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Hamiltonian-connected graphs

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Abstract

For a simple graph G, let $NCD(G) = \min\{|N(u) \cup N(v)| + d(w) : u, v, w \in V(G), uv \notin E(G), wv \text{ or } wu \notin E(G)\}$. In this paper, we prove that if $NCD(G) \geq |V(G)|$, then either G is Hamiltonian-connected, or G belongs to a well-characterized class of graphs. The former results by Dirac, Ore and Faudree et al. are extended.

Keywords: Hamiltonian path; Hamiltonian graphs; Hamiltonian-connected graphs; Neighborhood union; Degree condition

1. Introduction

Graphs considered in this paper are finite and simple. Undefined notations and terminologies can be found in [1]. In particular, we use $V(G)$, $E(G)$, $\kappa(G)$, $\delta(G)$ and $\alpha(G)$ to denote *the vertex set*, *the edge set*, *the connectivity*, *the minimum degree* and *the independence number* of G, respectively. If G is a graph and $u, v \in V(G)$, then a path in G from u to v is called a (u, v)-path of G. If $v \in V(G)$ and H is a subgraph of G, then $N_H(v)$ denotes the set of vertices in H that are adjacent to v in G. Thus, $d_H(v)$, the degree of v relative to H, is $|N_H(v)|$. We also write $d(v)$ for $d_G(v)$ and $N(v)$ for $N_G(v)$. If C and H are subgraphs of G, then $N_C(H) = \cup_{u \in V(H)} N_C(u)$, and $G - C$ denotes the subgraph of G induced by $V(G) - V(C)$. For vertices $u, v \in V(G)$, the distance between u and v, denoted by $d(u, v)$, is the length of a shortest (u, v)-path in G, or ∞ if no such path exists. Let $P_m = x_1 x_2 \cdots x_m$ denote a path of order m. Define $N^+_{P_m}(u) = \{x_{i+1} \in V(P_m) : x_i \in N_{P_m}(u)\}$ and $N^-_{P_m}(u) = \{x_{i-1} \in V(P_m) : x_i \in N_{P_m}(u)\}$. That means if $x_1 \in N_{P_m}(u)$, then $|N^-_{P_m}(u)| = |N_{P_m}(u)| - 1$ and if $x_m \in N_{P_m}(u)$, then $|N^+_{P_m}(u)| = |N_{P_m}(u)| - 1$.

For a graph G, define $NC(G) = \min\{|N(u) \cup N(v)| : u, v \in V(G), uv \notin E(G)\}$ and $NCD(G) = \min\{|N(u) \cup N(v)| + d(w) : u, v, w \in V(G), uv \notin E(G), wv \text{ or } wu \notin E(G)\}$.

Let G and H be two graphs. We use $G \cup H$ to denote the disjoint union of G and H and $G \bigvee H$ to denote the graph obtained from $G \cup H$ by joining every vertex of G to every vertex of H. We use K_n and K_n^c to denote the complete graph on n vertices and the empty graph on n vertices, respectively. Let G_n denote the family of all simple graphs of order n. For notational convenience, we also use G_n to denote a simple graph of order n. As an example,

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$G_2 \in \{K_2, K_2^c\}$. Define $G_2 : G_n$ to be the family of 2-connected graphs each of which is obtained from $G_2 \cup G_n$ by joining every vertex of G_2 to some vertices of G_n so that the resulting graph G satisfies $NCD(G) \geq |V(G)| = n+2$. For notational convenience, we also use $G_2 : G_n$ to denote a member in the family.

A graph G is *Hamiltonian* if it has a spanning cycle, and *Hamiltonian-connected* if for every pair of vertices $u, v \in V(G)$, G has a spanning (u, v)-path. There have been intensive studies on sufficient degree and/or neighborhood union conditions for Hamiltonian graphs and Hamiltonian-connected graphs. The following is a summary of these results that are related to our study.

Theorem 1.1. *Let G be a simple graph on n vertices.*

(i) *(Dirac, [2]). If $\delta(G) \geq n/2$, then G is Hamiltonian.*
(ii) *(Ore, [3]). If $d(u) + d(v) \geq n$ for each pair of nonadjacent vertices $u, v \in V(G)$, then G is Hamiltonian.*
(iii) *(Faudree et al., [4]). If G is 3-connected, and if $NC(G) \geq (2n+1)/3$, then G is Hamiltonian-connected.*
(iv) *(Faudree et al., [5]). If G is 2-connected, and if $NC(G) \geq n$, then G is Hamiltonian.*
(v) *(Wei, [6]). If G is a 2-connected, and if $\min\{d(u) + d(v) + d(w) - |N(u) \cap N(v) \cap N(w)| : u, v, w \in V(G), uv, vw, wu \notin E(G)\} \geq n+1$, then G is Hamiltonian-connected with some well-characterized exceptional graphs.*

Motivated by the results above, this paper aims to investigate the Hamiltonian and Hamiltonian-connected properties of graphs with relatively large $NCD(G)$. The main theorem is the following.

Theorem 1.2. *If G is a 2-connected graph with n vertices and if $NCD(G) \geq n$, then one of the following must hold:*

(i) *G is Hamiltonian-connected,*
(ii) $G \in \{G_2 : (K_s \cup K_h), G_{n/2} \bigvee K_{n/2}^c, G_2 : (K_s \cup K_h \cup K_t), G_3 \bigvee (K_s \cup K_h \cup K_t)\}$.

Let $G = G_2 : (K_s \cup K_h \cup K_t)$, and let x be a vertex in K_s and y a vertex in K_h. Then $d(x) + d(y) < |V(G)|$. Also, $G_3 \bigvee (K_s \cup K_h \cup K_t)$ satisfies the condition that $d(x) + d(y) \geq n$ for any two nonadjacent vertices x, y if and only if $s = h = t = 1$. Thus Corollary 1.3 below follows from Theorem 1.2 immediately and it extends Theorem 1.1(ii).

Corollary 1.3. *If G is a graph of order n satisfying $d(x) + d(y) \geq n$ for every pair of nonadjacent vertices $x, y \in V(G)$, then G is Hamiltonian-connected or $G \in \{G_2 : (K_s \cup K_h), G_{n/2} \bigvee K_{n/2}^c\}$.*

Since none of $G_2 : (K_s \cup K_h)$, $G_{n/2} \bigvee K_{n/2}^c$, $G_2 : (K_s \cup K_h \cup K_t)$ and $G_3 \bigvee (K_s \cup K_h \cup K_t)$ satisfies the condition that $d(x) + d(y) \geq n+1$ for every pair of nonadjacent vertices x, y, Theorem 1.2 also implies the following result of Ore [4].

Corollary 1.4 (*Ore, [7]*). *If G is a 2-connected graph of order n satisfying $d(x) + d(y) \geq n+1$ for every pair of nonadjacent vertices $x, y \in V(G)$, then G is Hamiltonian-connected.*

As $G_2 : (K_s \cup K_h)$, $G_{n/2} \bigvee K_{n/2}^c$ and $G_3 \bigvee (K_s \cup K_h \cup K_t)$ are all Hamiltonian, Theorem 1.2 implies the following Theorem 1.5.

Theorem 1.5. *If G is a 2-connected graph with n vertices such that $NCD(G) \geq n$, then G is Hamiltonian or $G \in \{G_2 : (K_s \cup K_h \cup K_t)\}$.*

Clearly, Theorem 1.5 extends Theorem 1.1(iv). Note that for any graph G, $NCD(G) \geq NC(G) + \delta(G)$. Moreover, if $G = K_3 \bigvee (K_s \cup K_h \cup K_t)$ and if $\max\{s, h, t\} \neq \min\{s, h, t\}$, then $NC(G) + \delta(G) \leq |V(G)| - 1$. Thus Theorem 1.2 also implies the following result.

Corollary 1.6. *If G is a 2-connected graph with n vertices such that $NC(G) + \delta(G) \geq n$, then G is Hamiltonian-connected or $G \in \{G_2 : (K_s \cup K_h), G_{n/2} \bigvee K_{n/2}^c, G_2 : (K_s \cup K_h \cup K_t), G_3 \bigvee (K_{(n-3)/3} \cup K_{(n-3)/3} \cup K_{(n-3)/3})\}$.*

2. Proof of Theorem 1.2

For a path $P_m = x_1x_2 \cdots x_m$, we use $[x_i, x_j]$ to denote the section $x_ix_{i+1} \cdots x_j$ of the path P_m if $i < j$, and to denote the section $x_ix_{i-1} \cdots x_j$ of the path P_m if $i > j$. For notational convenience, we also use $[x_i, x_j]$ to denote the vertex set of this path. If P_1 is an (x, y)-path and P_2 is a (y, z)-path in a graph G such that $V(P_1) \cap V(P_2) = \{y\}$, then P_1P_2 denotes the (x, z)-path of G induced by $E(P_1) \cup E(P_2)$.

Let G be a 2-connected graph on n vertices such that

$$NCD(G) \geq n. \quad (1)$$

We shall assume that G is not Hamiltonian-connected to show that Theorem 1.2(ii) must hold. Thus there exist $x, y \in V(G)$ such that G does not have a spanning (x, y)-path. Let

$$P_m = x_1x_2 \cdots x_m \text{ be a longest } (x, y)\text{-path in } G, \quad (2)$$

where $x_1 = x$ and $x_m = y$. Since P_m is not a Hamiltonian path, $G - P_m$ has at least one component.

Lemma 2.1. *Suppose that H is a component of $G - P_m$. Then each of the following holds.*

(i) *$\forall i$ with $1 < i < m$, if $x_i \in N_{P_m}(H) \setminus \{x_1, x_m\}$, then $x_{i+1} \notin N_{P_m}(H)$ and $x_{i-1} \notin N_{P_m}(H)$; if $x_1 \in N_{P_m}(H)$, then $x_2 \notin N_{P_m}(H)$, and if $x_m \in N_{P_m}(H)$, then $x_{m-1} \notin N_{P_m}(H)$.*

(ii) *If $x_i, x_j \in N_{P_m}(H)$ with $1 \leq i < j < m$, then $x_{i+1}x_{j+1} \notin E(G)$; if $x_i, x_j \in N_{P_m}(H)$ with $1 < i < j \leq m$, then $x_{i-1}x_{j-1} \notin E(G)$. Consequently, both $N^+_{P_m}(H)$ and $N^-_{P_m}(H)$ are independent sets.*

(iii) *Let $x_i, x_j \in N_{P_m}(H)$ with $1 \leq i < j < m$. If $x_tx_{j+1} \in E(G)$ for some vertex $x_t \in [x_{j+2}, x_m]$, then $x_{t-1}x_{i+1} \notin E(G)$ and $x_{t-1} \notin N_{P_m}(H)$; if $x_tx_{j+1} \in E(G)$ for some vertex $x_t \in [x_{i+1}, x_j]$, then $x_{t+1}x_{i+1} \notin E(G)$.*

(iii)′ *Let $x_i, x_j \in N_{P_m}(H)$ with $1 < i < j \leq m$. If $x_tx_{i-1} \in E(G)$ for some vertex $x_t \in [x_1, x_{i-2}]$, then $x_{t+1}x_{j-1} \notin E(G)$ and $x_{t+1} \notin N_{P_m}(H)$; if $x_tx_{i-1} \in E(G)$ for some vertex $x_t \in [x_{i+1}, x_j]$, then $x_{t-1}x_{j-1} \notin E(G)$.*

(iv) *If $x_i, x_j \in N_{P_m}(H)$ with $1 \leq i < j < m$, then no vertex of $G - (V(P_m) \cup V(H))$ is adjacent to both x_{i+1} and x_{j+1}; if $x_i, x_j \in N_{P_m}(H)$ with $1 < i < j \leq m$, then no vertex of $G - (V(P_m) \cup V(H))$ is adjacent to both x_{i-1} and x_{j-1}.*

(v) *Suppose that $u \in V(H)$ and $\{x_1, x_m\} \subseteq N_{P_m}(u)$. If $x_i, x_j \in N_{P_m}(H)$ with $1 \leq i < j < m$, then for any $v \in V(G) \setminus (N^+_{P_m}(H) \cup \{u\})$, $vx_{i+1} \in E(G)$ or $vx_{j+1} \in E(G)$; if $x_i, x_j \in N_{P_m}(H)$ with $1 < i < j \leq m$, then for any $v \in V(G) \setminus (N^-_{P_m}(H) \cup \{u\})$, $vx_{i-1} \in E(G)$ or $vx_{j-1} \in E(G)$.*

Proof. (i), (ii) and (iv) follow immediately from the assumption that P_m is a longest (x_1, x_m)-path in G. It remains to show that (iii) and (v) must hold. Since $x_i, x_j \in N_{P_m}(H)$, $\exists x'_i, x'_j \in V(H)$ such that $x_ix'_i, x_jx'_j \in E(G)$. Let P' denote an (x'_i, x'_j)-path in H.

(iii) Suppose that the first part of (iii) fails. Then there exists a vertex $x_t \in \{x_{j+2}, x_{j+3}, \ldots, x_m\}$ such that $x_tx_{j+1} \in E(G)$ and $x_{t-1}x_{i+1} \in E(G)$. Then $[x_1, x_i]P'\,[x_j, x_{i+1}]\,[x_{t-1}, x_{j+1}][x_t, x_m]$ is a longer (x_1, x_m)-path, contrary to (2). Hence $x_tx_{j+1} \notin E(G)$. Next we assume that x_{t-1} is adjacent to some vertex $x'_{t-1} \in V(H)$. Let P'' denote an (x'_{t-1}, x'_j)-path in H. Then $[x_1, x_j]P''[x_{t-1}, x_{j+1}][x_t, x_m]$ is a longer (x_1, x_m)-path, contrary to (2). The proof for (iii)′ is similar, and so it is omitted.

(v) For vertices $x_i, x_j \in N_{P_m}(H)$ with $1 \leq i < j < m$, by Lemma 2.1(i), we have $x_{i+1} \notin N(u)$, $x_{j+1} \notin N(u)$ and by Lemma 2.1(ii), we have $x_{i+1}x_{j+1} \notin E(G)$. By (2), $N(v_{i+1}) \cap (N^+_{P_m}(H) \cup \{u\}) = \emptyset$ and $N(v_{j+1}) \cap (N^+_{P_m}(H) \cup \{u\}) = \emptyset$, and so $N(v_{i+1}) \cup N(v_{j+1}) \subseteq V(G) - (N^+_{P_m}(H) \cup \{u\})$. Furthermore, $d(u) \leq |N_{P_m}(H)| = |N^+_{P_m}(H) \cup \{u\}|$. It follows that $|N(v_{i+1}) \cup N(v_{j+1})| + d(u) \leq |V(G)| - |N^+_{P_m}(H) \cup \{u\}| + d(u) \leq n$. Since $x_{i+1}x_{j+1} \notin E(G), ux_{i+1} \notin E(G), ux_{j+1} \notin E(G)$, by (1), $|N(v_{i+1}) \cup N(v_{j+1})| + d(u) \geq n$ and so we have $N(v_{i+1}) \cup N(v_{j+1}) = V(G) - (N^+_{P_m}(H) \cup \{u\})$, which implies $\forall v \in V(G) \setminus (N^+_{P_m}(H) \cup \{u\})$, $vx_{i+1} \in E(G)$ or $vx_{j+1} \in E(G)$. Similarly, if $x_i, x_j \in N_{P_m}(H)$ with $1 < i < j \leq m$, then for any $v \in V(G) \setminus (N^-_{P_m}(H) \cup \{u\})$, $vx_{i-1} \in E(G)$ or $vx_{j-1} \in E(G)$. This proves (v). □

Lemma 2.2. *Each of the following holds.*

(i) *If there is a component H of $G - P_m$ such that $N_{P_m}(H) = \{x_1, x_m\}$, then $G[\{x_2, x_3, \ldots, x_{m-1}\}]$ is a complete subgraph.*
(ii) *If $N_{P_m}(G - P_m) = \{x_1, x_m\}$, then $G - P_m$ has at most 2 components.*
(iii) *If $N_{P_m}(G - P_m) = \{x_1, x_m\}$, then every component of $G - P_m$ is a complete subgraph.*
(iv) *If $N_{P_m}(G - P_m) = \{x_1, x_m\}$, then $G \in \{G_2 : (K_s \cup K_h), G_2 : (K_s \cup K_h \cup K_t)\}$.*

Proof. (i) Suppose, to the contrary, that $G[\{x_2, x_3, \ldots, x_{m-1}\}]$ is not a complete subgraph. Then there exist $x_i, x_j \in \{x_2, x_3, \ldots, x_{m-1}\}$ such that $x_i x_j \notin E(G)$. Since $N_{P_m}(G - P_m) = \{x_1, x_m\}$, then $(N(x_i) \cup N(x_j)) \cap (V(H) \cup \{x_i, x_j\}) = \emptyset$ and so $|N(x_i) \cup N(x_j)| \leq |V(G) \setminus V(H)| - |\{x_i, x_j\}|$. Let $u \in V(H)$. Then $ux_i \notin E(G)$ and $ux_j \notin E(G)$. Furthermore, we have $d(u) \leq |V(H) \setminus \{u\}| + |\{x_1, x_m\}|$, and so $|N(x_i) \cup N(x_j)| + d(u) \leq |V(G) \setminus V(H)| - |\{x_i, x_j\}| + |V(H) \setminus \{u\}| + |\{x_1, x_m\}| \leq n - 1$, contrary to (1).

(ii) Suppose that $G - P_m$ has at least three components H_1, H_2 and H_3. Let $u \in V(H_1)$ and $v \in V(H_2)$. Then $uv \notin E(G)$. Since $N_{P_m}(G - P_m) = \{x_1, x_m\}$, then we have $ux_2 \notin E(G), vx_2 \notin E(G)$. Again by $N_{P_m}(G - P_m) = \{x_1, x_m\}$, we have $N(u) \cup N(v) \subseteq (V(H_1) - \{u\}) \cup (V(H_2) - \{v\}) \cup \{x_1, x_m\}$ and $N(x_2) \subseteq V(P_m) - \{x_2\}$ and so $|N(u) \cup N(v)| + d(x_2) \leq |V(H_1) \setminus \{u\}| + |V(H_2) \setminus \{v\}| + |\{x_1, x_m\}| + |V(P_m) \setminus \{x_2\}| = |V(H_1)| + |V(H_2)| + |V(P_m)| - 1 \leq n - 1$, contrary to (1).

(iii) Let H be a component of $G - P_m$ such that $u, v \in V(H)$ but $uv \notin E(H)$. Since $N_{P_m}(G - P_m) = \{x_1, x_m\}$, then $ux_2 \notin E(G)$ and $vx_2 \notin E(G)$ and $N(u) \cup N(v) \subseteq (V(H) - \{u, v\}) \cup \{x_1, x_m\}$. Thus $|N(u) \cup N(v)| + d(x_2) \leq |V(H) \setminus \{u, v\}| + |\{x_1, x_m\}| + |V(P_m) \setminus \{x_2\}| \leq n - 1$, contrary to (1).

(iv) The statement follows from (ii) and (iii). □

Lemma 2.3. *Let H be a component of $G - P_m$ such that $N_{P_m}(H) = \{x_1, x_i, x_m\}$ and $u \in V(H)$. Then each of the following holds:*

(i) *If there are $x_p, x_q \in V(P_m) \setminus N_{P_m}(H)$ such that $x_p x_q \notin E(G)$, then for any vertex $v \in V(G - H) \setminus \{x_p, x_q\}$, either $x_p v \in E(G)$ or $x_q v \in E(G)$.*
(ii) *$G[\{x_2, x_3, \ldots, x_{i-1}\}]$ and $G[\{x_{i+1}, x_{i+2}, \ldots, x_{m-1}\}]$ are complete subgraphs.*
(iii) *If $G - P_m = H = \{u\}$, then $G \in \{G_3 \bigvee (K_1 \cup K_h \cup K_t)\}$.*

Proof. (i) Let $x_p, x_q \in V(P_m) \setminus N_{P_m}(H)$ such that $x_p x_q \notin E(G)$. Then $ux_p \notin E(G)$ and $ux_q \notin E(G)$. Suppose, to the contrary, that there is $v_k \in V(G - H) \setminus \{x_p, x_q\}$ such that $x_p x_k \notin E(G)$ and $x_q x_k \notin E(G)$. Then we have $|N(x_p) \cup N(x_q)| + d(u) \leq |V(G)| - |V(H)| - |\{x_p, x_q, x_k\}| + d(u) = |V(G)| - |V(H)| \leq n - 1$, contrary to (1).

(ii) To prove that $G[\{x_2, x_3, \ldots, x_{i-1}\}]$ is a complete subgraph, we need to prove the following claims.

Claim 1: $v_2 v_k \in E(G)$ for any $i - 1 \geq k \geq 4$; $v_{i-1} v_l \in E(G)$ for any $3 \geq l \geq i - 3$.

We prove that $v_2 v_k \in E(G)$ for any $i - 1 \geq k \geq 4$ by induction on $(i - 1) - k$. First, we prove $x_2 x_{i-1} \in E(G)$, that is, the case when $(i - 1) - k = 0$. Suppose, to the contrary, that $x_2 x_{i-1} \notin E(G)$. Since $x_{i+1} \in V(P_m) \setminus \{x_2, x_{i-1}\}$, then by (i), either $x_{i+1} x_2 \in E(G)$ or $x_{i+1} x_{i-1} \in E(G)$. By Lemma 2.1(ii), $x_{i+1} x_2 \notin E(G)$ and so $x_{i+1} x_{i-1} \in E(G)$. Similarly, we must have $x_{m-1} x_2 \in E(G)$. Since every vertex in $\{x_{i+2}, x_{i+3}, \ldots, x_{m-1}\}$ must be adjacent to either x_2 or x_{i-1}, then there exist two vertices $x_h, x_{h+1} \in \{x_{i+1}, x_{i+2}, \ldots, x_{m-1}\}$ such that x_h, x_{h+1} are adjacent to x_2, x_{i-1} (or x_{i-1}, x_2), respectively. It follows that G has a longer (x_1, x_m)-path $x_1 u[x_i, x_{t-1}][x_2, x_{i-1}][x_t, x_m]$ (or $x_1 u[x_i, x_{t-1}][x_{i-1}, x_2][x_t, x_m]$), contrary to (2). This shows that $x_2 x_{i-1} \in E(G)$. Now suppose that $x_2 x_k \in E(G)$ for any $k \geq s > 4$. We need to prove that $x_2 x_{s-1} \in E(G)$. Suppose, to the contrary, that $x_2 x_{s-1} \notin E(G)$. Since $x_{i+1} \in V(P_m) \setminus \{x_2, x_{s-1}\}$, by (i), either $x_{i+1} x_2 \in E(G)$ or $x_{i+1} x_{s-1} \in E(G)$. By Lemma 2.1(ii), $x_2 x_{i+1} \notin E(G)$ and so $x_{i+1} x_{s-1} \in E(G)$. Thus G has a longer (x_1, x_m)-path $x_1 u[x_i, x_s][x_2, x_{s-1}][x_{i+1}, x_m]$, contrary to (2). Hence $x_2 x_{s-1} \in E(G)$ and so $v_2 v_k \in E(G)$ for any $i - 1 \geq k \geq 4$ by induction. Similarly, we can inductively prove that $v_{i-1} v_l \in E(G)$ for any $3 \leq l \leq i - 3$.

Claim 2: $x_p x_q \in E(G)$ for any $2 \leq p < q \leq i - 1$.

By Claim 1, $v_2 v_k \in E(G)$ for any $i - 1 \geq k \geq 4$ and $v_{i-1} v_l \in E(G)$ for any $3 \geq l \geq i - 3$.

Now suppose that for any $2 \leq p < p'$ and $i - 1 \geq q > q'$, where $p < p' < q' < q$, we have $x_p x_k \in E(G)$ for any $2 \leq k \leq i - 1$ and $x_q x_l \in E(G)$ for any $2 \leq l \leq i - 1$. We want to prove that $x_{p'} x_{q'} \in E(G)$. Suppose, to the contrary, that $x_{p'} x_{q'} \notin E(G)$. Since $x_{i+1} \in V(P_m) \setminus \{x_{p'}, x_{q'}\}$, by (i), either $x_{i+1} x_{p'} \in E(G)$ or $x_{i+1} x_{q'} \in E(G)$. If $x_{i+1} x_{p'} \in E(G)$, then G has a longer (x_1, x_m)-path $x_1 u[x_i, x_{p'+1}][x_2, x_{p'}][x_{i+1}, x_m]$ and if $x_{i+1} x_{q'} \in E(G)$, then G

has a longer (x_1, x_m)-path $x_1u[x_i, x_{q'+1}][x_2, x_{q'}][x_{i+1}, x_m]$, contrary to (2) in either case. Hence $x_{p'}x_{q'} \in E(G)$ and so $x_px_q \in E(G)$ for any $2 \le p < q \le i-1$ by induction.

By Claim 2, $G[\{x_2, x_3, \ldots, x_{i-1}\}]$ is a complete subgraph.

Similarly, $G[\{x_{i+1}, x_{i+2}, \ldots, x_{m-1}\}]$ is also a complete subgraph.

(iii) To prove (iii), we consider the following cases.

Case 1. There exists a vertex $x_t \in \{x_2, x_3, \ldots, x_{i-1}\}$ adjacent to some vertex $x_h \in \{x_{i+1}, x_{i+2}, \ldots, x_{m-1}\}$.

Let $L = \min\{|\{x_2, x_3, \ldots, x_{i-1}\}|, |\{x_{i+1}, x_{i+2}, \ldots, x_{m-1}\}|\}$. First suppose that $L = 1$. Without loss of generality, let $|\{x_2, x_3, \ldots, x_{i-1}\}| = 1$, that is $i = 3$. If $x_h \ne x_{m-1}$, then G has a Hamiltonian (x_1, x_m) path $x_1ux_3x_2[x_h, x_4][x_{h+1}, x_m]$, contrary to (2). Thus $x_h = x_{m-1}$. Since $x_1, x_3 \in N_{P_m}(u)$, then by Lemma 2.1(ii), we have $x_2x_4 \notin E(G)$ and so $x_{m-1} \ne x_4$. Since $x_2x_4 \notin E(G)$, then by (i), either $x_2x_m \in E(G)$ or $x_4x_m \in E(G)$. If $x_2x_m \in E(G)$, then G has a Hamiltonian (x_1, x_m) path $x_1u[x_3, x_{m-1}]x_2x_m$ and if $x_4x_m \in E(G)$, then G has a Hamiltonian (x_1, x_m) path $x_1ux_3x_2[x_{m-1}, x_4]x_m$, contrary to (2) in either case.

Hence we must have $L \ge 2$. If $x_t \notin \{x_2, x_{i-1}\}$ or $x_h \notin \{x_{i+1}, x_{m-1}\}$, then by the facts that $G[\{x_2, x_3, \ldots, x_{i-1}\}]$ and $G[\{x_{i+1}, x_{i+2}, \ldots, x_{m-1}\}]$ are complete subgraphs, G has a Hamiltonian (x_1, x_m) path $x_1u[x_i, x_{t+1}][x_{t-1}, x_2]x_t[x_h, x_{i+1}][x_{h+1}, x_m]$, contrary to (2). Now let $x_t \in \{x_2, x_{i-1}\}$ and $x_h \in \{x_{i+1}, x_{m-1}\}$. Since $x_2, x_{i+1} \in N^+_{P_m}(u)$ and $x_{i-1}, x_{m-1} \in N^-_{P_m}(u)$, then by Lemma 2.1(ii), $x_2x_{i+1} \notin E(G)$ and $x_{i-1}x_{m-1} \notin E(G)$. Then either $x_{i-1}x_{i+1} \in E(G)$ or $x_2x_{m-1} \in E(G)$. First assume that $x_{i-1}x_{i+1} \in E(G)$. If $x_{i-2}x_{i+2} \notin E(G)$, then by (i), either $x_ix_{i-2} \in E(G)$, whence $x_1ux_ix_{i-2}[x_{i-3}, x_2]x_{i-1}x_{i+1}[x_{i+2}, x_m]$ is a Hamiltonian (x_1, x_m)-path or $x_ix_{i+2} \in E(G)$, whence $[x_1, x_{i-1}]x_{i+1}[x_{i+3}, x_{m-1}]x_{i+2}x_iux_m$ is a Hamiltonian (x_1, x_m) path, contrary to (2) in either case. If $x_{i-2}x_{i+2} \in E(G)$, then $x_2 = x_{i-2}$ and $x_{i+2} = x_{m-1}$ and so $i = 4$, $m = 7$. Then G has a Hamiltonian (x_1, x_m) path $x_1x_2x_6x_5x_3x_4ux_7$, contrary to (2).

Now assume that $x_2x_{m-1} \in E(G)$. If $x_3x_{m-2} \in E(G)$, then $3 = i-1$ and $m-2 = i+1$, that is $i = 4$, $m = 7$. Then G has a Hamiltonian (x_1, x_m) path $x_1ux_4x_5x_3x_2x_6x_7$, contrary to (2). If $x_3x_{m-2} \notin E(G)$, by (i), either $x_3x_m \in E(G)$, whence G has a Hamiltonian (x_1, x_m)-path $x_1u[x_i, x_{m-1}]x_2[x_4, x_{i-1}]x_3x_m$ or $x_{m-2}x_m \in E(G)$, whence G has a Hamiltonian (x_1, x_m)-path $x_1u[x_i, x_2]x_{m-1}[x_{m-3}, x_{i+1}]x_{m-2}x_m$, contrary to (2) in either case.

Case 2. There is no vertex in $\{x_2, x_3, \ldots, x_{i-1}\}$ adjacent to a vertex in $\{x_{i+1}, x_{i+2}, \ldots, x_{m-1}\}$.

Since $N_{P_m}(u) = \{x_1, x_i, x_m\}$, then $ux_h \notin E(G)$ and by Lemma 2.1(i), $x_2u \notin E(G)$. By the assumption of Case 2, $x_2x_h \notin E(G)$ and $N(x_2) \cup N(u) \subseteq \{x_1, x_3, x_4, \ldots, x_i, x_m\}$ and for any $x_h \in \{x_{i+1}, x_{i+2}, \ldots, x_{m-1}\}$, $N(x_h)\{x_1, x_i, x_{i+1}, \ldots, x_{h-1}, x_{h+1}, x_{m-1}, x_m\}$. Then by (1), we have $n \le |N(x_2) \cup N(u)| + d(x_h) \le |\{x_1, x_3, \ldots, x_i, x_m\}| + |\{x_1, x_i, x_{i+1}, \ldots, x_{h-1}, x_{h+1}, x_{m-1}x_m\}| \le n$. Thus x_h must be adjacent to every vertex in $N_{P_m}(u)$. Since x_h is arbitrary, every vertex in $\{x_{i+1}, x_{i+2}, \ldots, x_m\}$ must be adjacent to every vertex in $N_{P_m}(u) = \{x_1, x_i, x_m\}$. Similarly, every vertex in $\{x_2, x_3, \ldots, x_{i-1}\}$ must be adjacent to every vertex in $N_{P_m}(u) = \{x_1, x_i, x_m\}$. This implies $G \in \{G_3 \bigvee (K_1 \cup K_h \cup K_t)\}$. □

Lemma 2.4. *Suppose that $V(G - P_m) = \{u\}$, $d(u) \ge 4$ and $\{x_1, x_m\} \subseteq N_G(u)$. Then $G \in \{G_{n/2} \bigvee K^c_{n/2}\}$.*

Proof. Without loss of generality, let $N_G(u) = \{x_1, x_i, x_j, \ldots, x_r, x_m\}$, where $1 < i < j \le r < m$. Then $j = r$ if $d(u) = 4$.

Case 1. $x_2x_{m-1} \in E(G)$.

Since $x_{m-2} \in V(P_m) \setminus N^-_{P_m}(u)$ and $1 < i < j < m$, then by Lemma 2.1(v), either $x_{i-1}x_{m-2} \in E(G)$ or $x_{j-1}x_{m-2} \in E(G)$. Without loss of generality, suppose $x_{i-1}x_{m-2} \in E(G)$. Then $x_1u[x_i, x_{m-2}][x_{i-1}, x_2]x_{m-1}x_m$ is a Hamiltonian (x_1, x_m)-path, a contradiction.

Case 2. $x_2x_{m-1} \notin E(G)$.

Then we consider two subcases $x_{r+1} \ne x_{m-1}$ and $x_{r+1} = x_{m-1}$.

Subcase 2.1. $x_{r+1} \ne x_{m-1}$.

Since $x_{m-1} \in V(P_m) \setminus N^+_{P_m}(u)$ and $1 < i < m$, then by Lemma 2.1(v), either $x_2x_{m-1} \in E(G)$ or $x_{i+1}x_{m-1} \in E(G)$. By the assumption of case 2, $x_2x_{m-1} \notin E(G)$ and so we must have $x_{i+1}x_{m-1} \in E(G)$. Since $x_{r+1} \in V(P_m) \setminus N^-_{P_m}(u)$ and $1 < i < j < m$, by Lemma 2.1(v), $x_{r+1}x_{i-1} \in E(G)$ or $x_{r+1}x_{j-1} \in E(G)$ (if $d(u) = 4$, then $j = r$). Then we consider the following two subcases.

Subcase 2.1.1 $x_{r+1}x_{i-1} \in E(G)$.

Since $x_i \in V(P_m) \setminus N^-_{P_m}(u)$ and $1 < j < m$, then by Lemma 2.1(v), either $x_ix_{j-1} \in E(G)$, whence G has a Hamiltonian (x_1, x_m)-path $[x_1, x_i][x_{j-1}, x_{i+1}]x_{m-1}[x_{i-2}, x_j]ux_m$ or $x_ix_{m-1} \in E(G)$, whence G has a Hamiltonian (x_1, x_m)-path $[x_1, x_{i-1}][x_{r+1}, x_{m-1}]$ $[x_i, x_r]ux_m$, contrary to (2) in either case.

Subcase 2.1.2. $x_{r+1}x_{j-1} \in E(G)$.

Since $x_{r+2} \in V(P_m) \setminus N^+_{P_m}(u)$ and $1 < i < m$, by Lemma 2.1(v), either $x_{r+2}x_2 \in E(G)$, whence by the fact that $x_{r+1}x_{j-1} \in E(G)$, G has a Hamiltonian (x_1, x_m)-path $x_1u[x_j, x_{r+1}][x_{j-1}, x_2][x_{r+2}, x_m]$, or $x_{r+2}x_{i+1} \in E(G)$, whence G has a Hamiltonian (x_1, x_m)-path $[x_1, x_i]u[x_j, x_{r+1}][x_{j-1}, x_{i+1}][x_{r+2}, x_m]$, contrary to (2) in either case.

Subcase 2.2 $x_{r+1} = x_{m-1}$.

Note that both $x_{r+1} = x_{m-1} \in N^+_{P_m}(u)$ and $x_{r+1} = x_{m-1} \in N^-_{P_m}(u)$. Let $x_i, x_j \in N_{P_m}(u)$ be such that $N_{P_m}(u) \cap \{x_{i+1}, x_{i+2}, \ldots, x_{j-1}\} = \emptyset$, then we claim that $x_{i+1} = x_{j-1}$.

Otherwise, since $x_{i+1} \in V(P_m) \setminus N^-_{P_m}(u)$ and $1 < i < m$, then by Lemma 2.1(v), $x_{i-1}x_{i+1} \in E(G)$ or $x_{m-1}x_{i+1} \in E(G)$. Since $x_{r+1} = x_{m-1}$, then $x_{i+1}x_{m-1} \notin E(G)$ and so $x_{i+1}x_{i-1} \in E(G)$. Since $x_{i+2} \in V(P_m) \setminus N^+_{P_m}(u)$ and $1 < i < r < m$, then by Lemma 2.1(v), $x_{i+2}x_2 \in E(G)$, whence G has a Hamiltonian (x_1, x_m)-path $x_1ux_ix_{i+1}[x_{i-1}, x_2][x_{i+2}, x_m]$, or $x_{i+2}x_{m-1} \in E(G)(x_{i+2}x_{r+1} \in E(G))$, whence G has a Hamiltonian (x_1, x_m)-path $[x_1, x_{i-1}]x_{i+1}x_iu[x_r, x_{i+2}]x_{r+1}x_m$, contrary to (2) in either case. Therefore, $N_{P_m}(u) = \{x_1, x_3, x_5, x_7, \ldots, x_{n-1}\}$. Since P_m is a longest (x_1, x_m)-path, then $\{u, x_2, x_4, x_6 \ldots, \ldots, x_{n-2}\}$ is an independent set. Since for any $x_p, x_q \in \{x_2, x_4, x_6 \ldots, \ldots, x_{n-2}\}$, we have $n \le |N(x_p) \cup N(x_q)| + d(u) \le |\{x_1, x_3, x_5, x_7, \ldots, x_{n-1}\}| + d(u) = n$, then every vertex in $\{x_2, x_4, x_6 \ldots, \ldots, x_{n-2}\}$ must be adjacent to every vertex in $\{x_1, x_3, x_5, x_7, \ldots, x_{n-1}\}$. Thus we can get $G \in \{G_{n/2} \bigvee K^c_{n/2}\}$. □

Lemma 2.5. *Suppose that for any $u \in V(G - P_m)$, both $\{x_1, x_m\} \subseteq N_{P_m}(u)$ and $N_{P_m}(G - P_m) \ne \{x_1, x_m\}$. If there exists a component H of $G - P_m$ such that $|V(H)| \ge 2$, then $G \in \{G_3 \bigvee (K_s \cup K_h \cup K_t)\}$.*

Proof. Without loss of generality, let $N_{P_m}(H) = \{x_1, x_i, x_j, \ldots, x_r, x_m\}$.

Claim 1: $|N_{P_m}(H)| = 3$.

Otherwise, since G is a 2-connected graph, then $|N_{P_m}(H)| = 2$ or $|N_{P_m}(H)| \ge 4$. If $|N_{P_m}(H)| = 2$, then $N_{P_m}(H) = \{x_1, x_m\}$. By Lemma 2.2(i), $G[\{x_2, x_3, \ldots, x_{m-1}\}]$ is a complete subgraph. Since $N_{P_m}(G - P_m) \ne \{x_1, x_m\}$ and G is 2-connected, then $G - P_m$ has a component S such that $x_i \in N_{P_m}(S) \setminus \{x_1, x_m\}$ and $x_j \in N_{P_m}(S)$. Without loss of generality, suppose that $1 < i < j \le m$. Since $x_i, x_j \in N_{P_m}(H)$, $\exists x'_i, x'_j \in V(H)$ such that $x_ix'_i, x_jx'_j \in E(G)$. Let P' denote an (x'_i, x'_j)-path in H. Hence G has a longer (x_1, x_m)-path $[x_1, x_{i-1}][x_{i+1}, x_{j-1}]x_iP'[x_j, x_m]$, contrary to (2). Now suppose $|N_{P_m}(H)| \ge 4$ and $u \in V(H)$. Let $v \in V(H) \setminus \{u\}$. By Lemma 2.1(v), $vx_2 \in E(G)$ or $vx_{i+1} \in E(G)$. Since $x_1 \in N_{P_m}(v)$, then by Lemma 2.1(i), $x_2 \notin N_{P_m}(v)$ and so $x_{i+1}v \in E(G)$. Since $|N_{P_m}(H)| \ge 4$, then there is $x_j \in N_{P_m}(H) \setminus \{x_1, x_i, x_m\}$. By the same argument, we have $x_{j+1}v \in E(G)$ and so $[x_1, x_i]u[x_j, x_{i+1}]v[x_{j+1}, x_m]$ is a longer (x_1, x_m)-path, contrary to (2).

Let $N_{P_m}(H) = \{x_1, x_i, x_m\}$. By Lemma 2.3(ii), we have the following Claim 2.

Claim 2: $G[\{x_2, x_3, \ldots, x_{m-1}\}]$ and $G[\{x_{i+1}, x_{i+2}, \ldots, x_{m-1}\}]$ are all complete subgraphs.

Since G is 2-connected and $|V(H)| \ge 2$, then there are $x'_1, x'_i \in V(H)$ such that $x'_1 \ne x'_i$ and $x_1x'_1, x_ix'_i \in E(G)$ or there are $x''_i, x''_m \in V(H)$ such that $x''_i \ne x''_m$ and $x_ix''_i, x_mx''_m \in E(G)$. Without loss of generality, suppose there are $x'_1, x'_i \in V(H)$ such that $x'_1 \ne x'_i$ and $x_1x'_1, x_ix'_i \in E(G)$. Let P' denote an (x'_1, x'_i)-path in H.

Claim 3: $G - P_m$ is a connected subgraph.

Otherwise, let S be another component of $G - P_m$. By Lemma 2.3(i), every vertex in S must be adjacent to one of x_2 and x_{i+1}. Since every vertex in S is adjacent to x_1, by Lemma 2.1(i), no vertex in S can be adjacent to x_2 and so every vertex in S must be adjacent to x_{i+1}. If $x_2x_{i+2} \in E(G)$, then we can get a longer (x_1, x_m)-path $x_1P'[x_i, x_2][x_{i+2}, x_m]$, contrary to (2). Then we have $x_2x_{i+2} \notin E(G)$. By Lemma 2.3(i) and Lemma 2.1(i) again, every vertex in S must be adjacent to x_{i+2}, contradicting Lemma 2.1(i).

Claim 4: H is a complete subgraph.

Otherwise, let $u, v \in V(H)$ such that $uv \notin E(G)$. Then we have $|N(x_2) \cup N(x_{i+1})| + d(u) \le |V(P_m)| + |V(H)| - |\{x_2, x_{i+1}, u, v\}| + |N_{P_m}(H)| \le n - 1$, contrary to (1).

Claim 5: For any $u \in V(H)$, u must be adjacent to every vertex of $N_{P_m}(H)$.

Otherwise, there exists $u \in V(H)$ such that $ux_i \notin E(G)$. Then $|N(x_2) \cup N(x_{i+1})| + d(u) \le |V(P_m) \setminus \{x_2, x_{i+1}\}| + |V(H) \setminus \{u\}| + |N_{P_m}(H) \setminus \{x_i\}| \le n - 1$, contrary to (1). Similarly, for every vertex u in $\{x_2, x_3, \ldots, x_{i-1}\}$ or $\{x_{i+1}, x_{i+2}, \ldots, x_{m-1}\}$, u must be adjacent to every vertex in $N_{P_m}(H) = \{x_1, x_i, x_m\}$. Then by Claims 1–5, we have $G \in \{G_3 \bigvee (K_s \cup K_h \cup K_t)\}$. □

Proof of Theorem 1.2. Let G be a 2-connected graph such that (1) holds. Suppose that G is not Hamiltonian-connected and so we may assume that there exist $x, y \in V(G)$ such that G has no Hamiltonian (x, y)-path and such that (2) holds. We want to show that $G \in \{G_2 : (K_s \cup K_h), G_{n/2} \bigvee K^c_{n/2}, G_2 : (K_s \cup K_h \cup K_t), G_3 \bigvee (K_s \cup K_h \cup K_t)\}$. We consider the following cases.

Case 1. There exists a vertex u in $G - P_m$ such that ux_1 or $ux_m \notin E(G)$.

Without loss of generality, suppose $ux_m \notin E(G)$. let G^* be the component of $G - P_m$ containing u. Since G is 2-connected, then $|N_{P_m}(G^*)| \geq 2$.

Subcase 1.1. $|N_{P_m}(G^*)| \geq 3$.

In this case, there exist two distinct vertices $x_{i+1}, x_{j+1} \in N^+P_m(G^*)$ such that $x_{i+1}x_{j+1} \notin E(G)$. Then we have the following claim.

Claim: For any vertex $v \in N_{G-P_m}(u) \cup N^+_{P_m}(u)$, $vx_{i+1} \notin E(G)$ and $vx_{j+1} \notin E(G)$.

By Lemma 2.1(ii), for any vertex $v \in N^+P_m(u)$, $vx_{i+1} \notin E(G)$ and $vx_{j+1} \notin E(G)$. Now suppose there is $v \in N_{G-P_m}(u)$ such that $vx_{i+1} \in E(G)$ or $vx_{j+1} \in E(G)$. Without loss of generality, suppose that $vx_{i+1} \in E(G)$. Since $x_i \in N_{P_m}(G^*)$, $\exists x'_i \in V(G^*)$ such that $x_i x'_i \in E(G)$. Let P' denote an (x'_i, v)-path in G^*. Then we get a longer (x_1, x_m)-path $[x_1, x_i]P_1[x_{i+1}, x_m]$, contrary to (2).

Since $x_{i+1}, x_{j+1} \in N^+P_m(G^*)$, by Lemma 2.1(i), $ux_{i+1} \notin E(G)$and $ux_{j+1} \notin E(G)$. By the above Claim, we have $|N(x_{i+1}) \cup N(x_{j+1})| \leq |V(G)| - |N_{G-P_m}(u) \cup N^+_{P_m}(u)| - |\{u\}|$. Since $|N^+_{P_m}(u)| = |N_{P_m}(u)|$, then $|N_{G-P_m}(u) \cup N^+_{P_m}(u)| = |N_{G-P_m}(u) \cup N_{P_m}(u)| = |N(u)|$ and so $|N(x_{i+1}) \cup N(x_{j+1})| \leq |V(G)| - |N(u)| - |\{u\}| = n - |N(u)| - 1$, which implies $|N(x_{i+1}) \cup N(x_{j+1})| + d(u) \leq n - 1$, contrary to (1).

Subcase 1.2. $|N_{P_m}(G^*)| = 2$.

If $N_{P_m}(G^*) \neq \{x_1, x_m\}$, then by the argument similar to that in above Subcase 1.1, we can obtain a contradiction. Then we have $N_{P_m}(G^*) = \{x_1, x_m\}$. By Lemma 2.2(i), $G[\{x_2, x_3, \ldots, x_{m-1}\}]$ is complete subgraph.

If there exists a vertex $x_i \in V(P_m) \setminus \{x_1, x_m\}$ satisfying x_i is adjacent to some vertex of $G - P_m$, then there exists a component H of $G - P_m - G^*$ such that x_i is adjacent to some vertex of H. Since G is 2-connected, then there exist $x_{i+1}, x_{j+1} \in N^+_{P_m}(H)$ or $x_{i-1}, x_{j-1} \in N^-_{P_m}(H)$. Since $G[\{x_2, x_3, \ldots, x_{m-1}\}]$ is a complete subgraph, then $x_{i+1}x_{j+1}$ and $x_{i-1}x_{j-1} \in E(G)$, contrary to Lemma 2.1(ii). Then we have $N_{P_m}(G - P_m) = \{x_1, x_m\}$. By Lemma 2.2(iv), we have $G \in \{G_2 : (K_s \cup K_h), G_2 : (K_s \cup K_h \cup K_t)\}$.

Case 2. For any vertex u in $G - P_m$, u is adjacent to x_1 and x_m.

If $N_{P_m}(G - P_m) = \{x_1, x_m\}$, by Lemma 2.2(iv), we have $G \in \{G_2 : (K_s \cup K_h), G_2 : (K_s \cup K_h \cup K_t)\}$. In the following, we suppose that $N_{P_m}(G - P_m) \neq \{x_1, x_m\}$. Then there exists a component G^* of $G - P_m$ such that $N_{P_m}(G^*) \cap (V(P_m) \setminus \{x_1, x_m\}) \neq \emptyset$.

Subcase 2.1. $|V(G - P_m)| = |\{u\}| = 1$.

Since u is adjacent to x_1 and x_m and $N_{P_m}(u) \cap (V(P_m) \setminus \{x_1, x_m\}) \neq \emptyset$, then $d(u) \geq 3$. If $d(u) = 3$, then by Lemma 2.3(iii), $G \in \{G_3 \bigvee (K_1 \cup K_h \cup K_t)\}$. If $d(u) \geq 4$, then by Lemma 2.4, $G \in \{G_{n/2} \bigvee K^c_{n/2}\}$.

Subcase 2.2. $|V(G - P_m)| \geq 2$.

If there exists a component H of $G - P_m$ such that $|V(H)| \geq 2$, then by Lemma 2.5, $G \in \{G_3 \bigvee (K_s \cup K_h \cup K_t)\}$. Now we suppose that for every component H of $G - P_m$, $|V(H)| = 1$.

Claim: For any vertex $u \in V(G - P_m)$, $N_{P_m}(u) \leq 3$.

Otherwise, let $N_{P_m}(u) \geq 4$ and $N_{P_m}(u) = \{x_1, x_i, x_j, \ldots, x_m\}$ with $1 < i < j < m$. Since $|V(G - P_m)| \geq 2$, there exists a vertex $v \in V(G - P_m) \setminus \{u\}$. By Lemma 2.1(v), $vx_2 \in E(G)$ or $vx_{i+1} \in E(G)$. Since $x_1 \in N_{P_m}(v)$, then by Lemma 2.1(i), $vx_2 \notin E(G)$ and so $vx_{i+1} \in E(G)$. Similarly, $vx_{j+1} \in E(G)$, contrary to Lemma 2.1(iv).

Since $N_{P_m}(G^*) \cap (V(P_m) \setminus \{x_1, x_m\}) \neq \emptyset$, then there exists $v \in V(G - P_m)$ such that $|N_{P_m}(v)| = 3$. Without loss of generality, let $N_{P_m}(v) = \{x_1, x_i, x_m\}$. Let $w \in V(G - P_m) \setminus \{v\}$. By Lemma 2.1(v), either $wx_2 \in E(G)$ or $wx_{i+1} \in E(G)$. Since $x_1 \in N_{P_m}(w)$, then $wx_2 \notin E(G)$ and so $wx_{i+1} \in E(G)$. Similarly, $wx_{i-1} \in E(G)$. Then $x_{i-1}, x_{i+1}, x_1, x_m \in N_{P_m}(w)$, namely, $|N_{P_m}(w)| \geq 4$, contrary to the claim that for any vertex $u \in V(G - P_m)$, $N_{P_m}(u) \leq 3$. □

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