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Existence theory for anti-periodic boundary value problems of fractional differential equations and inclusions

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ABSTRACT

This paper studies the existence of solutions for nonlinear fractional differential equations and inclusions of order $q \in (3, 4]$ with anti-periodic boundary conditions. In the case of inclusion problem, the existence results are established for convex as well as nonconvex multivalued maps. Our results are based on some fixed point theorems, Leray–Schauder degree theory, and nonlinear alternative of Leray–Schauder type. Some illustrative examples are discussed.

1. Introduction

The subject of fractional calculus has recently gained much momentum and a variety of problems involving differential equations and inclusions of fractional order have been addressed by several researchers. Fractional differential equations appear naturally in a number of fields such as physics, polymer rheology, regular variation in thermodynamics, biophysics, blood flow phenomena, aerodynamics, electro-dynamics of complex medium, viscoelasticity, Bode analysis of feedback amplifiers, capacitor theory, electrical circuits, electro-analytical chemistry, biology, control theory, fitting of experimental data, etc. [1–4]. For some recent work on fractional differential equations and inclusions, see [5–13] and the references therein.

Anti-periodic boundary value problems occur in the mathematical modeling of a variety of physical processes and have recently received considerable attention. For examples and details of anti-periodic boundary conditions, see [14–18] and the references therein.

In this paper, we discuss some existence results for anti-periodic boundary value problems of differential equations and inclusions of fractional order $q \in (3, 4]$. Precisely, we consider the following problems:

$$\begin{cases} {}^cD^q x(t) = f(t, x(t)), \quad t \in [0, T],\ T > 0,\ 3 < q \le 4, \\ x(0) = -x(T), \qquad x'(0) = -x'(T), \qquad x''(0) = -x''(T), \qquad x'''(0) = -x'''(T), \end{cases} \tag{1.1}$$

where ${}^cD^q$ denotes the Caputo fractional derivative of order q, f is a given continuous function, and

$$\begin{cases} {}^cD^q x(t) \in F(t, x(t)), \quad t \in [0, T],\ T > 0,\ 3 < q \le 4, \\ x(0) = -x(T), \qquad x'(0) = -x'(T), \qquad x''(0) = -x''(T), \qquad x'''(0) = -x'''(T). \end{cases} \tag{1.2}$$

In (1.2), $F : [0, T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map, $\mathcal{P}(\mathbb{R})$ is the family of all subsets of $\mathbb{R}$.

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This paper is organized as follows. In Section 3, we discuss the existence of solutions for problem (1.1) by applying some well known fixed point theorems and the Leray–Schauder degree theory. Section 4 deals with some existence results for the inclusion problem (1.2) involving convex as well as nonconvex multivalued maps. These results are based on the nonlinear alternative of Leray–Schauder type, a selection theorem due to Bressan and Colombo and a fixed point theorem due to Covitz and Nadler. The methods used in this paper are standard, however their exposition in the framework of problems (1.1) and (1.2) is new.

2. Preliminaries

Definition 2.1. For a continuous function $g : [0, \infty) \to \mathbb{R}$, the Caputo derivative of fractional order q is defined as

$$ {}^cD^q g(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} g^{(n)}(s) \mathrm{d}s, \quad n-1 < q < n, \ n = [q]+1, $$

where $[q]$ denotes the integer part of the real number q.

Definition 2.2. The Riemann–Liouville fractional integral of order q is defined as

$$ I^q g(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{g(s)}{(t-s)^{1-q}} \mathrm{d}s, \quad q > 0, $$

provided the integral exists.

Lemma 2.1 ([3]). *For $q > 0$, the general solution of the fractional differential equation ${}^cD^q x(t) = 0$ is given by*

$$ x(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}, $$

where $c_i \in \mathbb{R}, \ i = 0, 1, 2, \ldots, n-1 (n = [q]+1)$.

In view of Lemma 2.1, it follows that

$$ I^{q\,c}D^q x(t) = x(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}, \tag{2.1} $$

for some $c_i \in \mathbb{R}, \ i = 0, 1, 2, \ldots, n-1 \ (n = [q]+1)$.

To study the nonlinear problems (1.1) and (1.2), we need the following lemma.

Lemma 2.2. *For any $\sigma \in C[0, T]$, the unique solution of the boundary value problem*

$$ \begin{cases} {}^cD^q x(t) = \sigma(t), & 0 < t < T, \ 3 < q \le 4, \\ x(0) = -x(T), \quad x'(0) = -x'(T), & x''(0) = -x''(T), \quad x'''(0) = -x'''(T) \end{cases} \tag{2.2} $$

is

$$ x(t) = \int_0^T G(t,s) \sigma(s) \mathrm{d}s, $$

where $G(t, s)$ is Green's function given by

$$ G(t,s) = \begin{cases} \dfrac{(t-s)^{q-1} - \frac{1}{2}(T-s)^{q-1}}{\Gamma(q)} + \dfrac{(T-2t)(T-s)^{q-2}}{4\Gamma(q-1)} + \dfrac{t(T-t)(T-s)^{q-3}}{4\Gamma(q-2)} \\ \quad + \dfrac{(6t^2T - 4t^3 - T^3)(T-s)^{q-4}}{48\Gamma(q-3)}, \quad 0 < s < t < T, \\ -\dfrac{(T-s)^{q-1}}{2\Gamma(q)} + \dfrac{(T-2t)(T-s)^{q-2}}{4\Gamma(q-1)} + \dfrac{t(T-t)(T-s)^{q-3}}{4\Gamma(q-2)} \\ \quad + \dfrac{(6t^2T - 4t^3 - T^3)(T-s)^{q-4}}{48\Gamma(q-3)}, \quad 0 < t < s < T. \end{cases} \tag{2.3} $$

Proof. Using (2.1), for some constants $b_1, b_2, b_3, b_4 \in \mathbb{R}$, we have

$$ x(t) = I^q \sigma(t) - b_1 - b_2 t - b_3 t^2 - b_4 t^3 = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \sigma(s) \mathrm{d}s - b_1 - b_2 t - b_3 t^2 - b_4 t^3. \tag{2.4} $$

Applying the boundary conditions for problem (2.2) in (2.4), we find that

$$b_1 = \frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)}\sigma(s)\mathrm{d}s - \frac{T}{4}\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)}\sigma(s)\mathrm{d}s + \frac{T^3}{48}\int_0^T \frac{(T-s)^{q-4}}{\Gamma(q-3)}\sigma(s)\mathrm{d}s,$$

$$b_2 = \frac{1}{2}\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)}\sigma(s)\mathrm{d}s - \frac{T}{4}\int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)}\sigma(s)\mathrm{d}s,$$

$$b_3 = \frac{1}{4}\int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)}\sigma(s)\mathrm{d}s - \frac{T}{8}\int_0^T \frac{(T-s)^{q-4}}{\Gamma(q-3)}\sigma(s)\mathrm{d}s$$

$$b_4 = \frac{1}{12}\int_0^T \frac{(T-s)^{q-4}}{\Gamma(q-3)}\sigma(s)\mathrm{d}s.$$

Substituting the values of b_1, b_2, b_3 and b_4 in (2.4), we obtain

$$\begin{aligned} x(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}\sigma(s)\mathrm{d}s - \frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)}\sigma(s)\mathrm{d}s + \frac{(T-2t)}{4}\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)}\sigma(s)\mathrm{d}s \\ &\quad + \frac{t(T-t)}{4}\int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)}\sigma(s)\mathrm{d}s + \frac{(6t^2T-4t^3-T^3)}{48}\int_0^T \frac{(T-s)^{q-4}}{\Gamma(q-3)}\sigma(s)\mathrm{d}s \\ &= \int_0^T G(t,s)\sigma(s)\mathrm{d}s, \end{aligned}$$

where $G(t,s)$ is given by (2.3). This completes the proof. □

Observe that

$$\begin{cases} |(T-2t)(T-s)^{q-4}| \le |(T-t)(T-s)^{q-4}| \le |(T-t)^{q-3}|, & t < s, \\ |(T-2t)(T-s)^{q-4}| \le |(T-t)(T-s)^{q-4}| \le |(T-s)^{q-3}|, & t \ge s. \end{cases} \tag{2.5}$$

Remark 2.1. It is worthwhile to note that the first two terms in $G(t,s)$ correspond to the anti-periodic fractional boundary value problem of order $q \in (1,2]$ [15] and the first three terms in $G(t,s)$ correspond to the anti-periodic problem of fractional order $q \in (2,3]$ [16]. The consideration of the anti-periodic problem of fractional order $q \in (3,4]$ gives rise to four terms in $G(t,s)$. Thus, it can easily be inferred that the contributions due to lower-order anti-periodic fractional boundary value problems preserve their form in Green's function $G(t,s)$ for fractional anti-periodic problems of higher order and can be sorted out accordingly. Moreover, Green's function $G(t,s)$ for the fourth-order anti-periodic boundary value problem of ordinary differential equations can be obtained by taking $q = 4$ in (2.3), which is a new result and is given by

$$G(t,s) = \frac{1}{2}\begin{cases} \dfrac{2(t-s)^3-(T-s)^3}{3!} + \dfrac{t(T-t)(T-s)^{q-3}}{2} \\ \quad + \dfrac{(T-2t)}{4}\left((T-s)^2 + \dfrac{(2t^2-2tT-T^2)}{6}\right), & 0 < s < t < T, \\ -\dfrac{(T-s)^3}{3!} + \dfrac{t(T-t)(T-s)^{q-3}}{2} \\ \quad + \dfrac{(T-2t)}{4}\left((T-s)^2 + \dfrac{(2t^2-2tT-T^2)}{6}\right), & 0 < t < s < T. \end{cases} \tag{2.6}$$

3. Existence results

Let $\mathcal{C} = C([0,T],\mathbb{R})$ denote the Banach space of all continuous functions from $[0,T] \to \mathbb{R}$ endowed with the norm defined by $\|x\| = \sup\{|x(t)|, t \in [0,T]\}$.

Now we state some known results which are needed to prove the existence of solutions for (1.1).

Theorem 3.1 (*[19]*). *Let X be a Banach space. Assume that Ω is an open bounded subset of X with $\theta \in \Omega$ and let $T : \overline{\Omega} \to X$ be a completely continuous operator such that*

$$\|Tu\| \le \|u\|, \quad \forall u \in \partial\Omega.$$

Then T has a fixed point in $\overline{\Omega}$.

Theorem 3.2 (*[19]*). *Let M be a closed convex and nonempty subset of a Banach space X. Let A, B be the operators such that* (i) *$Ax + By \in M$ whenever $x, y \in M$;* (ii) *A is compact and continuous;* (iii) *B is a contraction mapping. Then there exists $z \in M$ such that $z = Az + Bz$.*

Define an operator $\mathcal{G} : \mathcal{C} \to \mathcal{C}$ as

$$\begin{aligned}(\mathcal{G}x)(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s,x(s))\mathrm{d}s - \frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f(s,x(s))\mathrm{d}s \\ &\quad + \frac{(T-2t)}{4}\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f(s,x(s))\mathrm{d}s + \frac{t(T-t)}{4}\int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)} f(s,x(s))\mathrm{d}s \\ &\quad + \frac{(6t^2T-4t^3-T^3)}{48}\int_0^T \frac{(T-s)^{q-4}}{\Gamma(q-3)} f(s,x(s))\mathrm{d}s, \quad t\in[0,T]. \end{aligned} \tag{3.1}$$

Observe that problem (1.1) has a solution if and only if the operator $\mathcal{G}$ has a fixed point.

Lemma 3.1. *The operator $\mathcal{G} : \mathcal{C} \to \mathcal{C}$ is completely continuous.*

Proof. Let $\Omega \subset \mathcal{C}$ be bounded. Then, $\forall t \in [0,T]$, $x \in \Omega$, there exists a positive constant L_1 such that $|f(t,x)| \le L_1$. Thus, we have

$$\begin{aligned}|(\mathcal{G}x)(t)| &\le \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} |f(s,x(s))|\mathrm{d}s + \frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} |f(s,x(s))|\mathrm{d}s \\ &\quad + \frac{1}{4}|T-2t|\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} |f(s,x(s))|\mathrm{d}s + \frac{1}{4}|t(T-t)|\int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)} |f(s,x(s))|\mathrm{d}s \\ &\quad + \frac{1}{48}|6t^2T-4t^3-T^3|\int_0^T \frac{(T-s)^{q-4}}{\Gamma(q-3)} |f(s,x(s))|\mathrm{d}s \\ &\le L_1\Bigg[\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\mathrm{d}s + \frac{1}{2\Gamma(q)}\int_0^T (T-s)^{q-1}\mathrm{d}s \\ &\quad + \frac{|T-2t|}{4\Gamma(q-1)}\int_0^T (T-s)^{q-2}\mathrm{d}s + \frac{|t(T-t)|}{4\Gamma(q-2)}\int_0^T (T-s)^{q-3}\mathrm{d}s \\ &\quad + \frac{1}{48\Gamma(q-3)}|6t^2T-4t^3-T^3|\int_0^T (T-s)^{q-4}\mathrm{d}s\Bigg] \\ &\le L_1\left[\frac{T^q}{2\Gamma(q+1)}\left(3+\frac{q(q^2+11)}{24}\right)\right] = L_2, \end{aligned} \tag{3.2}$$

which implies that $\|(\mathcal{G}x)\| \le L_2$. Furthermore,

$$\begin{aligned}|(\mathcal{G}x)'(t)| &= \int_0^t \frac{(t-s)^{q-2}}{\Gamma(q-1)} |f(s,x(s))|\mathrm{d}s + \frac{1}{2}\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} |f(s,x(s))|\mathrm{d}s \\ &\quad + \frac{|T-2t|}{4}\int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)} |f(s,x(s))|\mathrm{d}s + \frac{|t(T-t)|}{4}\int_0^T \frac{(T-s)^{q-4}}{\Gamma(q-3)} |f(s,x(s))|\mathrm{d}s \\ &\le L_1\Bigg[\int_0^t \frac{(t-s)^{q-2}}{\Gamma(q-1)}\mathrm{d}s + \frac{1}{2}\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)}\mathrm{d}s \\ &\quad + \frac{|T-2t|}{4}\int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)}\mathrm{d}s + \frac{|t(T-t)|}{4}\int_0^T \frac{(T-s)^{q-4}}{\Gamma(q-3)}\mathrm{d}s\Bigg] \\ &\le L_1\left[\frac{T^{q-1}}{2\Gamma(q)}\left(3+\frac{(q-1)(q+2)}{8}\right)\right] = L_3. \end{aligned} \tag{3.3}$$

Hence, for $t_1, t_2 \in [0,T]$, we have

$$|(\mathcal{G}x)(t_2) - (\mathcal{G}x)(t_1)| \le \int_{t_1}^{t_2} |(\mathcal{G}x)'(s)|\mathrm{d}s \le L_3(t_2-t_1).$$

This implies that $\mathcal{G}$ is equicontinuous on $[0,T]$. Thus, by the Arzelá–Ascoli theorem, the operator $\mathcal{G} : \mathcal{C} \to \mathcal{C}$ is completely continuous. $\square$

Theorem 3.3. *Let $f : [0,T] \times \mathbb{R} \to \mathbb{R}$, and $\lim_{x\to 0} \frac{f(t,x)}{x} = 0$. Then problem (1.1) has at least one solution.*

Proof. Since $\lim_{x\to 0}\frac{f(t,x)}{x}=0$, there exists a constant $r>0$ such that $|f(t,x)|\le\delta|x|$ for $0<|x|<r$, where $\delta>0$ is such that

$$\max_{t\in[0,T]}\left\{\frac{2|t^q|+T^q}{2\Gamma(q+1)}+\frac{|T-2t|T^{q-1}}{4\Gamma(q)}+\frac{|t(T-t)|T^{q-2}}{4\Gamma(q-1)}+\frac{|6t^2T-4t^3-T^3|T^{q-3}}{48\Gamma(q-2)}\right\}\delta\le 1. \tag{3.4}$$

Define $\Omega_1=\{x\in\mathcal{C}\mid\|x\|<r\}$ and take $x\in\mathcal{C}$ such that $\|x\|=r$, that is, $x\in\partial\Omega_1$. By Lemma 3.1, we know that $\mathcal{G}$ is completely continuous and

$$|\mathcal{G}x(t)|\le\max_{t\in[0,T]}\left\{\frac{2|t^q|+T^q}{2\Gamma(q+1)}+\frac{|T-2t|T^{q-1}}{4\Gamma(q)}+\frac{|t(T-t)|T^{q-2}}{4\Gamma(q-1)}+\frac{|6t^2T-4t^3-T^3|T^{q-3}}{48\Gamma(q-2)}\right\}\delta\|x\|. \tag{3.5}$$

Thus, in view of (3.4), we obtain $\|\mathcal{G}x\|\le\|x\|$, $x\in\partial\Omega_1$. Hence, by Theorem 3.1, the operator $\mathcal{G}$ has at least one fixed point, which in turn implies that problem (1.1) has at least one solution. □

Example 3.1. Consider the problem

$$\begin{cases}{}^cD^qx(t)=(5+x^3(t))^{\frac{1}{2}}+2(t+1)(x-\sin x(t))-\sqrt{5}, & 0<t<1,\\ x(0)=-x(1),\quad x'(0)=-x'(1),\quad x''(0)=-x''(1),\quad x'''(0)=-x'''(1),\end{cases} \tag{3.6}$$

where $3<q\le 4$, and $T=1$.

It can easily be verified that all the assumptions of Theorem 3.3 hold. Consequently, the conclusion of Theorem 3.3 implies that problem (3.6) has at least one solution.

Our next existence result is based on Krasnoselskii's fixed point theorem [19].

Theorem 3.4. *Assume that $f:[0,T]\times\mathbb{R}\to\mathbb{R}$ is a jointly continuous function and the following assumptions hold:*

(A_1) $|f(t,x)-f(t,y)|\le L|x-y|,\ \forall t\in[0,T],\ x,y\in X;$
(A_2) $|f(t,x)|\le\mu(t),\ \forall(t,x)\in[0,1]\times X$, *and* $\mu\in L^1([0,T],R^+)$.

Then the anti-periodic boundary value problem (1.1) has at least one solution on $[0,T]$ *if*

$$\frac{LT^q}{\Gamma(q+1)}\left(1+\frac{q(q^2+11)}{24}\right)<1.$$

Proof. Let us fix

$$r\ge\frac{T^{q-1}\|\mu\|_{L^1}}{\Gamma(q)}\left(\frac{3}{2}+\frac{(q-1)[(q-1)^2+11]}{48}\right),$$

and consider $B_r=\{x\in C:\|x\|\le r\}$. We define the operators Φ and Ψ on B_r as

$$(\Phi x)(t)=\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}f(s,x(s))ds,$$

$$\begin{aligned}(\Psi x)(t)\ =\ &-\frac{1}{2}\int_0^T\frac{(T-s)^{q-1}}{\Gamma(q)}f(s,x(s))ds+\frac{1}{4}(T-2t)\int_0^T\frac{(T-s)^{q-2}}{\Gamma(q-1)}f(s,x(s))ds\\&+\frac{1}{4}(t(T-t))\int_0^T\frac{(T-s)^{q-3}}{\Gamma(q-2)}f(s,x(s))ds+\frac{(T-2t)(2t^2-2tT-T^2)}{48}\int_0^T\frac{(T-s)^{q-4}}{\Gamma(q-3)}f(s,x(s))ds.\end{aligned}$$

For $x,y\in B_r$, by virtue of (2.5), we find that

$$\|\Phi x+\Psi y\|\le\frac{T^{q-1}\|\mu\|_{L^1}}{\Gamma(q)}\left(\frac{3}{2}+\frac{(q-1)[(q-1)^2+11]}{48}\right)\le r.$$

Thus, $\Phi x+\Psi y\in B_r$. It follows from assumption (A_1) that Ψ is a contraction mapping for $\frac{LT^q}{\Gamma(q+1)}\left(1+\frac{q(q^2+11)}{24}\right)<1$. Continuity of f implies that the operator Φ is continuous. Also, Φ is uniformly bounded on B_r as

$$\|\Phi x\|\le\frac{T^{q-1}\|\mu\|_{L^1}}{\Gamma(q)}.$$

Now we prove the compactness of the operator Φ. In view of (A_1), we define

$$\sup_{(t,x)\in[0,T]\times B_r}\|f(t,x)\|=f_{\max}<\infty.$$

Then, for $t_1, t_2 \in [0, T]$, we have

$$\begin{aligned}\|(\Phi x)(t_1) - (\Phi x)(t_2)\| &= \left\| \frac{1}{\Gamma(q)} \int_0^{t_1} [(t_2 - s)^{q-1} - (t_1 - s)^{q-1}] f(s, x(s)) \mathrm{d}s + \int_{t_1}^{t_2} (t_2 - s)^{q-1} f(s, x(s)) \mathrm{d}s \right\| \\ &\le \frac{f_{\max}}{\Gamma(q+1)} |2(t_2 - t_1)^q + t_1^q - t_2^q|,\end{aligned}$$

which is independent of x and tends to zero as $t_2 \to t_1$. So Φ is relatively compact on B_r. Hence, by the Arzelá–Ascoli theorem, Φ is compact on B_r. Thus all the assumptions of theorem by Theorem 3.2 are satisfied. Therefore, the conclusion of Theorem 3.2 applies and the anti-periodic fractional boundary value problem (1.1) has at least one solution on $[0, T]$. This completes the proof. □

Now we prove an existence and uniqueness result by means of the Banach contraction principle.

Theorem 3.5. *Assume that $f : [0, T] \times X \to X$ is a jointly continuous function satisfying the condition*

$$\|f(t, x) - f(t, y)\| \le L\|x - y\|, \quad \forall t \in [0, T],\ x, y \in X$$

with

$$L \le \frac{\Gamma(q+1)}{T^q \left(3 + \frac{q(q^2+11)}{24}\right)}.$$

Then the anti-periodic boundary value problem (1.1) has a unique solution.

Proof. Setting $\sup_{t\in[0,T]} |f(t, 0)| = M$ and selecting $r \ge \frac{MT^q}{\Gamma(q+1)} \left(3 + \frac{q(q^2+11)}{24}\right)$, we show that $\mathcal{G}B_r \subset B_r$, where $B_r = \{x \in C : \|x\| \le r\}$. For $x \in B_r$, we have

$$\begin{aligned}\|(\mathcal{G}x)(t)\| &\le \max_{t\in[0,T]} \Bigg[\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} |f(s, x(s))| \mathrm{d}s + \frac{1}{2} \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} |f(s, x(s))| \mathrm{d}s \\ &\quad + \frac{1}{4} |T - 2t| \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} |f(s, x(s))| \mathrm{d}s + \frac{1}{4} |t(T-t)| \int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)} |f(s, x(s))| \mathrm{d}s \\ &\quad + \frac{1}{48} |6t^2T - 4t^3 - T^3| \int_0^T \frac{(T-s)^{q-4}}{\Gamma(q-3)} |f(s, x(s))| \mathrm{d}s \Bigg] \\ &\le \max_{t\in[0,T]} \Bigg[\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} (|f(s, x(s)) - f(s, 0)| + |f(s, 0)|) \mathrm{d}s \\ &\quad + \frac{1}{2} \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} (|f(s, x(s)) - f(s, 0)| + |f(s, 0)|) \mathrm{d}s \\ &\quad + \frac{1}{4} |T - 2t| \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} (|f(s, x(s)) - f(s, 0)| + |f(s, 0)|) \mathrm{d}s \\ &\quad + \frac{1}{4} |t(T-t)| \int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)} (|f(s, x(s)) - f(s, 0)| + |f(s, 0)|) \mathrm{d}s \\ &\quad + \frac{1}{48} |6t^2T - 4t^3 - T^3| \int_0^T \frac{(T-s)^{q-4}}{\Gamma(q-3)} (|f(s, x(s)) - f(s, 0)| + |f(s, 0)|) \mathrm{d}s \Bigg] \\ &\le (Lr + M) \max_{t\in[0,T]} \Bigg[\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \mathrm{d}s + \frac{1}{2\Gamma(q)} \int_0^T (T-s)^{q-1} \mathrm{d}s \\ &\quad + \frac{|T - 2t|}{4\Gamma(q-1)} \int_0^T (T-s)^{q-2} \mathrm{d}s + \frac{|t(T-t)|}{4\Gamma(q-2)} \int_0^T (T-s)^{q-3} \mathrm{d}s \\ &\quad + \frac{1}{48\Gamma(q-3)} |6t^2T - 4t^3 - T^3| \int_0^T (T-s)^{q-4} \mathrm{d}s \Bigg] \\ &\le (Lr + M) \left[\frac{T^q}{2\Gamma(q+1)} \left(3 + \frac{q(q^2+11)}{24}\right) \right] \le r.\end{aligned}$$

Now, for $x, y \in \mathrm{C}$ and for each $t \in [0, T]$, we obtain

$$\begin{aligned}
\|(\mathcal{G}x)(t) - (\mathcal{G}y)(t)\| &\leq \max_{t\in[0,T]}\Bigg[\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}\|f(s,x(s)) - f(s,y(s))\|\mathrm{d}s \\
&\quad + \frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)}\|f(s,x(s)) - f(s,y(s))\|\mathrm{d}s \\
&\quad + \frac{1}{4}|T-2t|\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)}\|f(s,x(s)) - f(s,y(s))\|\mathrm{d}s \\
&\quad + \frac{1}{4}|t(T-t)|\int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)}\|f(s,x(s)) - f(s,y(s))\|\mathrm{d}s \\
&\quad + \frac{1}{48}|6t^2T - 4t^3 - T^3|\int_0^T \frac{(T-s)^{q-4}}{\Gamma(q-3)}\|f(s,x(s)) - f(s,y(s))\|\mathrm{d}s\Bigg] \\
&\leq L\|x-y\|\max_{t\in[0,T]}\Bigg[\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\mathrm{d}s + \frac{1}{2\Gamma(q)}\int_0^T (T-s)^{q-1}\mathrm{d}s \\
&\quad + \frac{|T-2t|}{4\Gamma(q-1)}\int_0^T (T-s)^{q-2}\mathrm{d}s + \frac{|t(T-t)|}{4\Gamma(q-2)}\int_0^T (T-s)^{q-3}\mathrm{d}s \\
&\quad + \frac{1}{48\Gamma(q-3)}|6t^2T - 4t^3 - T^3|\int_0^T (T-s)^{q-4}\mathrm{d}s\Bigg] \\
&\leq \frac{LT^q}{2\Gamma(q+1)}\left(3 + \frac{q(q^2+11)}{24}\right)\|x-y\| = \Lambda_{L,T,q}\|x-y\|,
\end{aligned}$$

where $\Lambda_{L,T,q} = \frac{LT^q}{2\Gamma(q+1)}\left(3 + \frac{q(q^2+11)}{24}\right)$, which depends only on the parameters involved in the problem. As $\Lambda_{L,T,q} < 1$, $\mathcal{G}$ is a contraction. Thus, the conclusion of the theorem follows by the contraction mapping principle (Banach fixed point theorem). □

Example 3.2. Consider the following anti-periodic fractional boundary value problem

$$\begin{cases} {}^cD^{\frac{7}{2}}x(t) = \dfrac{1}{(t+2)^3}\dfrac{\|x\|}{1+\|x\|}, \quad t \in [0,2], \\ x(0) = -x(2), \qquad x'(0) = -x'(2), \qquad x''(0) = -x''(2), \qquad x'''(0) = -x'''(2), \end{cases} \tag{3.7}$$

where $q = 7/2$, and $T = 2$. Clearly, $L = \frac{1}{8}$ as $\|f(t,x) - f(t,y)\| \leq \frac{1}{8}\|x-y\|$. Further,

$$\frac{LT^q}{\Gamma(q+1)}\left(3 + \frac{q(q^2+11)}{24}\right) = 0.7769875842 < 1.$$

Thus, all the assumptions of Theorem 3.5 are satisfied. Hence, the fractional boundary value problem (3.7) has a unique solution on $[0, 2]$.

The last result of this section is based on the Leray–Schauder degree theory.

Theorem 3.6. *Let $f : [0,T] \times \mathbb{R} \to \mathbb{R}$. Assume that there exist constants $0 \leq \kappa < \frac{1}{\delta}$, where*

$$\delta = \frac{T^q}{\Gamma(q+1)}\left(\frac{3}{2} + \frac{q(q^2+11)}{48}\right)$$

and $M > 0$ such that $|f(t,x)| \leq \kappa|x| + M$ for all $t \in [0,T]$, $x \in \mathbb{R}$. Then the boundary value problem (1.1) has at least one solution.

Proof. Lets us define a fixed point problem by

$$x = \mathcal{G}x, \tag{3.8}$$

where $\mathcal{G}$ is defined by (3.1). Then we just need to prove the existence of at least one solution $x \in C[0,T]$ satisfying (3.8). Define a suitable ball $B_R \subset C[0,T]$ with radius $R > 0$ as

$$B_R = \{x \in C[0,T] : \max_{t\in[0,T]}|x(t)| < R\},$$

where R will be fixed later. Then, it is sufficient to show that $\mathcal{G}x : \overline{B}_R \to C[0, T]$ satisfies

$$x \neq \lambda \mathcal{G}x, \quad \forall x \in \partial B_R \text{ and } \forall\, \lambda \in [0, T]. \tag{3.9}$$

Let us set

$$H(\lambda, x) = \lambda \mathcal{G}x, \quad x \in C(\mathbb{R})\ \lambda \in [0, 1].$$

Then, by the Arzelá–Ascoli theorem, $h_\lambda(x) = x - H(\lambda, x) = x - \lambda \mathcal{G}x$ is completely continuous. If (3.9) is true, then the following Leray–Schauder degrees are well defined and by the homotopy invariance of topological degree, it follows that

$$\begin{aligned} \deg(h_\lambda, B_R, 0) &= \deg(I - \lambda \mathcal{G}x, B_R, 0) = \deg(h_1, B_R, 0) \\ &= \deg(h_0, B_R, 0) = \deg(I, B_R, 0) = 1 \neq 0, \quad 0 \in B_r, \end{aligned}$$

where I denotes the unit operator. By the nonzero property of the Leray–Schauder degree, $h_1(t) = x - \lambda \mathcal{G}x = 0$ for at least one $x \in B_R$. In order to prove (3.9), we assume that $x = \lambda \mathcal{G}x$ for some $\lambda \in [0, T]$ and for all $t \in [0, T]$ so that

$$\begin{aligned} |x(t)| = |\lambda \mathcal{G}x(t)| &\leq \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} |f(s, x(s))| ds + \frac{1}{2} \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} |f(s, x(s))| ds \\ &\quad + \frac{1}{4} |T - 2t| \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} |f(s, x(s))| ds + \frac{1}{4} |t(T-t)| \int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)} |f(s, x(s))| ds \\ &\quad + \frac{1}{48} |6t^2 T - 4t^3 - T^3| \int_0^T \frac{(T-s)^{q-4}}{\Gamma(q-3)} |f(s, x(s))| ds \\ &\leq (\kappa |x| + M) \Bigg[\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} ds + \frac{1}{2\Gamma(q)} \int_0^T (T-s)^{q-1} ds \\ &\quad + \frac{|T-2t|}{4\Gamma(q-1)} \int_0^T (T-s)^{q-2} ds + \frac{|t(T-t)|}{4\Gamma(q-2)} \int_0^T (T-s)^{q-3} ds \\ &\quad + \frac{1}{48\Gamma(q-3)} |6t^2 T - 4t^3 - T^3| \int_0^T (T-s)^{q-4} ds \Bigg] \\ &\leq \frac{(\kappa |x| + M) T^q}{\Gamma(q+1)} \left(\frac{3}{2} + \frac{q(q^2+11)}{48} \right) \\ &= (\kappa |x| + M) \delta \end{aligned}$$

which, on taking norm ($\sup_{t \in [0,T]} |x(t)| = \|x\|$) and solving for $\|x\|$, yields

$$\|x\| \leq \frac{M\delta}{1 - \kappa\delta}.$$

Letting $R = \frac{M\delta}{1-\kappa\delta} + 1$, (3.9) holds. This completes the proof. $\square$

Remark 3.1. The new existence results for a class of fourth-order nonlinear differential equations with anti-periodic boundary conditions can be obtained as a special case by taking $q = 4$ in the results of this section.

4. Existence results for multivalued maps

In this section, we discuss the existence of solutions for the anti-periodic boundary value problem of differential inclusion of fractional order $q \in (3, 4]$ given by (1.2). First of all, we recall some basic concepts of multivalued maps [20–22].

For a normed space $(X, \|.\|)$, let $P_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is closed}\}$, $P_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ is bounded}\}$, $P_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact}\}$, and $P_{cp,c}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact and convex}\}$. A multivalued map $G : X \to \mathcal{P}(X)$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$. The map G is bounded on bounded sets if $G(\mathbb{B}) = \cup_{x \in \mathbb{B}} G(x)$ is bounded in X for all $\mathbb{B} \in P_b(X)$ (i.e. $\sup_{x \in \mathbb{B}}\{\sup\{|y| : y \in G(x)\}\} < \infty$). G is called upper semi-continuous (u.s.c.) on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and if for each open set N of X containing $G(x_0)$, there exists an open neighborhood $\mathcal{N}_0$ of x_0 such that $G(\mathcal{N}_0) \subseteq N$. G is said to be completely continuous if $G(\mathbb{B})$ is relatively compact for every $\mathbb{B} \in P_b(X)$. If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph, i.e., $x_n \to x_*$, $y_n \to y_*$, $y_n \in G(x_n)$ imply $y_* \in G(x_*)$. G has a fixed point if there is $x \in X$ such that $x \in G(x)$. The fixed point set of the multivalued operator G will be denoted by *Fix* G. A multivalued map $G : [0, T] \to P_{cl}(\mathbb{R})$ is said to be measurable if for every $y \in \mathbb{R}$, the function

$$t \longmapsto d(y, G(t)) = \inf\{|y - z| : z \in G(t)\}$$

is measurable.

Let $C([0,T])$ denote a Banach space of continuous functions from $[0,T]$ into $\mathbb{R}$ with the norm $\|x\| = \sup_{t\in[0,T]}|x(t)|$. Let $L^1([0,T],\mathbb{R})$ be the Banach space of measurable functions $x:[0,T]\to\mathbb{R}$ which are Lebesgue integrable and normed by $\|x\|_{L^1} = \int_0^T |x(t)|\mathrm{d}t$.

Definition 4.1. A multivalued map $G:[0,T]\to\mathcal{P}(\mathbb{R})$ with nonempty compact convex values is said to be measurable if for any $x\in\mathbb{R}$, the function

$$t \longmapsto d(x,F(t)) = \inf\{|x-z| : z\in F(t)\}$$

is measurable.

Definition 4.2. A multivalued map $F:[0,T]\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$ is said to be Carathéodory if

(i) $t\longmapsto F(t,x)$ is measurable for each $x\in\mathbb{R}$;
(ii) $x\longmapsto F(t,x)$ is upper semi-continuous for almost all $t\in[0,T]$;

Further a Carathéodory function F is called L^1-Carathéodory if

(iii) for each $\alpha>0$, there exists $\varphi_\alpha\in L^1([0,T],\mathbb{R}^+)$ such that

$$\|F(t,x)\| = \sup\{|v| : v\in F(t,x)\} \le \varphi_\alpha(t)$$

for all $\|x\|_\infty\le\alpha$ and for a.e. $t\in[0,T]$.

Let E be a Banach space, X a nonempty closed subset of E and $G:X\to\mathcal{P}(E)$ a multivalued operator with nonempty closed values. G is lower semi-continuous (l.s.c.) if the set $\{x\in X : G(x)\cap B\neq\emptyset\}$ is open for any open set B in E. Let A be a subset of $[0,T]\times\mathbb{R}$. A is $\mathcal{L}\otimes\mathcal{B}$ measurable if A belongs to the σ-algebra generated by all sets of the form $\mathcal{J}\times D$, where $\mathcal{J}$ is Lebesgue measurable in $[0,T]$ and D is Borel measurable in $\mathbb{R}$. A subset A of $L^1([0,T],\mathbb{R})$ is decomposable if for all $u, v\in A$ and $\mathcal{J}\subset[0,T]$ measurable, the function $u\chi_{\mathcal{J}} + v\chi_{J-\mathcal{J}}\in A$, where $\chi_{\mathcal{J}}$ stands for the characteristic function of $\mathcal{J}$.

Definition 4.3. If $F:[0,T]\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$ is a multivalued map with nonempty compact values and $u(.)\in C([0,T],\mathbb{R})$, then the set of selections of $F(.,.)$, denoted by $S_{F,u}$, is of lower semi-continuous type if

$$S_{F,u} = \{w\in L^1([0,T],\mathbb{R}) : w(t)\in F(t,u(t)) \text{ for a.e. } t\in[0,T]\}$$

is lower semi-continuous with nonempty closed and decomposable values.

Let (X,d) be a metric space associated with the norm $|\cdot|$. The Pompeiu–Hausdorff distance of the closed subsets $A, B\subset X$ is defined by

$$d_H(A,B) = \max\{d^*(A,B), d^*(B,A)\},\qquad d^*(A,B) = \sup\{d(a,B) : a\in A\},$$

where $d(x,B) = \inf_{y\in B} d(x,y)$.

Definition 4.4. A multivalued operator N on X with nonempty values in X is called

(a) γ-Lipschitz if and only if there exists $\gamma>0$ such that

$$d_H(N(x),N(y)) \le \gamma d(x,y),\quad \text{for each } x,\ y\in X;$$

(b) a contraction if and only if it is γ-Lipschitz with $\gamma<1$.

The following lemmas will be used in what follows.

Lemma 4.1 ([23]). *Let X be a Banach space. Let $F:[0,T]\times\mathbb{R}\to\mathcal{P}_{cp,c}(X)$ be an L^1-Carathéodory multivalued map and let Θ be a linear continuous mapping from $L^1([0,T],X)$ to $C([0,T],X)$. Then the operator*

$$\Theta\circ S_F : C([0,T],X)\to P_{cp,c}(C([0,T],X)),\quad x\mapsto(\Theta\circ S_F)(x) = \Theta(S_{F,x})$$

is a closed graph operator in $C([0,T],X)\times C([0,T],X)$.

Lemma 4.2 ([24]). *Let Y be a separable metric space and let $N:Y\to\mathcal{P}(L^1([0,T],\mathbb{R}))$ be a lower semi-continuous multivalued map with closed decomposable values. Then $N(.)$ has a continuous selection, i.e., there exists a continuous mapping (single valued) $g:Y\to L^1([0,T],\mathbb{R})$ such that $g(y)\in N(y)$ for every $y\in Y$.*

Lemma 4.3 ([25]). *Let (X,d) be a complete metric space. If $N:X\to P_{cl}(X)$ is a contraction, then Fix $N\neq\emptyset$.*

Theorem 4.1. *Assume that*

(H_1) *$F:[0,T]\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$ is Carathéodory and has convex values;*
(H_2) *there exists a continuous nondecreasing function $\psi:[0,\infty)\to(0,\infty)$ and a function $p\in L^1([0,T],\mathbb{R}^+)$ such that*

$$\|F(t,x)\|_{\mathcal{P}} := \sup\{|y| : y\in F(t,x)\} \le p(t)\psi(\|x\|_\infty)\quad \text{for each } (t,x)\in[0,T]\times\mathbb{R};$$

(H_3) *there exists a number $M > 0$ such that*

$$\frac{M}{\gamma\psi(M)\|p\|_{L^1}} > 1, \tag{4.1}$$

where

$$\gamma = \frac{T^{q-1}}{\Gamma(q)}\left(\frac{3}{2} + \frac{(q-1)[(q-1)^2+11]}{48}\right).$$

Then the boundary value problem (1.2) has at least one solution on $[0, T]$.

Proof. Define an operator

$$\Omega(x) = \left\{ \begin{array}{l} h \in C([0,T],\mathbb{R}): \\ h(t) = \left\{ \begin{array}{l} \displaystyle\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)\mathrm{d}s - \frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f(s)\mathrm{d}s \\ \displaystyle\quad + \frac{(T-2t)}{4}\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f(s)\mathrm{d}s \\ \displaystyle\quad + \frac{t(T-t)}{4}\int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)} f(s)\mathrm{d}s \\ \displaystyle\quad + \frac{(6t^2T-4t^3-T^3)}{48}\int_0^T \frac{(T-s)^{q-4}}{\Gamma(q-3)} f(s)\mathrm{d}s, \; t \in [0,T] \end{array} \right. \end{array} \right\}$$

for $f \in S_{F,x}$. We will show that Ω satisfies the assumptions of the nonlinear alternative of the Leray–Schauder type. The proof consists of several steps. As a first step, we show that $\Omega(x)$ is convex for each $x \in C([0, T], \mathbb{R})$. For that, let $h_1, h_2 \in \Omega(x)$. Then there exist $f_1, f_2 \in S_{F,x}$ such that for each $t \in [0, T]$, we have

$$\begin{aligned} h_i(t) = {} & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f_i(s)\mathrm{d}s - \frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f_i(s)\mathrm{d}s + \frac{(T-2t)}{4}\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f_i(s)\mathrm{d}s \\ & + \frac{t(T-t)}{4}\int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)} f(s,x(s))\mathrm{d}s + \frac{(6t^2T-4t^3-T^3)}{48}\int_0^T \frac{(T-s)^{q-4}}{\Gamma(q-3)} f_i(s)\mathrm{d}s, \quad i = 1, 2. \end{aligned}$$

Let $0 \le \omega \le 1$. Then, for each $t \in [0, T]$, we have

$$\begin{aligned} [\omega h_1 + (1-\omega)h_2](t) = {} & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} [\omega f_1(s) + (1-\omega) f_2(s)]\mathrm{d}s - \frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} [\omega f_1(s) + (1-\omega) f_2(s)]\mathrm{d}s \\ & + \frac{(T-2t)}{4}\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} [\omega f_1(s) + (1-\omega) f_2(s)]\mathrm{d}s + \frac{t(T-t)}{4}\int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)} \\ & \times f(s,x(s))\mathrm{d}s + \frac{(6t^2T-4t^3-T^3)}{48}\int_0^T \frac{(T-s)^{q-4}}{\Gamma(q-3)} [\omega f_1(s) + (1-\omega) f_2(s)]\mathrm{d}s, \quad i = 1, 2. \end{aligned}$$

Since $S_{F,x}$ is convex (F has convex values), it follows that $\omega h_1 + (1 - \omega)h_2 \in \Omega(x)$.

Next, we show that $\Omega(x)$ maps bounded sets into bounded sets in $C([0, T], \mathbb{R})$. For a positive number r, let $B_r = \{x \in C([0, T], \mathbb{R}) : \|x\|_\infty \le r\}$ be a bounded set in $C([0, T], \mathbb{R})$. Then, for each $h \in \Omega(x)$, $x \in B_r$, there exists $f \in S_{F,x}$ such that

$$\begin{aligned} h(t) = {} & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)\mathrm{d}s - \frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f(s)\mathrm{d}s \\ & + \frac{(T-2t)}{4}\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f(s)\mathrm{d}s + \frac{t(T-t)}{4}\int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)} f(s)\mathrm{d}s \\ & + \frac{(6t^2T-4t^3-T^3)}{48}\int_0^T \frac{(T-s)^{q-4}}{\Gamma(q-3)} f(s)\mathrm{d}s. \end{aligned}$$

Then

$$\begin{aligned} |h(t)| = {} & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} |f(s)|\mathrm{d}s + \frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} |f(s)|\mathrm{d}s \\ & + \frac{1}{4}|T-2t|\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} |f(s)|\mathrm{d}s + \frac{1}{4}|t(T-t)|\int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)} |f(s)|\mathrm{d}s \end{aligned}$$

$$+\frac{1}{48}|(T-2t)(2t^2-2tT-T^2)|\int_0^T\frac{(T-s)^{q-4}}{\Gamma(q-3)}|f(s)|ds$$
$$\le\frac{T^{q-1}\psi(M)}{\Gamma(q)}\left(\frac{3}{2}+\frac{(q-1)[(q-1)^2+11]}{48}\right)\int_0^T|p(s)|ds.$$

Thus,

$$\|h\|_\infty\le\frac{T^{q-1}\psi(M)}{\Gamma(q)}\left(\frac{3}{2}+\frac{(q-1)[(q-1)^2+11]}{48}\right)\int_0^T|p(s)|ds,$$

where we have used (2.5), (H_2) and (H_3).

Now we show that Ω maps bounded sets into equicontinuous sets of $C([0,T],\mathbb{R})$. Let $t',t''\in[0,T]$ with $t'<t''$ and $x\in B_r$, where B_r is a bounded set of $C([0,T],\mathbb{R})$. In view of (H_3), For each $h\in\Omega(x)$, we obtain

$$\begin{aligned}|h(t'')-h(t')| &= \Bigg|\frac{1}{\Gamma(q)}\int_0^{t''}(t''-s)^{q-1}f(s)ds-\frac{1}{\Gamma(q)}\int_0^{t'}(t'-s)^{q-1}f(s)ds\\
&\quad-\frac{(t''-t')}{2}\int_0^T\frac{(T-s)^{q-2}}{\Gamma(q-1)}f(s)ds+\frac{(t''-t')[T-t''-t']}{4}\int_0^T\frac{(T-s)^{q-3}}{\Gamma(q-2)}f(s)ds\\
&\quad+\frac{(t''-t')[3T(t''+t')-2(t''^2+t''t'+t'^2)]}{24}\int_0^T\frac{(T-s)^{q-4}}{\Gamma(q-3)}f(s)ds\Bigg|\\
&\le\frac{1}{\Gamma(q)}\int_0^{t'}|(t''-s)^{q-1}-(t'-s)^{q-1}|\,|f(s)|ds\\
&\quad+\frac{1}{\Gamma(q)}\int_{t'}^{t''}|(t''-s)^{q-1}||f(s)|ds+\left|\frac{(t''-t')}{2}\right|\int_0^T\frac{(T-s)^{q-2}}{\Gamma(q-1)}|f(s)|ds\\
&\quad+\left|\frac{(t''-t')[T-t''-t']}{4}\right|\int_0^T\frac{(T-s)^{q-3}}{\Gamma(q-2)}|f(s)|ds\\
&\quad+\left|\frac{(t''-t')[3T(t''+t')-2(t''^2+t''t'+t'^2)]}{24}\right|\int_0^T\frac{(T-s)^{q-4}}{\Gamma(q-3)}|f(s)|ds\\
&\le\frac{1}{\Gamma(q)}\int_0^{t'}|(t''-s)^{q-1}-(t'-s)^{q-1}|\,|p(s)\psi(M)|ds\\
&\quad+\frac{1}{\Gamma(q)}\int_{t'}^{t''}|(t''-s)^{q-1}||p(s)\psi(M)|ds+\left|\frac{(t''-t')}{2}\right|\int_0^T\frac{(T-s)^{q-2}}{\Gamma(q-1)}|p(s)\psi(M)|ds\\
&\quad+\left|\frac{(t''-t')[T-t''-t']}{4}\right|\int_0^T\frac{(T-s)^{q-3}}{\Gamma(q-2)}|p(s)\psi(M)|ds\\
&\quad+\left|\frac{(t''-t')[3T(t''+t')-2(t''^2+t''t'+t'^2)]}{24}\right|\int_0^T\frac{(T-s)^{q-4}}{\Gamma(q-3)}|p(s)\psi(M)|ds.\end{aligned}$$

Obviously the right hand side of the above inequality tends to zero independently of $x\in B_{r'}$ as $t''-t'\to 0$. As Ω satisfies the above three assumptions, it follows by the Arzelá–Ascoli theorem that $\Omega: C([0,T],\mathbb{R})\to\mathcal{P}(C([0,T],\mathbb{R}))$ is completely continuous.

In our next step, we show that Ω has a closed graph. Let $x_n\to x_*$, $h_n\in\Omega(x_n)$ and $h_n\to h_*$. Then we need to show that $h_*\in\Omega(x_*)$. Associated with $h_n\in\Omega(x_n)$, there exists $f_n\in S_{F,x_n}$ such that for each $t\in[0,T]$,

$$\begin{aligned}h_n(t)&=\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}f_n(s)ds-\frac{1}{2}\int_0^T\frac{(T-s)^{q-1}}{\Gamma(q)}f_n(s)ds+\frac{(T-2t)}{4}\int_0^T\frac{(T-s)^{q-2}}{\Gamma(q-1)}f_n(s)ds\\
&\quad+\frac{t(T-t)}{4}\int_0^T\frac{(T-s)^{q-3}}{\Gamma(q-2)}f_n(s)ds+\frac{(6t^2T-4t^3-T^3)}{48}\int_0^T\frac{(T-s)^{q-4}}{\Gamma(q-3)}f_n(s)ds.\end{aligned}$$

Thus we have to show that there exists $f_*\in S_{F,x_*}$ such that for each $t\in[0,T]$,

$$\begin{aligned}h_*(t)&=\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}f_*(s)ds-\frac{1}{2}\int_0^T\frac{(T-s)^{q-1}}{\Gamma(q)}f_*(s)ds+\frac{(T-2t)}{4}\int_0^T\frac{(T-s)^{q-2}}{\Gamma(q-1)}f_*(s)ds\\
&\quad+\frac{t(T-t)}{4}\int_0^T\frac{(T-s)^{q-3}}{\Gamma(q-2)}f_*(s)ds+\frac{(6t^2T-4t^3-T^3)}{48}\int_0^T\frac{(T-s)^{q-4}}{\Gamma(q-3)}f_*(s)ds.\end{aligned}$$

Let us consider the continuous linear operator $\Theta : L^1([0,T],\mathbb{R}) \to C([0,T],\mathbb{R})$ given by

$$f \mapsto \Theta(f)(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)\mathrm{d}s - \frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f(s)\mathrm{d}s + \frac{(T-2t)}{4}\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f(s)\mathrm{d}s + \frac{t(T-t)}{4}\int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)} f(s)\mathrm{d}s + \frac{(6t^2T-4t^3-T^3)}{48}\int_0^T \frac{(T-s)^{q-4}}{\Gamma(q-3)} f(s)\mathrm{d}s.$$

Observe that

$$\begin{aligned}\|h_n(t)-h_*(t)\| = \Bigg\| &\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}(f_n(s)-f_*(s))\mathrm{d}s - \frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)}(f_n(s)-f_*(s))\mathrm{d}s \\ &+ \frac{(T-2t)}{4}\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)}(f_n(s)-f_*(s))\mathrm{d}s + \frac{t(T-t)}{4}\int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)}(f_n(s)-f_*(s))\mathrm{d}s \\ &+ \frac{(6t^2T-4t^3-T^3)}{48}\int_0^T \frac{(T-s)^{q-4}}{\Gamma(q-3)}(f_n(s)-f_*(s))\mathrm{d}s\Bigg\| \to 0\end{aligned}$$

as $n \to \infty$. Thus, it follows by Lemma 4.1 that $\Theta \circ S_F$ is a closed graph operator. Further, we have $h_n(t) \in \Theta(S_{F,x_n})$. Since $x_n \to x_*$, we have

$$h_*(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f_*(s)\mathrm{d}s - \frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f_*(s)\mathrm{d}s + \frac{(T-2t)}{4}\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f_*(s)\mathrm{d}s + \frac{t(T-t)}{4}\int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)} f_*(s)\mathrm{d}s + \frac{(6t^2T-4t^3-T^3)}{48}\int_0^T \frac{(T-s)^{q-4}}{\Gamma(q-3)} f_*(s)\mathrm{d}s,$$

for some $f_* \in S_{F,x_*}$.

Finally, we discuss a priori bounds on solutions. Let x be a solution of (1.2). Then there exists $f \in L^1([0,T],\mathbb{R})$ with $f \in S_{F,x}$ such that, for $t \in [0,T]$, we have

$$x(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)\mathrm{d}s - \frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f(s)\mathrm{d}s + \frac{(T-2t)}{4}\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f(s)\mathrm{d}s + \frac{t(T-t)}{4}\int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)} f(s)\mathrm{d}s + \frac{(T-2t)(2t^2-2tT-T^2)}{48}\int_0^T \frac{(T-s)^{q-4}}{\Gamma(q-3)} f(s)\mathrm{d}s.$$

For each $t \in [0,T]$, using (H_2) and (2.5), we obtain

$$\begin{aligned}|x(t)| &\le \frac{T^{q-1}}{\Gamma(q)}\left(\frac{3}{2} + \frac{(q-1)[(q-1)^2+11]}{48}\right)\int_0^T f(s)\mathrm{d}s \\ &\le \gamma\psi(\|x\|_\infty)\int_0^T p(s)\mathrm{d}s.\end{aligned}$$

Consequently, we have

$$\frac{\|x\|_\infty}{\gamma\psi(\|x\|_\infty)\|p\|_{L^1}} \le 1.$$

In view of (H_3), there exists M such that $\|x\|_\infty \ne M$. Let us set

$$U = \{x \in C([0,T],\mathbb{R}) : \|x\|_\infty < M+1\}.$$

Note that the operator $\Omega : \overline{U} \to \mathcal{P}(C([0,T],\mathbb{R}))$ is upper semi-continuous and completely continuous. From the choice of U, there is no $x \in \partial U$ such that $x \in \mu\Omega(x)$ for some $\mu \in (0,1)$. Consequently, by the nonlinear alternative of the Leray–Schauder type [26], we deduce that Ω has a fixed point $x \in \overline{U}$ which is a solution of problem (1.2). This completes the proof. □

As a next result, we study the case when F is not necessarily convex valued. Our strategy to deal with these problems is based on the nonlinear alternative of the Leray–Schauder type together with the selection theorem of Bressan and Colombo [24] for lower semi-continuous maps with decomposable values.

Theorem 4.2. *Assume that (H_2), (H_3) and the following conditions hold:*

(H_4) *$F : [0,T]\times\mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a nonempty compact-valued multivalued map such that*
   (a) *$(t,x) \longmapsto F(t,x)$ is $\mathcal{L}\otimes\mathcal{B}$ measurable,*
   (b) *$x \longmapsto F(t,x)$ is lower semi-continuous for each $t \in [0,T]$;*

(H_5) *for each* $\sigma > 0$, *there exists* $\varphi_\sigma \in L^1([0,T],\mathbb{R}^+)$ *such that*

$$\|F(t,x)\| = \sup\{|y| : y \in F(t,x)\} \leq \varphi_\sigma(t) \quad \textit{for all } \|x\|_\infty \leq \sigma \textit{ and for a.e. } t \in [0,T].$$

Then the boundary value problem (1.2) *has at least one solution on* $[0,T]$.

Proof. It follows from (H_4) and (H_5) that F is of l.s.c. type. Then from Lemma 4.2, there exists a continuous function $f : C([0,T],\mathbb{R}) \to L^1([0,T],\mathbb{R})$ such that $f(x) \in \mathcal{F}(x)$ for all $x \in C([0,T],\mathbb{R})$.

Consider the problem

$$\begin{cases} {}^cD^q x(t) = f(x(t)), \quad t \in [0,T],\ T > 0,\ 3 < q \leq 4, \\ x(0) = -x(T), \qquad x'(0) = -x'(T), \qquad x''(0) = -x''(T), \qquad x'''(0) = -x'''(T). \end{cases} \tag{4.2}$$

Observe that if $x \in C^2([0,T])$ is a solution of (4.2), then x is a solution to problem (1.2). In order to transform problem (4.2) into a fixed point problem, we define the operator $\overline{\Omega}$ as

$$\begin{aligned} \overline{\Omega}x(t) = {} & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(x(s))ds - \frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f(x(s))ds + \frac{(T-2t)}{4}\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f(x(s))ds \\ & + \frac{t(T-t)}{4}\int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)} f(x(s))ds + \frac{(6t^2T - 4t^3 - T^3)}{48}\int_0^T \frac{(T-s)^{q-4}}{\Gamma(q-3)} f(x(s))ds. \end{aligned}$$

It can easily be shown that $\overline{\Omega}$ is continuous and completely continuous. The remaining part of the proof is similar to that of Theorem 4.1. So we omit it. This completes the proof. □

Now we prove the existence of solutions for problem (1.2) with a nonconvex-valued right hand side by applying a fixed point theorem for a multivalued map due to Covitz and Nadler [25].

Theorem 4.3. *Assume that the following conditions hold:*

(H_6) $F : [0,T] \times \mathbb{R} \to P_{cp}(\mathbb{R})$ *is such that* $F(.,x) : [0,T] \to P_{cp}(\mathbb{R})$ *is measurable for each* $x \in \mathbb{R}$.

(H_7) $d_H(F(t,x),F(t,\bar{x})) \leq m(t)|x-\bar{x}|$ *for almost all* $t \in [0,T]$ *and* $x,\bar{x} \in \mathbb{R}$ *with* $m \in L^1([0,T],\mathbb{R}^+)$ *and* $d(0,F(t,0)) \leq m(t)$ *for almost all* $t \in [0,T]$.

Then the boundary value problem (1.2) *has at least one solution on* $[0,T]$ *if*

$$\frac{T^{q-1}}{\Gamma(q)}\left(\frac{3}{2} + \frac{(q-1)[(q-1)^2+11]}{48}\right) < 1.$$

Proof. Observe that the set $S_{F,x}$ is nonempty for each $x \in C([0,T],\mathbb{R})$ by assumption (H_6), so F has a measurable selection (see [27, Theorem III.6]). Now we show that the operator Ω satisfies the assumptions of Lemma 4.3. To show that $\Omega(x) \in P_{cl}((C[0,T],\mathbb{R}))$ for each $x \in C([0,T],\mathbb{R})$, let $\{u_n\}_{n\geq 0} \in \Omega(x)$ be such that $u_n \to u (n \to \infty)$ in $C([0,T],\mathbb{R})$. Then $u \in C([0,T],\mathbb{R})$ and there exists $v_n \in S_{F,x}$ such that, for each $t \in [0,T]$,

$$\begin{aligned} u_n(t) = {} & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} v_n(s)ds - \frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} v_n(s)ds + \frac{(T-2t)}{4}\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} v_n(s)ds \\ & + \frac{t(T-t)}{4}\int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)} v_n(s)ds + \frac{(6t^2T - 4t^3 - T^3)}{48}\int_0^T \frac{(T-s)^{q-4}}{\Gamma(q-3)} v_n(s)ds. \end{aligned}$$

As F has compact values, we pass onto a subsequence to obtain that v_n converges to v in $L^1([0,T],\mathbb{R})$. Thus, $v \in S_{F,x}$ and for each $t \in [0,T]$,

$$\begin{aligned} u_n(t) \to u(t) = {} & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} v(s)ds - \frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} v(s)ds + \frac{(T-2t)}{4}\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} v(s)ds \\ & + \frac{t(T-t)}{4}\int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)} v(s)ds + \frac{(6t^2T - 4t^3 - T^3)}{48}\int_0^T \frac{(T-s)^{q-4}}{\Gamma(q-3)} v(s)ds. \end{aligned}$$

Hence, $u \in \Omega(x)$.

Next we show that there exists $\tau < 1$ such that

$$d_H(\Omega(x),\Omega(\bar{x})) \leq \tau\|x-\bar{x}\|_\infty \quad \text{for each } x,\bar{x} \in C([0,T],\mathbb{R}).$$

Let $x, \bar{x} \in C([0,T],\mathbb{R})$ and $h_1 \in \Omega(x)$. Then there exists $v_1(t) \in F(t, x(t))$ such that, for each $t \in [0, T]$,

$$\begin{aligned} h_1(t) = {} & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} v_1(s)\mathrm{d}s - \frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} v_1(s)\mathrm{d}s + \frac{(T-2t)}{4}\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} v_1(s)\mathrm{d}s \\ & + \frac{t(T-t)}{4}\int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)} v_1(s)\mathrm{d}s + \frac{(6t^2T-4t^3-T^3)}{48}\int_0^T \frac{(T-s)^{q-4}}{\Gamma(q-3)} v_1(s)\mathrm{d}s. \end{aligned}$$

By (H_7), we have

$$d_H(F(t,x), F(t,\bar{x})) \le m(t)|x(t) - \bar{x}(t)|.$$

So, there exists $w \in F(t, \bar{x}(t))$ such that

$$|v_1(t) - w| \le m(t)|x(t) - \bar{x}(t)|, \quad t \in [0, T].$$

Define $V : [0, T] \to \mathcal{P}(\mathbb{R})$ by

$$V(t) = \{w \in \mathbb{R} : |v_1(t) - w| \le m(t)|x(t) - \bar{x}(t)|\}.$$

Since the multivalued operator $V(t) \cap F(t, \bar{x}(t))$ is measurable [27, Proposition III.4], there exists a function $v_2(t)$ which is a measurable selection for V. So $v_2(t) \in F(t, \bar{x}(t))$ and for each $t \in [0, T]$, we have $|v_1(t) - v_2(t)| \le m(t)|x(t) - \bar{x}(t)|$.

For each $t \in [0, T]$, let us define

$$\begin{aligned} h_2(t) = {} & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} v_2(s)\mathrm{d}s - \frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} v_2(s)\mathrm{d}s + \frac{(T-2t)}{4}\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} v_2(s)\mathrm{d}s \\ & + \frac{t(T-t)}{4}\int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)} v_2(s)\mathrm{d}s + \frac{(6t^2T-4t^3-T^3)}{48}\int_0^T \frac{(T-s)^{q-4}}{\Gamma(q-3)} v_2(s)\mathrm{d}s. \end{aligned}$$

Thus, for each $t \in [0, T]$, it follows by (2.5) that

$$\begin{aligned} |h_1(t) - h_2(t)| \le {} & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} |v_1(s) - v_2(s)|\mathrm{d}s - \frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} |v_1(s) - v_2(s)|\mathrm{d}s \\ & + \frac{(T-2t)}{4}\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} |v_1(s) - v_2(s)|\mathrm{d}s + \frac{t(T-t)}{4}\int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)} |v_1(s) - v_2(s)|\mathrm{d}s \\ & + \frac{(6t^2T-4t^3-T^3)}{48}\int_0^T \frac{(T-s)^{q-4}}{\Gamma(q-3)} |v_1(s) - v_2(s)|\mathrm{d}s \\ \le {} & \frac{T^{q-1}}{\Gamma(q)}\left(\frac{3}{2} + \frac{(q-1)[(q-1)^2+11]}{48}\right)\int_0^T m(s)\|x - \bar{x}\|\mathrm{d}s. \end{aligned}$$

Hence,

$$\|h_1(t) - h_2(t)\|_\infty \le \frac{T^{q-1}}{\Gamma(q)}\left(\frac{3}{2} + \frac{(q-1)[(q-1)^2+11]}{48}\right)\|m\|_{L^1}\|x - \bar{x}\|_\infty.$$

Analogously, interchanging the roles of x and $\bar{x}$, we obtain

$$\begin{aligned} d_H(\Omega(x), \Omega(\bar{x})) & \le \tau\|x - \bar{x}\|_\infty \\ & \le \frac{T^{q-1}}{\Gamma(q)}\left(\frac{3}{2} + \frac{(q-1)[(q-1)^2+11]}{48}\right)\|m\|_{L^1}\|x - \bar{x}\|_\infty. \end{aligned}$$

Since Ω is a contraction, it follows by Lemma 4.3 that Ω has a fixed point x which is a solution of (1.2). This completes the proof. □

Example 4.1. Consider the anti-periodic fractional inclusion boundary value problem given by

$$\begin{cases} {}^cD^{15/4}x(t) \in F(t, x(t)), & t \in [0, 1], \\ x(0) = -x(1), \quad x'(0) = -x'(1), & x''(0) = -x''(1), \quad x'''(0) = -x'''(1), \end{cases} \tag{4.3}$$

where $T = 1$, $q = 15/4$, and $F : [0, 1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map given by

$$x \to F(t,x) = \left[\frac{x^3}{x^3+3} + t^3 + 3, \frac{x}{x+1} + t + 1\right].$$

For $f \in F$, we have

$$|f| \leq \max\left(\frac{x^3}{x^3+3} + t^3 + 3, \frac{x}{x+1} + t + 1\right) \leq 5, \quad x \in \mathbb{R}.$$

Thus,

$$\|F(t,x)\|_{\mathcal{P}} := \sup\{|y| : y \in F(t,x)\} \leq 5 = p(t)\psi(\|x\|_\infty), \quad x \in \mathbb{R},$$

with $p(t) = 1$, $\psi(\|x\|_\infty) = 5$. Further, using the condition

$$\frac{\Gamma(q)M}{T^{q-1}\psi(M)\|p\|_L^1}\left(\frac{3}{2} + \frac{(q-1)[(q-1)^2+11]}{48}\right)^{-1} > 1,$$

we find that $M > \frac{625}{176\Gamma(3/4)}$. Clearly, all the conditions of Theorem 4.1 are satisfied. So there exists at least one solution of problem (4.3) on [0, 1].

Remark 4.1. The new existence results for a class of fourth-order nonlinear differential inclusions with anti-periodic boundary conditions follow as a special case by taking $q = 4$ in the results of this section.

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