are the boundary values of four harmonic functions which define the full elastic solution inside the domain, and the unknown boundary values of stresses or displacements on proper parts of the boundary.

On the basis of the obtained results, it is inferred that the tangential stress component on the fixed part of the boundary has a singularity at each of the two separation points, thought to be of logarithmic type. A tentative form for the singular solution is proposed to calculate the full solution in bulk directly from the given boundary conditions using the well-known Boundary Collocation Method. It is shown that this addition substantially decreases the error in satisfying the boundary conditions on some interval not containing the singular points.

The obtained results are discussed and boundary curves for unknown functions are provided, as well as three-dimensional plots for quantities of practical interest. The efficiency of the used

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numerical schemes is discussed, in what concerns the number of boundary nodes needed to calculate the approximate solution.

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1. Introduction

The plane problem of the linear Theory of Elasticity has received considerable attention long ago as being a simplified alternative to the more realistic three-dimensional problems of practical interest. A large class of two-dimensional problems has been tackled using various analytical techniques. Due to the increasing mathematical difficulties encountered in the theoretical studies of problems involving arbitrary boundary shapes or complicated boundary conditions, many purely numerical techniques have been developed in the past few decades, which rely on finite difference or finite element techniques. In both methods, the natural boundary of the body is usually replaced by an outer polygonal shape which involves a multitude of corner points and necessarily adds or deletes parts to the region occupied by the body. This, in turn, necessitates the application of boundary conditions on artificial boundaries, a fact that introduces additional inaccuracies into the solution. Minimizing the error requires large computing times.

Some of the disadvantages of the numerical techniques are overcome by the use of alternative, semi-analytical treatments based on Boundary Integral Formulations of the problem. Such approaches are usually classified under the general title of Boundary Integral Methods. They have the advantage of reducing the volume of calculations by considering, at one stage, only the boundary values of the unknown functions and then using them to find the complete solution in bulk. An extensive account of integral equation methods in potential theory and in elastostatics may be found in [3] and [4]. Natroshvili et al. [5] give a brief review of boundary integral methods as applied to the theory of micropolar elasticity. Constanda [6] investigates the use of integral equations of the first kind in plane elasticity. Atluri and Zhu [7] present a meshless local Petrov-Galerkin approach for solving problems of elastostatics. Sladek et al. [8] and Rui et al. [9] present meshless boundary integral methods for 2D elastodynamic problems. Elliotis et al. [10] present a boundary integral method for solving problems involving the biharmonic equation with crack singularities. Li et al. [11] present a numerical solution for models of linear elastostatics involving crack singularities.

The solution of plane problems of elasticity for isotropic media with mixed boundary conditions is a difficult task. Boundary methods may be useful in providing such solutions, especially when the geometry of the domain boundary is not simple. Several papers deal with such problems, either for Laplace's equation or for the biharmonic equation. Shmegera [12] finds exact solutions of non-stationary contact problems of elastodynamics for a half-plane with friction condition in the contact zone in a closed form. A new method of solution based on the use of Radon transform is used. Schiavone [13] presents integral solutions of mixed problems in plane strain elasticity with microstructure. Haller-Dintelman et al. [14] consider three-dimensional elliptic model problems for heterogeneous media, including mixed boundary conditions. Helsing [15] studies Laplace's equation under mixed boundary conditions and their solution by an integral equation method. Problems of elasticity are also considered. Lee et al. [17,16] study singular solutions at corners and cracks in linear elastostatics under mixed boundary conditions. Explicit solutions are obtained. Khuri [18] outlines a general method for finding well-posed boundary value problems for linear equations of mixed elliptic and hyperbolic type, which extends previous techniques. This method is then used to study a particular class of fully non-linear mixed type equations.

Abou-Dina and Ghaleb [1,2] proposed a method to deal with the static, plane problems of elasticity in stresses for homogeneous isotropic media occupying simply connected regions. The method relies on the representation of the biharmonic stress function in terms of two harmonic functions and on the well-known integral representation of harmonic functions expressed in real variables. This method was applied to a number of examples with boundary conditions of the first, or of the second type only, but the case of mixed conditions was not considered. Constanda [19] discusses Kupradze's method of approximate solution in linear elasticity. The same author [20] explains the advantages and convenience of the use of real variables due to its generality in dealing with the different forms of the boundary, unlike the approach based on the use of complex variables "where the essential ingredients of the solution must be constructed in full for every individual situation".

In the present paper, we propose a semi-analytical scheme for the solution of a mixed boundary-value problem of plane, linear elasticity for homogeneous, isotropic elastic bodies occupying a domain bounded by an ellipse. Part of the boundary is subjected to a given pressure, and the remaining part of the boundary is fixed. The initial problem with mixed boundary conditions is replaced by two subproblems with homogeneous boundary conditions, one of each type, having a common solution. Following the scheme presented in [1], the equations for each of these two subproblems are reduced to a system of boundary integral equations which are then discretized in the usual way, and the problem at this stage is reduced to the solution of a rectangular system of linear algebraic equations. The obtained results are discussed and graphs are given. In particular, we put in evidence the singular behavior of the tangential stress component at the two separation boundary points. A singular solution is proposed and used to obtain the solution in bulk by the Boundary Collocation Method. Three-dimensional plots are provided. The efficiency of the used scheme is discussed. All figures were produced using Mathematica 7.0 software.

2. Problem formulation and basic equations

Consider an infinitely long cylinder of elliptical normal cross-section from an isotropic, homogeneous, elastic material. A system of orthogonal Cartesian coordinates is used, with origin 0 at the center of the ellipse, x-axis along the major axis of the ellipse. The parametric equations of the boundary C may be taken as:

$$x = a\cos(\theta), \qquad y = b\sin(\theta), \quad 0 \leqslant \theta < 2\pi, \tag{1}$$

$2a$ and $2b$ being respectively the lengths of the major and the minor axes of the ellipse, while θ denotes the eccentric angle of a general point on the ellipse. For dimension analysis purposes, the half-length of the major axis is taken to be the characteristic length, i.e. a is taken to be equal to unity, $a = 1$. Also, we take $b = 0.5$.

Let τ be the unit vector tangent in the positive sense associated with C and $\mathbf{n}$ the unit outwards normal to C at any arbitrary point. One has

$$\tau = \frac{\dot{x}}{\omega}\mathbf{i} + \frac{\dot{y}}{\omega}\mathbf{j}, \qquad \mathbf{n} = \frac{\dot{y}}{\omega}\mathbf{i} - \frac{\dot{x}}{\omega}\mathbf{j}. \tag{2}$$

The general equations of the linear theory of elasticity for a homogeneous and isotropic material are well established and may found in standard references. In what follows, we shall quote these equations as presented in [1] without proof, to be used throughout the text. In the absence of body forces, the equations of equilibrium are automatically satisfied if the identically non-vanishing stress components are defined through the stress function U by the relations

$$\sigma_{xx} = \frac{\partial^2 U}{\partial y^2}, \quad \sigma_{yy} = \frac{\partial^2 U}{\partial x^2}, \quad \sigma_{xy} = -\frac{\partial^2 U}{\partial x \partial y}. \tag{3}$$

With respect to polar coordinates, the stress components are:

$$\sigma_{rr} = \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2}, \quad \sigma_{\theta\theta} = \frac{\partial^2 U}{\partial r^2}, \quad \sigma_{r\theta} = \frac{1}{r^2}\frac{\partial U}{\partial \theta} - \frac{1}{r}\frac{\partial^2 U}{\partial r \partial \theta}. \tag{4}$$

In Cartesian coordinates, Hooke's law reads

$$\sigma_{xx} = \frac{vE}{(1+v)(1-2v)}\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right] + \frac{E}{(1+v)}\frac{\partial u}{\partial x}, \tag{5}$$

$$\sigma_{yy} = \frac{vE}{(1+v)(1-2v)}\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right] + \frac{E}{(1+v)}\frac{\partial v}{\partial y}, \tag{6}$$

$$\sigma_{xy} = \frac{E}{2(1+v)}\left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right], \tag{7}$$

where u and v are the displacement components along the axes, E and v are Young's modulus and Poisson's ratio, respectively, for the considered elastic medium.

The compatibility condition for the solution of Eqs. (5)–(7) for the displacement components leads to the following homogeneous biharmonic equation for the stress function U:

$$\Delta^2 U = 0. \tag{8}$$

The function U solving Eq. (8) is

$$U = x\phi + y\phi^c + \psi, \tag{9}$$

where ϕ and ψ are two harmonic functions, the superscript "c" denotes the harmonic conjugate and $\overline{D}$ is the closure of D. Since the boundary integral representation is to be used, it seems adequate to suppose from the outset that the function ϕ and ψ and their conjugates belong to the class of functions $C^2(D)$. The following representation for the mechanical displacement components may be easily deduced:

$$\frac{E}{1+v}u = -\frac{\partial U}{\partial x} + 4(1-v)\phi, \tag{10}$$

and

$$\frac{E}{1+v}\upsilon = -\frac{\partial U}{\partial y} + 4(1-v)\phi^c. \tag{11}$$

In terms of the harmonic functions ϕ, ϕ^c and ψ, the stress and the displacement components are expressed as follows:

$$\sigma_{xx} = x\frac{\partial^2 \phi}{\partial y^2} + 2\frac{\partial \phi^c}{\partial y} + y\frac{\partial^2 \phi^c}{\partial y^2} + \frac{\partial^2 \psi}{\partial y^2}, \tag{12}$$

$$\sigma_{yy} = x\frac{\partial^2 \phi}{\partial x^2} + 2\frac{\partial \phi}{\partial x} + y\frac{\partial^2 \phi^c}{\partial x^2} + \frac{\partial^2 \psi}{\partial x^2}, \tag{13}$$

$$\sigma_{xy} = -x\frac{\partial^2 \phi}{\partial x \partial y} - y\frac{\partial^2 \phi^c}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial x \partial y} \tag{14}$$

and

$$\frac{E}{1+v}u = (3-4v)\phi - x\frac{\partial \phi}{\partial x} - y\frac{\partial \phi^c}{\partial x} - \frac{\partial \psi}{\partial x}, \tag{15}$$

$$\frac{E}{1+v}\upsilon = (3-4v)\phi^c - x\frac{\partial \phi}{\partial y} - y\frac{\partial \phi^c}{\partial y} - \frac{\partial \psi}{\partial y}. \tag{16}$$

3. Boundary integral representation of the basic equations

In what follows, we present the boundary integral representation of the basic equations and boundary conditions to be used in the sequel. We closely follow the guidelines of [1].

3.1. Boundary integral representation of harmonic functions

Let $f \in C^2(\overline{D})$ be harmonic in D. We use the well-known integral representation for f at an arbitrary field point (x,y) in D in terms of the boundary values of the function f and its complex conjugate f^c in the form:

$$f(x,y) = \frac{1}{2\pi}\oint_s \left[f(\acute{s})\frac{\partial}{\partial \acute{n}}\ln R + f^c(\acute{s})\frac{\partial}{\partial \acute{s}}\ln R\right]d\acute{s}, \tag{17}$$

where R is the distance between the point (x, y) in D and the current integration point $(x(\acute{s}), y(\acute{s}))$ on S. The representation of the conjugate function is given by

$$f^c(x,y) = \frac{1}{2\pi}\oint_s \left[f^c(\acute{s})\frac{\partial}{\partial \acute{n}}\ln R - f(\acute{s})\frac{\partial}{\partial \acute{s}}\ln R\right]d\acute{s}. \tag{18}$$

The integral representations (17) and (18) for the harmonic functions f and f^c replace the usual Cauchy-Riemann conditions

$$\frac{\partial f}{\partial x} = \frac{\partial f^c}{\partial y} \quad \text{and} \quad \frac{\partial f}{\partial y} = -\frac{\partial f^c}{\partial x}. \tag{19}$$

When the point (x, y) tends to a boundary point $(x(s), y(s))$, relation (17) yields

$$f(s) = \frac{1}{\pi}\oint_s \left[f(\acute{s})\frac{\partial}{\partial \acute{n}}\ln R + f^c(\acute{s})\frac{\partial}{\partial \acute{s}}\ln R\right]d\acute{s}. \tag{20}$$

Replacing $\left(\frac{\partial}{\partial \acute{n}}\right) lnR$ by $\left(\frac{\partial}{\partial \acute{s}}\right)\Theta$ in (17), (18) and their boundary version (20), where Θ is the complex conjugate of lnR, it is readily seen that these integral relations are invariant under the transformation of parameter from the arc length s to any other suitable parameter. This property makes the method more flexible.

3.2. Conditions for the uniqueness of the solution

Before dealing with each of the two above-mentioned fundamental problem, we first turn to the conditions to be satisfied in order to determine the unknown harmonic functions in an unambiguous manner. This is of primordial importance for any numerical treatment of the problem, for a proper use of the solving algorithm. We shall require the following supplementary conditions to be satisfied at the point Q_0(s = 0) of the boundary, in order to determine the totality of the arbitrary integration constants appearing throughout the solution process. These additional conditions have no physical implications on the throughout the problem:

(1) The vanishing of the function U and its first order partial derivatives at Q_0

$$U = \frac{\partial U}{\partial x} = \frac{\partial U}{\partial y} = 0,$$

or, equivalently,

$$U = \frac{\partial U}{\partial s} = \frac{\partial U}{\partial n} = 0,$$

which, in terms of the boundary values of the unknown harmonic functions, give

$$x(0)\phi(0) + y(0)\phi^c(0) + \psi(0) = 0, \tag{21}$$

$$x(0)\phi(\dot{0}) + y(0)\phi^c(\dot{0}) + \psi(\dot{0}) + x(\dot{0})\phi(0) + y(\dot{0})\phi^c(0) = 0, \tag{22}$$

$$x(0)\phi^c(\dot{0}) - y(0)\phi(\dot{0}) + \psi^c(\dot{0}) + y(\dot{0})\phi(0) - x(\dot{0})\phi^c(0) = 0. \tag{23}$$

(ii) The vanishing of the expression

$$x(0)\phi(0)^c - y(0)\psi(\dot{0}) + \phi^c(0) = 0. \tag{24}$$

This last additional condition amounts to determining the value of ψ^c at Q_0 and this is chosen to simplify the formulae.

3.3. Boundary conditions for the first fundamental problem of elasticity

In the first fundamental problem, we are given the force distribution on the boundary S of the domain D.

Let

$\mathbf{f} = f_x\mathbf{i} + f_y\mathbf{j} = f_\tau\tau + f_n\mathbf{n}$,

denote the external force per unit length of the boundary. Then, at a general boundary point Q, the stress vector

$\sigma_n = \mathbf{f}$,

or, in components,

$$\sigma_{xx}n_x + \sigma_{xy}n_y = f_x \quad \text{and} \quad \sigma_{xy}n_x + \sigma_{yy}n_y = f_y. \tag{25}$$

The stress function U at the boundary point Q.

$$\frac{\partial U}{\partial s}(s) = -\dot{x}(s)Y(s) + \dot{y}(s)X(s), \qquad \frac{\partial U}{\partial n}(s) = -\dot{y}(s)Y(s) - \dot{x}(s)X(s),$$

or, in terms of the unknown harmonic functions

$$x(s)\dot{\phi}(s) + y(s)\dot{\phi}^c(s) + \dot{\psi}(s) + \dot{x}(s)\phi(s) + \dot{y}(s)\phi^c(s) = -\dot{x}(s)Y(s) + \dot{y}(s)X(s) \tag{26}$$

and

$$x(s)\dot{\phi}^c(s) - y(s)\dot{\phi}(s) + \dot{\psi}^c(s) + \dot{y}(s)\phi(s) - \dot{x}(s)\phi^c(s) = -\dot{y}(s)Y(s) - \dot{x}(s)X(s). \tag{27}$$

3.4. Boundary conditions for the second fundamental problem of elasticity

In this problem, we are given the displacement vector on the boundary S of the domain D. Let this vector be denoted $\mathbf{d} = d_x\mathbf{i} + d_y\mathbf{j} = d_\tau\tau + d_n\mathbf{n}$.

Multiplying the restriction of expression (15) to the boundary S by $\dot{x}(s)$ and that of expression (16) by $\dot{y}(s)$ and adding, one gets

$$(3-4v)(\dot{x}(s)\phi(s) + \dot{y}(s)\phi^c(s)) - x(s)\dot{\phi}s - y(s)\dot{\phi}^c(s) - \dot{\psi}(s) = \frac{E}{1+v}(\dot{x}(s)d_x(s) + \dot{y}(s)d_y(s))\omega. \tag{28}$$

Similarly, if one multiplies the restriction of expression (15) to the boundary S by $\dot{y}(s)$ and that of expression (16) by $\dot{x}(s)$ and subtracting, one obtains

$$(3-4v)(\dot{y}(s)\phi(s) - \dot{x}(s)\phi^c(s)) - x(s)\dot{\phi}^c s + y(s)\dot{\phi}(s) - \dot{\psi}^c(s) = \frac{E}{1+v}(\dot{y}(s)d_x(s) - \dot{x}(s)d_y(s))\omega. \tag{29}$$

These last two relations may be conveniently rewritten as

$$(3-4v)(\dot{x}(s)\phi(s) + \dot{y}(s)\phi^c(s)) - x(s)\dot{\phi}s - y(s)\dot{\phi}^c(s) - \dot{\psi}(s) = \frac{E}{1+v}d_\tau(s)\omega \tag{30}$$

and

$$(3-4v)(\dot{y}(s)\phi(s) - \dot{x}(s)\phi^c(s)) - x(s)\dot{\phi}^c s + y(s)\dot{\phi}(s) - \dot{\psi}^c(s) = \frac{E}{1+v}d_n(s)\omega. \tag{31}$$

3.5. Boundary conditions for the third fundamental problem of elasticity

This is a problem with mixed boundary conditions. For definiteness, we shall restrict further considerations to the case where one half of the boundary has a prescribed pressure on it, while the other half of the boundary is fixed. This problem will be replaced by two subproblems, each with homogeneous boundary condition. The first subproblem is of the first kind. It involves the given known pressure on the same half of the boundary as the initial problem and an unknown stress on the other half. This stress is expressed through its normal and tangential components, respectively, denoted $\widetilde{\sigma_n}$, $\widetilde{\sigma_\tau}$.

The second subproblem is of the second type. It involves zero displacement on the same half of the boundary as the initial problem and an unknown displacement on the other half. This displacement is expressed through its normal and tangential components, respectively, denoted $\widetilde{u_n}$, $\widetilde{u_\tau}$.

In what follows, we shall apply this idea to solve the problem for the ellipse.

3.6. Calculation of the harmonic functions at internal points

Having determined the boundary values of the harmonic functions, formulae (17) and (18) may now be used to calculate the values of these functions at any point (x, y) inside the domain. For this, we write:

$$\begin{gathered} R = \sqrt{(x - x(s'))^2 + (y - y(s'))^2}, \\ \frac{\partial \ln(R)}{\partial n} = \mathbf{n} \cdot \nabla(\ln R)\, \frac{\partial \ln(R)}{\partial s} = \tau \cdot \nabla(\ln R). \end{gathered} \tag{32}$$

We can also proceed otherwise. In fact, if we write down expansions of the four harmonic functions involved in the calculations in terms of some adequately chosen basis, we can then determine the expansion coefficients using the well-known Boundary Collocation Method (BCM). This is in fact the method we have used to calculate the unknown functions in the circular domain. The expansions of the four basic harmonic functions in terms of polar harmonics are as follows:

$$\phi = \sum_{n=1}^{N} r^n (A_n \cos n\theta + B_n \sin n\theta), \quad \phi^c = \sum_{n=1}^{N} r^n (A_n \sin n\theta - B_n \cos n\theta), \tag{33}$$

$$\psi = \sum_{n=1}^{N} r^n (E_n \cos n\theta + D_n \sin n\theta), \quad \psi^c = \sum_{n=1}^{N} r^n (E_n \sin n\theta - D_n \cos n\theta), \tag{34}$$

while the stress function is

$$U = x\phi + y\phi^c + \psi \tag{35}$$

and the quantities of practical interest are:

$$\begin{aligned} 2\mu u = {} & \sum_{n=1}^{N} r^n (A_n \cos n\theta + B_n \sin n\theta) - \sum_{n=1}^{N} n r^n (A_n \cos(n-2)\theta \\ & + B_n \sin(n-2)\theta) - \sum_{n=1}^{N} n r^{n-1} (E_n \cos(n-1)\theta + D_n \\ & \times \sin(n-1)\theta), \end{aligned} \tag{36}$$

$$\begin{aligned} 2\mu v = {} & \sum_{n=1}^{N} r^n (A_n \sin n\theta - B_n \cos n\theta) - \sum_{n=1}^{N} n r^n (-A_n \sin(n-2)\theta \\ & + B_n \cos(n-2)\theta) - \sum_{n=1}^{N} n r^{n-1} (-E_n \sin(n-1)\theta + D_n \\ & \times \cos(n-1)\theta). \end{aligned} \tag{37}$$

The equations for the normal and the tangential stresses on any given element of area with unit normal (n_x, n_y) inside the body or on its boundary are given by the following formulae:

$$\begin{aligned} \sigma_n = {} & \sum_{n=1}^{N} r^{n-1} A_n (\cos(n-1)\theta((3n-n^2)n_r^2 + (n^2+n)n_\theta^2) + 2(n^2-n)n_r n_\theta \sin(n-1)\theta) \\ & + \sum_{n=1}^{N} r^{n-1} B_n (\sin(n-1)\theta((3n-n^2)n_r^2 + (n^2+n)n_\theta^2) + 2(n^2-n)n_r n_\theta \cos(n-1)\theta) \\ & + \sum_{n=1}^{N} r^{n-2} C_n (\cos n\theta((n-n^2)n_r^2 + (n^2-n)n - \theta^2) + 2(n^2-n)n_r n_\theta \sin n\theta) \\ & + \sum_{n=1}^{N} r^{n-2} D_n (\sin n\theta((n-n^2)n_r^2 + (n^2-n)n_\theta^2) + 2(n-n^2)n_r n_\theta \cos n\theta), \end{aligned} \tag{38}$$

$$\begin{aligned} \sigma_\tau = {} & \sum_{n=1}^{N} r^{n-1} A_n \cos(n \\ & - 1)\theta((-2n+2n^2)n_r n_\theta + (n^2-n)(n_r^2 - n_\theta^2) \sin(n-1)\theta) \\ & + \sum_{n=1}^{N} r^{n-1} B_n (\sin(n \\ & - 1)\theta((-2n+2n^2)n_r n_\theta + (n-n^2)(n_r^2 - n_\theta^2) \cos(n-1)\theta) \\ & + \sum_{n=1}^{N} r^{n-2} C_n (\cos n\theta(-2n+2n^2)n_r n_\theta + (n^2-n)n_\theta^2 + 2(n^2 \\ & - n)(n_r^2 - n_\theta^2) \sin n\theta) \\ & + \sum_{n=1}^{N} r^{n-2} D_n (\sin n\theta(-2n-2n^2)n_r n_\theta + (n-n^2)(n_r^2 - n_\theta^2) \cos n\theta), \end{aligned} \tag{39}$$

$$\begin{aligned} \sigma_{xx} = {} & \sum_{n=1}^{N} n(n-1)r^{n-1}(-A_n \cos(n-3)\theta - B_n \sin(n \\ & - 3)\theta) + 2\sum_{n=1}^{N} n r^{n-1}(A_n \cos(n-1)\theta + B_n \sin(n \\ & - 1)\theta) + \sum_{n=1}^{N} n(n-1)r^{n-2}(-E_n \cos(n-2)\theta + D_n \\ & \times \sin(n-2)\theta), \end{aligned} \tag{40}$$

$$\begin{aligned} \sigma_{yy} = {} & \sum_{n=1}^{N} n(n-1)r^{n-1}(A_n \cos(n-3)\theta + B_n \sin(n-3)\theta) \\ & + 2\sum_{n=1}^{N} n r^{n-1}(A_n \cos(n-1)\theta + B_n \sin(n-1)\theta) \\ & + \sum_{n=1}^{N} n(n-1)r^{n-2}(E_n \cos(n-2)\theta + D_n \sin(n \\ & - 2)\theta) \end{aligned} \tag{41}$$

and

$$\begin{aligned} \sigma_{xy} = {} & \sum_{n=1}^{N} n(n-1)r^{n-1}(-A_n \sin(n-1)\theta + B_n \cos(n \\ & - 1)\theta) - \sum_{n=1}^{N} n(n-1)r^{n-2}(-E_n \sin(n-2)\theta + D_n \\ & \times \cos(n-2)\theta). \end{aligned} \tag{42}$$

The relevant boundary relations are discretized in the usual way by considering a partition of the boundary. As a result, the actual boundary is replaced by a contour formed by broken lines. The differential and integral equations thus reduce to a rectangular system of linear algebraic equations which are solved by the Least Squares method. The convergence of the solution of the discretized system of equations to the solution of the initial problem was discussed elsewhere [20]. Here, we only notice the existence of removable singularities in the formulae of integral representation of harmonic functions. These are dealt with in the manner explained in [2]. Also, the tangential derivatives of the unknown harmonic functions have to be evaluated carefully as they can be a major source of error. We have calculated these derivatives using 31 points.

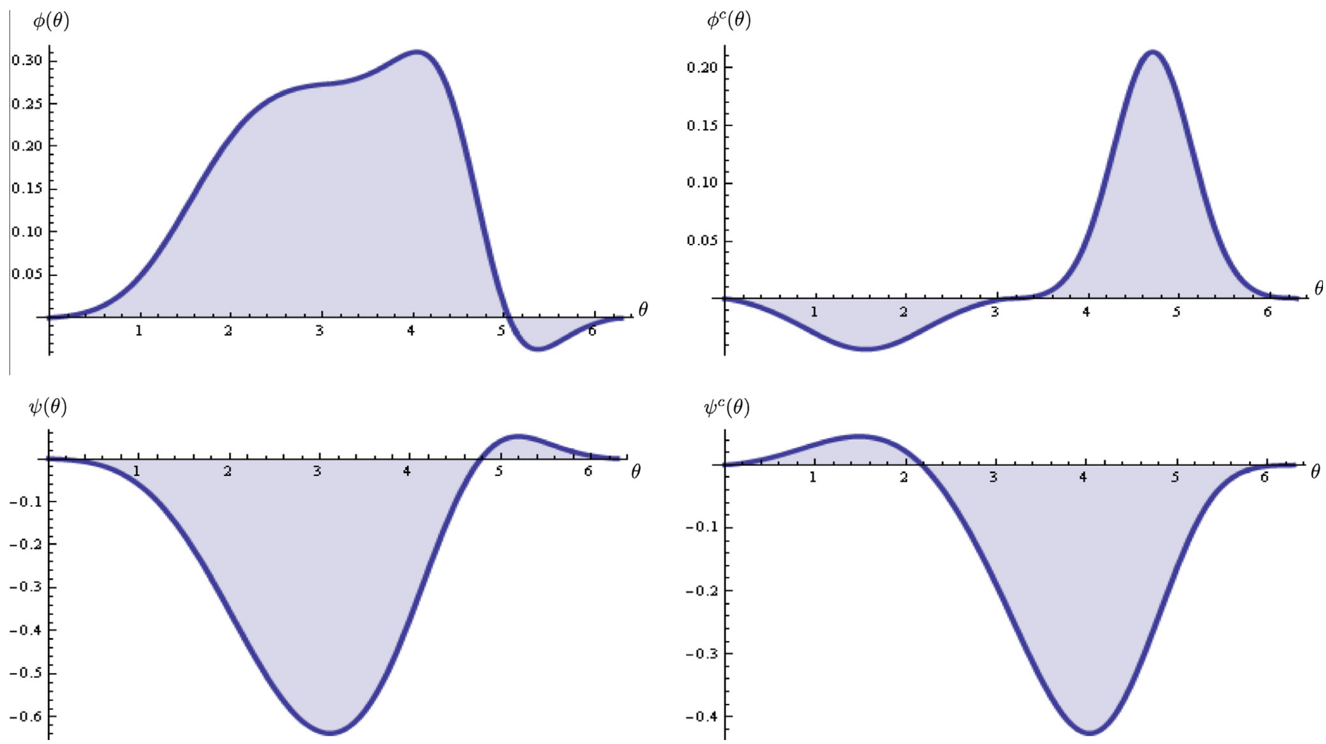

Figure 1 Boundary values of the basic harmonic functions for 220 nodes.

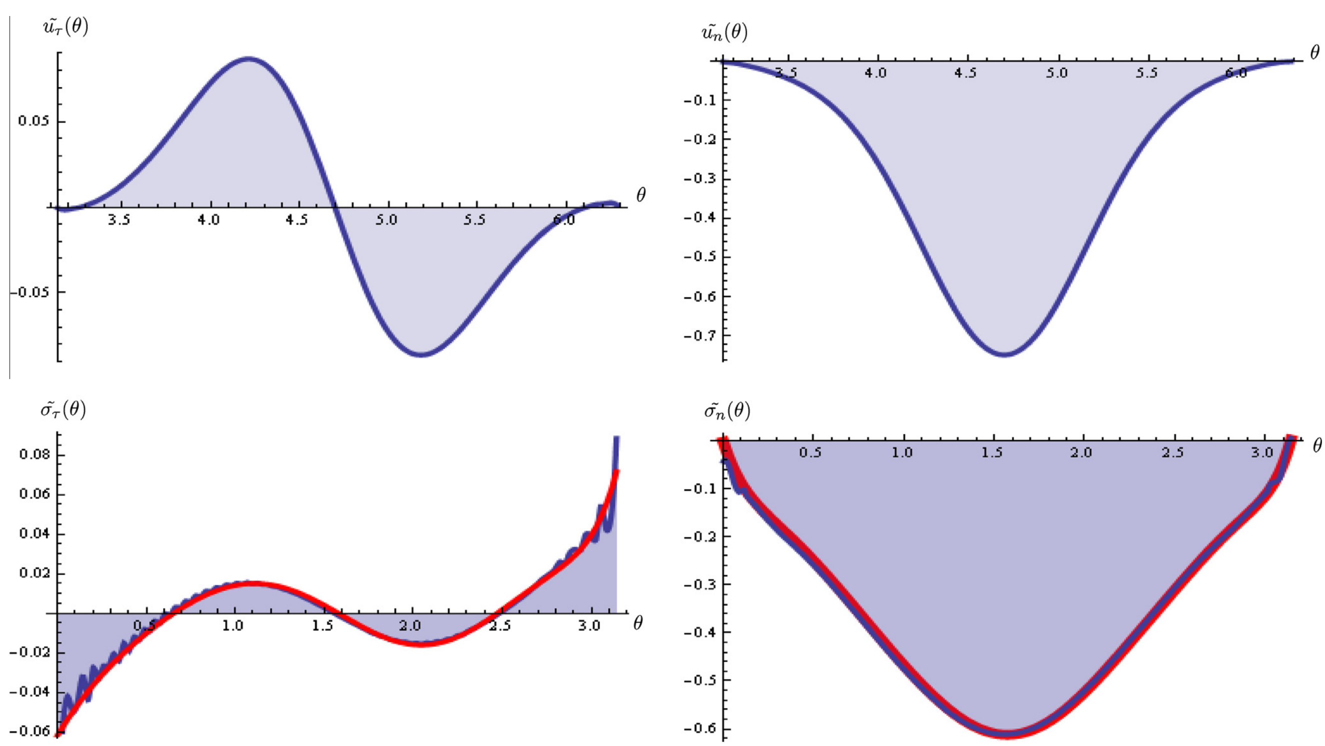

Figure 2 Boundary values of the normal and tangential force and displacement components for 220 nodes.

4. Numerical results and discussion

The force acting on one half of the boundary is a pressure of intensity f given by

$$f = -p_0(\sin\theta)^6, \qquad \pi < \theta \leqslant 2\pi. \tag{43}$$

The other half of the boundary is completely fixed:

$$u = v = 0, \qquad 0 < \theta \leqslant \pi. \tag{44}$$

For definiteness, we have taken $p_0 = 1$. The motivation for the above choice of the pressure is to make the pressure distribution tend to zero smoothly enough at both ends of its interval

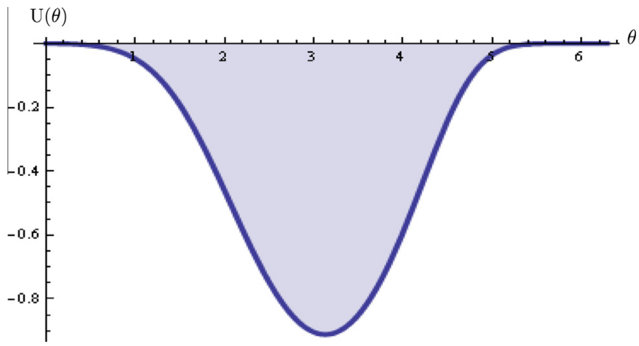

Figure 3 Boundary values of the stress function.

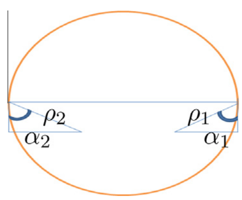

Figure 4 Singular points.

of definition, so as to reduce any potential conflict with the boundary condition prevailing on the other half of the boundary (zero displacements).

The above boundary integral equations were solved numerically, from which we have obtained the boundary values of the harmonic functions ϕ, ϕ^c, ψ, ψ^c, $\tilde{\sigma}_n$, $\tilde{\sigma}_\tau$, $\tilde{u}_n$, $\tilde{u}_\tau$ and, accordingly, of the stress function U on the boundary. In this procedure, it is important to specify the rule by which the nodal points are scattered on the boundary. Equidistant nodes are one variant, concentration of the nodes toward the singular points is another option. For the present case, the boundary was discretized by placing a number of nodal points distributed uniformly with respect to the angle θ on it as explained, 220 boundary nodes were needed in order to get the present results. The results are shown on Fig. 1.

As numerical experiments, we have considered the case of the circular boundary and found out that weak discontinuities appear on the curves of the basic harmonic functions at the boundary separation points. These discontinuities are not clearly defined on the figures for the presently considered case of the ellipse.

The displacement components converged rapidly to zero toward the boundary separation points. Perturbations appeared, however, when calculating the tangential stress component in the vicinity of these singular points. Increase in the number of nodes could not improve the situation, thus indicating a singular behavior. To put in evidence the nature of the singularities, curve fitting was used to reach smooth shapes of the curves. A logarithm was used at each separation point for curve fitting for the tangential stress component. This is shown on Fig. 2 (see Fig. 3).

5. On the singular solution

Based on the results of numerical experiments presented above, we propose a function ψ_s with boundary singularities at the separation points to be added to the function ψ, in order to get the required logarithmic behavior of the function σ_τ at the singular points $(\pm a, 0)$. Such a function was proposed by Abou-Dina and Ghaleb [21] in connection with the solution of some boundary-value problems for Laplace's equation in rectangular domains. Here, it has a somewhat different shape. To get the required singular behavior of the tangential stress component, one needs an analytic function of the type $z^2 log\, z$ to be added to the stress function, where z denotes a complex argument. Fig. 4 shows the emplacements of the singularities of function ψ_s, where local polar coordinates centered at the singular points have been used. A correct choice of direction of the polar axes at the singularities is necessary for obtaining good results.

The function ψ_s has the form:

$$\psi_s = \frac{1}{2\pi}[\rho_1^2(\sin 2\theta_1 \ln \rho_1 + \theta_1 \cos 2\theta_1) + \rho_2^2(\sin 2\theta_2 \ln \rho_2 + \theta_2 \times \cos 2\theta_2)],$$

where

$$\rho_1 = \sqrt{r^2 - 2ar\cos\theta + a^2}, \quad \rho_2 = \sqrt{r^2 + 2ar\cos\theta + a^2},$$
$$\theta_1 = \tan^{-1}\left(\frac{r\cos\theta - a}{-r\sin\theta}\right), \quad \theta_2 = \tan^{-1}\left(\frac{r\cos\theta + a}{-r\sin\theta}\right).$$

Figs. 5 and 6 show three-dimensional plots of the stress and the displacement components inside the ellipse after adding the singular solution. The sinusoidal form of the normal stress component on one-half of the boundary is clear, while the tangential stresses on the same portion of boundary show

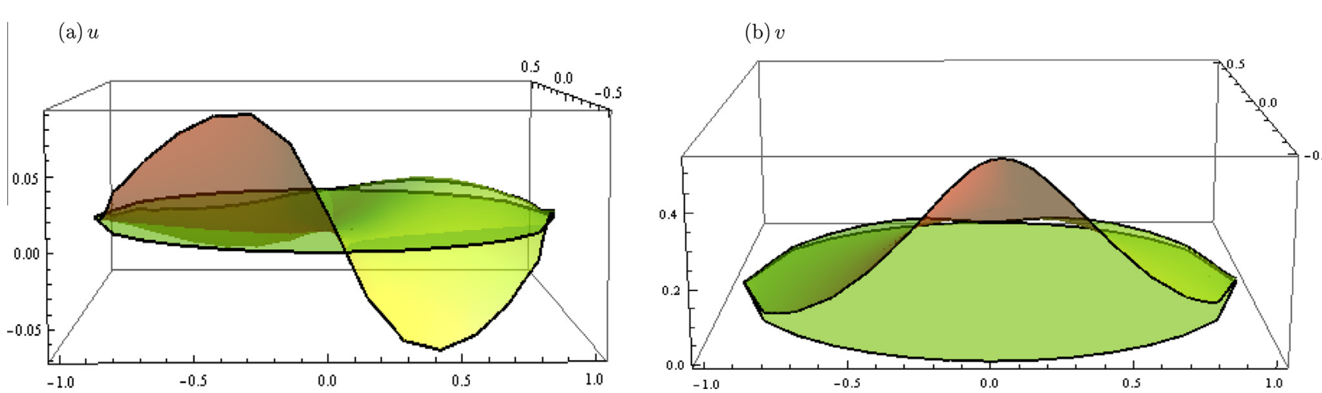

Figure 5 Displacement components inside the ellipse by BCM.

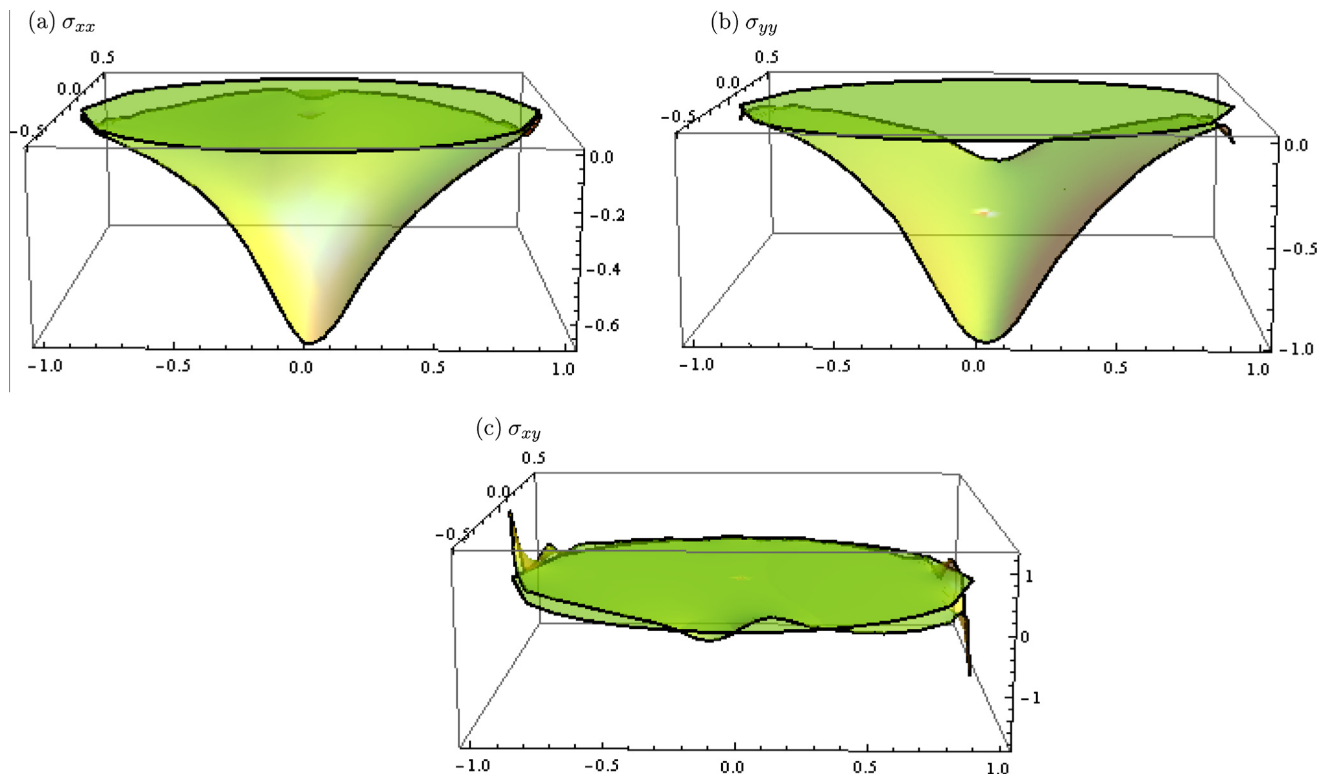

Figure 6 Stress components inside the ellipse by BCM.

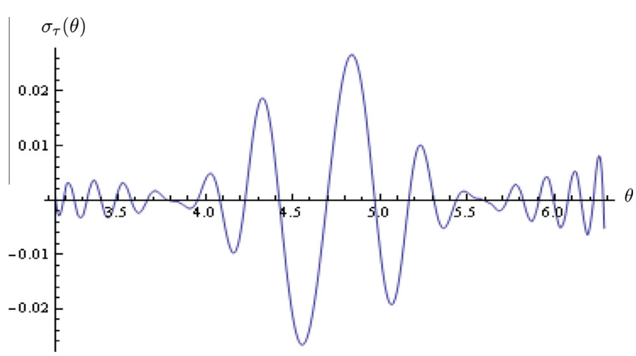

Figure 7 Boundary values of the tangential stress component without logarithm.

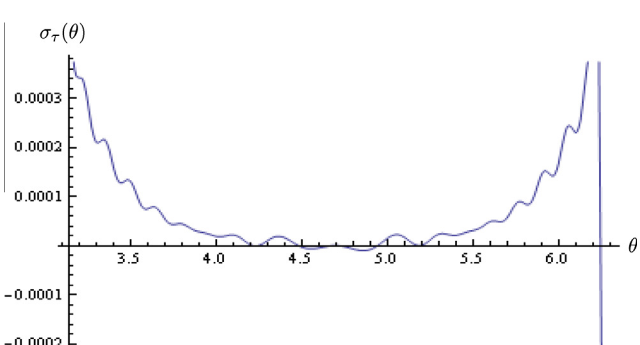

Figure 8 Boundary values of the tangential stress component with logarithm.

logarithmic behavior. For reference, we have also plotted a flat surface representing the cross-section of the ellipse.

Such rise in the tangential stress component means that the linear elastic model will fail to provide an accurate description of the problem in the vicinity of the boundary separation points. A more accurate study would require the consideration of regions of plastic deformation around these points.

Figs. 7 and 8 show the improvement that occurred in the evaluation of the tangential stress component on the boundary interval $3.17 \leqslant \theta \leqslant 6.25$ after including the singular solution. The maximal absolute error on this interval was reduced from $\simeq 3 \times 10^{-2}$ to $\simeq 3 \times 10^{-4}$.

6. Conclusions

We have considered a boundary-value problem of the plane theory of elasticity with mixed boundary conditions in the ellipse. Half of the boundary is subjected to a variable pressure, while the other half is completely fixed. The boundary pressure was chosen to decrease smoothly enough to zero toward the points of separation in order to reduce the possibility of non-existence of a solution. To get the solution on the boundary, the initial problem was replaced by two subproblems, each with homogeneous boundary condition of one type, having a common solution. The calculations on the boundary were performed using a known boundary integral technique involving harmonic functions only, including regularization at the nodes and a careful calculation of the derivatives of functions along the boundary. The boundary calculations indicated a logarithmic behavior of the tangential stress component on the fixed part of the boundary. The solution inside the domain was obtained by the Collocation Method directly using the prescribed boundary conditions. In solving the systems of linear algebraic equations arising from discretization, we have used packages named Least Squares and QR-Factorization techniques. Both yielded the same results. Each time, we verified that the obtained results satisfy the system of equations with high accuracy.

The numerical treatment within boundary integral methods of this type of problems requires a relatively large number of boundary nodes at which the unknowns are to be calculated. For the present case, 220 points could be reached without obtaining satisfactory results over the whole boundary. The reason for this is the presence of singular boundary points at the separation points of the boundary conditions. Increasing the number of points increased the accuracy of the results up to a certain level. In order to improve the solution, a singular term with logarithmic boundary singularity was added to the solution. The absolute errors in satisfying the boundary conditions on an interval not including the separation points could thus be reduced from $\simeq 3 \times 10^{-2}$ to $\simeq 3 \times 10^{-4}$. The obtained results indicate the need to introduce domains of possible plastic behavior of the material around the two boundary separation points.

Future work will involve more complicated shapes of the boundary and other types of mixed boundary conditions. In each case, the behavior of the solution near the points of separation will be investigated.

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