Approximate controllability of fractional order semilinear systems with bounded delay

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1. Introduction

Let $V$ and $\hat{V}$ be Hilbert spaces and $\mathcal{Z} = L_2([0, \tau]; V)$, $\mathcal{Z}_h = L_2([-h, \tau]; V)$ be the function spaces corresponding to $V$ and $Y = L_2([0, \tau]; \hat{V})$ be the function space corresponding to $\hat{V}$.

Consider the fractional order semilinear delay control system

$$
\begin{aligned}
\mathcal{C}D_{\alpha}^t x(t) &= Ax(t) + Bu(t) + f\left(t, x(t-h)\right), & t \in [0, \tau]; \\
\end{aligned}
$$

$$
x(t) = \varphi(t), & t \in [-h, 0]
$$

(1)

where $\mathcal{C}D_{\alpha}^t$ is the Caputo fractional derivative of order $\alpha$; $1/2 < \alpha < 1$. The state $x(t)$ takes its values in the space $V$; the control function $u(t)$ takes its values in the space $\hat{V}$; $A : D(A) \subseteq V \to V$ is
a closed linear operator with dense domain $D(A)$ and generates a $C_0$-semigroup $T(t)$; $B$ is a bounded linear operator from $Y$ to $Z$; the function $f : [0, \tau] \times V \rightarrow V$ is nonlinear and $\varphi \in C([-h, 0]; V)$.

Fractional order semilinear equations are abstract formulations for many problems arising in engineering and physics. The potential applications of fractional calculus are in diffusion process, electrical science, electrochemistry, viscoelasticity, control science, electro magnetic theory and several more. For more details see [1–10] and the references cited therein. In [11] Lyapunov–Krasovskii theorem for the stability of fractional order delay system has been proved.

Exact controllability for fractional order systems have been proved by many authors [12–15] and the boundary controllability is proved by Ahmed [16]. In these papers, the main tool used by the authors is to convert the controllability problem into a fixed point problem with the assumption that the controllability operator has an induced inverse on a quotient space. In [14–16], the authors made an assumption that the semigroup associated with the linear part is compact, to prove the controllability results. However, if the operator $B$ is compact or $C_0$-semigroup $T(t)$ is compact then the controllability operator is also compact and hence inverse of it does not exist if the state space $V$ is infinite dimensional [17]. Thus, the concept of exact controllability is too strong in infinite dimensional spaces and the approximate controllability is more appropriate.

The approximate controllability of the systems of integer order ($\alpha = 1, 2$) has been proved in [18–22] among others. However, there are only few papers which deal with the approximate controllability of fractional order system. In [23] Sakthivel et al. proved the approximate controllability by assuming that the $C_0$-semigroup $T(t)$ is compact and the nonlinear function is continuous and uniformly bounded. Recently, Sukavanam et al. [24] have proved some sufficient conditions for the approximate controllability of a fractional order system in which the nonlinear term depends on both state and control variables.

The main objective of this paper is to provide different sufficient conditions for the approximate controllability of fractional order semilinear systems with fixed delay. To prove the results we use the techniques similar to that of [20,25] with suitable modifications so as to be compatible with fractional order delay systems. The uniform boundedness of nonlinear function assumed by other authors is replaced by Lipschitz continuity.

The paper is organized as follows: in Section 2, we present some basic definitions and a lemma as preliminaries. In Section 3, the existence and uniqueness of the mild solution is proved. Sufficient conditions for the approximate controllability are proved in Section 4. In Section 5, two examples are given to illustrate the theory.

2. Preliminaries

**Definition 2.1.** A real function $f(t)$ is said to be in the space $C_{\alpha}$, $\alpha \in \mathbb{R}$, if there exists a real number $p > \alpha$, such that $f(t) = t^p g(t)$, where $g \in C[0, \infty[$ and it is said to be in the space $C_{\alpha}^m$ iff $f^{(m)} \in C_{\alpha}$, $m \in \mathbb{N}$.

**Definition 2.2.** If the function $f \in C_{\alpha}^{m-1}$ and $m$ is a positive integer then we can define the fractional derivative of $f(t)$ in the Caputo sense as

$$C_D_t^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-s)^{m-\alpha-1} f^{(m)}(s) \, ds, \quad m - 1 \leq \alpha < m.$$

**Definition 2.3.** (See [26,]) A function $x(\cdot) \in Z_h$ is said to be the mild solution of (1) if it satisfies

$$x(t) = \begin{cases} S_{\alpha}(t)\varphi(0) + \int_0^t (t-s)^{\alpha-1} T_{\alpha}(t-s) [Bu(s) + f(s, x(s-h))] \, ds, & t \in [0, \tau]; \\ \varphi(t), & t \in [-h, 0], \end{cases}$$

(2)
where \( S_\alpha(t)x = \int_0^\infty \phi_\alpha(\theta)T(t^\alpha \theta)x d\theta \) and \( T_\alpha(t)x = \alpha \int_0^\infty \theta \phi_\alpha(\theta)T(t^\alpha \theta)x d\theta \). Here \( \phi_\alpha(\theta) = \frac{1}{\alpha} \theta^{-1-1/\alpha} \times \psi_\alpha(\theta^{-1/\alpha}) \), Note that \( \phi_\alpha(\theta) \) satisfies the conditions of a probability density function defined on \((0, \infty)\), that is \( \phi_\alpha(\theta) \geq 0 \), and \( \int_0^\infty \phi_\alpha(\theta)d\theta = 1 \). Also the term \( \psi_\alpha(\theta) \) is defined as

\[
\psi_\alpha(\theta) = \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^{n+1} \theta^{-n+1} \frac{\Gamma(n\alpha+1)}{n!} \sin(n\pi \alpha), \quad \theta \in (0, \infty).
\]

Let \( x(t) \) be the state value of system (1) at time \( t \) corresponding to the control \( u \). The system (1) is said to be approximately controllable in time interval \([0, \tau]\), if for every desired final state \( \xi \) and \( \epsilon > 0 \) there exists a control function \( u \in Y \) such that the solution of (1) satisfies

\[
\|x(\tau) - \xi\| < \epsilon.
\]

**Definition 2.4.** The set \( K_\tau(f) = \{x(\tau) \in V : x(\cdot) \) is the mild solution of (1)\} is called the reachable set of the semilinear system (1). If \( f \equiv 0 \), then the system (1) is called the corresponding linear system and is denoted by \((1)^*\). In this case, \( K_\tau(0) \) denotes the reachable set of the linear system \((1)^*\).

**Definition 2.5.** The system (1) is said to be approximately controllable on \([0, \tau]\) if \( K_\tau(f) = V \), where \( K_\tau(f) \) denotes the closure of \( K_\tau(f) \). Clearly, the corresponding linear system \((1)^*\) is approximately controllable if \( K_\tau(0) = V \).

**Lemma 2.1.** (See [26].) For any fixed \( t \geq 0 \), \( S_\alpha(t) \) and \( T_\alpha(t) \) are bounded linear operators. Hence \( \|S_\alpha(t)x\| \leq M\|x\| \) and \( \|T_\alpha(t)x\| \leq \frac{M_\alpha}{t(t+\alpha)} \|x\| \) for all \( x \in V \), where \( M \) is a constant such that \( \|T(t)\| \leq M \) for all \( t \in [0, \tau] \).

### 3. Existence and uniqueness of mild solution

In this section we prove the existence and uniqueness of the mild solution of (1). To prove the result let us assume the following condition:

**\( \text{(H1)} \)** The nonlinear function \( f(t, x) \) satisfies the Lipschitz condition, i.e. there exists a positive constant \( l \) such that

\[
\|f(t, x) - f(t, y)\| \leq l\|x - y\|
\]

for all \( x, y \in V, 0 < t \leq \tau \).

**Theorem 3.1.** If the condition \( \text{(H1)} \) holds, the system (2) admits a unique mild solution in \( Z_h \) for each control function \( u(\cdot) \in Y \).

**Proof.** Let \( I_f = \max_{0 \leq t \leq \tau} \|f(t, 0)\| \) and \( B \leq \|S_\alpha(t)\| \leq M_B \). Define the mapping \( \Phi : L_2([-h, t_1]; V) \rightarrow L_2([-h, t_1]; V) \) as

\[
(\Phi x)(t) = \begin{cases} 
S_\alpha(t)\varphi(0) + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s)[Bu(s) + f(s, x(s-h))] ds, & \text{if } t \in [0, t_1]; \\
\varphi(t), & \text{if } t \in [-h, 0].
\end{cases}
\]

Now, if we are able to show that \( \Phi \) has a fixed point in the space \( L_2([-h, t_1]; V) \) then (2) is the mild solution on \([-h, t_1]\).

Let \( B_R = \{x(\cdot) \in L_2([-h, t_1]; V) : \|x\|_{L_2([-h, t_1]; V)} \leq R, \ x(0) = \varphi(0)\} \), which is bounded and closed subset of \( L_2([-h, t_1]; V) \). For any \( x(\cdot) \in B_R \), we have
Therefore, \( M \| \varphi(0) \| + \frac{M \alpha}{\Gamma(1 + \alpha)} \left[ M_B \int_0^t (t - s)^{\alpha - 1} \| u(s) \| \, ds \right. \]
\[ + l \int_0^t (t - s)^{\alpha - 1} \| x(s - h) \| \, ds \left. + l_f \int_0^t (t - s)^{\alpha - 1} \, ds \right] \leq M \| \varphi(0) \| + \frac{M \alpha}{\Gamma(1 + \alpha)} \left[ M_B \sqrt{\frac{t^{2\alpha - 1}}{2\alpha - 1}} \| u \|_y \right. \]
\[ + l \int_{-h}^{t-h} (t - h - \sigma)^{\alpha - 1} \| x(\sigma) \| \, d\sigma + l_f \int_0^t (t - s)^{\alpha - 1} \, ds \right] \leq M \| \varphi(0) \| + \frac{M \alpha}{\Gamma(1 + \alpha)} \left[ \sqrt{\frac{t^{2\alpha - 1}}{2\alpha - 1}} (M_B \| u \|_y + lR) + \frac{l_f t^\alpha}{\alpha} \right] \]
\[ \leq M \| \varphi(0) \| + \frac{M \alpha}{\Gamma(1 + \alpha)} \left[ \sqrt{\frac{t^{2\alpha - 1}}{2\alpha - 1}} (M_B \| u \|_y + lR) + \frac{l_f t^\alpha}{\alpha} \right] . \]

Now let \( M \| \varphi(0) \| + \frac{M \alpha}{\Gamma(1 + \alpha)} \left[ \sqrt{\frac{t^{2\alpha - 1}}{2\alpha - 1}} (M_B \| u \|_y + lR) + \frac{l_f t^\alpha}{\alpha} \right] < R. \) Then
\[ M \| \varphi(0) \| + \frac{M \alpha}{\Gamma(1 + \alpha)} \left[ \sqrt{\frac{t^{2\alpha - 1}}{2\alpha - 1}} (M_B \| u \|_y + lR) + \frac{l_f t^\alpha}{\alpha} \right] < R \left( 1 - \frac{M \alpha}{\Gamma(1 + \alpha)} \sqrt{\frac{t^{2\alpha - 1}}{2\alpha - 1}} \right). \]

The RHS will be positive, if
\[ t_1^{2\alpha - 1} < (2\alpha - 1) \left( \frac{\Gamma(\alpha)}{Ml} \right)^2 . \tag{3} \]

Therefore, \( \Phi \) maps the ball \( B_R \) of radius \( R \) into itself, when \( t_1 \) satisfies (3).

Next, we show that \( \Phi \) is a contraction on \( B_R \). For this, let us take \( x_1, x_2 \in B_R \), then we get
\[ \| (\Phi x_1)(t) - (\Phi x_2)(t) \| \leq \frac{M \alpha}{\Gamma(1 + \alpha)} \int_0^t (t - s)^{\alpha - 1} \left\| f(s, x_1(s - h)) - f(s, x_2(s - h)) \right\| \, ds \]
\[ \leq \frac{M l \alpha}{\Gamma(1 + \alpha)} \int_0^t (t - s)^{\alpha - 1} \left\| x_1(s - h) - x_2(s - h) \right\| \, ds \]
\[ \leq \frac{M l \alpha}{\Gamma(1 + \alpha)} \sqrt{\frac{t^{2\alpha - 1}}{2\alpha - 1}} \| x_1 - x_2 \|_{L_2([-h, t_1]; V)} \]
\[ \leq \frac{M l}{\Gamma(\alpha)} \sqrt{\frac{t^{2\alpha - 1}}{2\alpha - 1}} \| x_1 - x_2 \|_{L_2([-h, t_1]; V)}. \]
From condition (3), we conclude that $Φ$ is a contraction. Therefore, $Φ$ has a unique fixed point in $B_R$, so (2) is the mild solution on $[-h, t_1]$. Similarly, we can prove that (2) is the mild solution on an interval $[t_1, t_2]$, $t_1 < t_2$. Repeating the above process, we can show that (2) is the mild solution with the maximal existence interval $[-h, t^*]$, $t^* \leq \tau$. Next, we show that the mild solution is bounded by showing its boundedness in each subinterval $[(k-1)h, kh], k = 1, 2, \ldots, n$. If $t \in [-h, 0)$, then $x(t) = ψ(t)$. Hence it is bounded. If $t \in [0, h]$, then

\[
\|x(t)\| \leq M\|ϕ\| + \frac{Mα}{Γ(1+α)} \left[ M_B \int_0^t (t-s)^{α-1} \|u(s)\| ds + l_f \int_0^t (t-s)^{α-1} ds \right] + \int_0^t (t-s)^{α-1} \|x(s-h)\| ds.
\]

But $s \in [0, h]$ implies $(s-h) \in [-h, 0]$. Hence $x(s-h) = ψ(s)$ is bounded by the previous step. This implies that $x(t)$ is bounded in $[0, h]$. By repeating the same argument, we can show that the mild solution $x(t)$ is bounded in the intervals $[(k-1)h, kh], k = 1, 2, \ldots, n$. Thus we conclude that $x(\cdot)$ is well defined on $[-h, \tau]$, where $τ \in [(n-1)h, nh]$.

Finally, we prove the uniqueness of the mild solution. For this, let $x_1$ and $x_2$ be any two solutions of (2), if $t \in [-h, 0]$ then $x_1(t) = x_2(t) = ψ(t)$, implies the uniqueness of the mild solution in $[-h, 0]$. Next, if $t \in [0, h]$ then

\[
\|x_1(t) - x_2(t)\| \leq \frac{Mα}{Γ(1+α)} \int_0^t (t-s)^{α-1} \|f(s, x_1(s-h)) - f(s, x_2(s-h))\| ds \leq \frac{Mlα}{Γ(1+α)} \int_0^t (t-s)^{α-1} \|x_1(s-h) - x_2(s-h)\| ds.
\]

Since in this case $(s-h) \in [-h, 0]$ and the uniqueness of the mild solution is already proved in the previous interval $[-h, 0]$, $x_1(t) = x_2(t)$ for all $t \in [0, h]$. Similarly, we can prove the uniqueness of the mild solution in the successive intervals $[h, 2h], [2h, 3h]$ up to the interval $[(n-1)h, nh]$. Hence, $x_1(t) = x_2(t)$ for all $t \in [-h, τ]$. This completes the proof. □

4. Controllability of system (1)

Define the operator $F_h : Z_h \to Z$ as

\[ [F_hx](t) = f(t, x(t-h)), \quad 0 < t \leq \tau. \]

If $h = 0$, the operator $F_h$ is known as the Nemytskii operator of nonlinear function.
Also, we define the linear operator $L$ from $Z$ to $V$ by

$$Lp = \int_0^\tau (\tau - s)^{\alpha - 1} T_\alpha (\tau - s) p(s) \, ds.$$  

Let $N_0(L)$ be the null space of the operator $L$, which is a closed subspace in $Z$ and its orthogonal space is $N_0^\perp(L)$. Then $Z$ can be decomposed uniquely as $Z = N_0(L) \oplus N_0^\perp(L)$. Denote the range of operator $B$ by $R(B)$ and its closure by $\overline{R(B)}$.

**Assumptions.** We impose the following conditions to prove the results:

(H2) The $C_0$-semigroup is compact.
(H3) For each $p \in Z$ there exists a function $q \in \overline{R(B)}$ such that $Lp = Lq$.

Clearly, assumption (H3) implies that for any $p \in Z$ there exists a function $q \in \overline{R(B)}$ such that $L(p - q) = 0$. Hence $p - q = n \in N_0(L)$ which implies that $Z = N_0(L) \oplus \overline{R(B)}$. Therefore, we can define a linear and continuous mapping $P$ from $N_0^\perp(L)$ into $\overline{R(B)}$, as

$$\|Pv\| = \|q^*\| = \min \{ \|v\| : v \in \{u^* + N_0(L)\} \cap \overline{R(B)} \}.$$  

From (H3) it follows that for each $u^* \in N_0^\perp(L)$, the set $\{u^* + N_0(L)\} \cap \overline{R(B)}$ is not empty and each $z \in Z$ has a unique decomposition $z = n + q^*$. Thus the operator $P$ is well defined. Moreover, $\|P\| \leq C$ for some constant $C$ [25].

**Lemma 4.1.** (See [27].) For each $z \in Z$ and corresponding $n \in N_0(L)$, the following inequality holds

$$\|n\|_Z \leq (1 + C) \|z\|_Z.$$  

Define the operator $K : Z \to Z$ as

$$Kz(t) = \int_0^t (t - s)^{\alpha - 1} T_\alpha (t - s) z(s) \, ds.$$  

Let $M_0$ be the subspace of $Z_h$ such that

$$M_0 = \left\{ m \in Z_h : m(t) = (Kn)(t), n \in N_0(L), \begin{array}{ll} 0 < t \leq \tau; & m(t) = 0, \quad 0 \leq t \leq 0 \end{array} \right\}.$$  

Note that $m(\tau) = 0$, for all $m \in M_0$.

For each mild solution $x(\cdot)$ of linear system (1)* with control $u$, we can define an operator $f_x : M_0 \to M_0$ as

$$f_x(m) = \begin{cases} Kn, & 0 < t \leq \tau; \\ 0, & -h < t \leq 0. \end{cases}$$
where \( n \) is given by the unique decomposition

\[
F_h(x + m) = n + q, \quad n \in N_0(L), \quad q \in \overline{R(B)}.
\]  

First we prove the approximate controllability of the corresponding linear system \((1)^*\). Then the approximate controllability of fractional order semilinear system \((1)\) is proved.

**Theorem 4.1.** Under assumption \((H3)\), the fractional order linear system \((1)^*\) is approximately controllable i.e.

\[
K_\tau (0) = V.
\]

**Proof.** Since the domain \( D(A) \) of the operator \( A \) is dense in \( V \), it is sufficient to prove that \( D(A) \subset K_\tau (0) \). To prove this, let us take \( \xi \in D(A) \), then \( \xi - S_\alpha(t)\varphi(0) \in D(A) \). It can be seen that there exists some \( p \in C^1[0, t; V] \) such that \( \eta = \int_0^t (\tau - s)^{\alpha-1}T_\alpha(\tau - s)p(s)\, ds \), where \( \eta = \xi - S_\alpha(t)\varphi(0) \).

The assumption \((H3)\) implies that there exists a function \( q \in \overline{R(B)} \) such that following equality holds

\[
\eta = \int_0^t (\tau - s)^{\alpha-1}T_\alpha(\tau - s)q(s)\, ds.
\]

Since \( q \in \overline{R(B)} \), for a given \( \epsilon > 0 \) there exists a control function \( u_\epsilon \) in \( Y \) such that

\[
\|Bu_\epsilon - q\| < \left( \frac{M\alpha}{\Gamma(1+\alpha)}\sqrt{\frac{\tau^{2\alpha-1}}{2\alpha - 1}} \right)^{-1} \epsilon \quad \text{for } 1/2 < \alpha < 1.
\]

Put \( \eta_\epsilon = \int_0^t (\tau - s)^{\alpha-1}T_\alpha(\tau - s)Bu_\epsilon(s)\, ds \). Since \( \eta_\epsilon = \xi_\epsilon - S_\alpha(\tau)\varphi(0) \), then

\[
\|\xi_\epsilon - \xi| = \|\eta_\epsilon - \eta_\epsilon\| \leq \int_0^t (\tau - s)^{\alpha-1}\left\|T_\alpha(\tau - s)\right\|\|Bu_\epsilon(s) - q(s)\|\, ds
\]

\[
\leq \frac{M\alpha}{\Gamma(1+\alpha)}\sqrt{\frac{\tau^{2\alpha-1}}{2\alpha - 1}}\|Bu_\epsilon - q\| < \epsilon.
\]

Since \( \epsilon \) is arbitrary, we infer that \( K_\tau (0) \subset D(A) \). The denseness of the domain \( D(A) \) in \( V \) implies the approximate controllability of the linear system \((1)^*\). \( \square \)

**Lemma 4.2.** Under the assumptions \((H1)\) and \((H2)\), the operator \( f_x \) has a fixed point \( m_0 \) in the set \( M_0 \) if

\[
\frac{M\tau^\alpha (1+C)}{\Gamma(1+\alpha)} < 1.
\]  

**Proof.** Let \( B_r = \{z \in M_0| \|z\| \leq r\} \) for some positive number \( r \). First, we show that \( f_x \) maps \( B_r \) into itself. If it not true, then for each positive number \( r \), there exists a function \( m \in B_r \), such that \( f_x(m) \) is not the element of \( B_r \), i.e. \( \|f_x(m)\| > r \). On the other hand, from \((H1)\) and Lemma 4.1, we have
\[ r < \| f_x(m) \| = \| Kn \| \leq \int_0^t (t-s)^{\alpha-1} \| T_\alpha(t-s) \| \| n(s) \| \, ds \]

\[ \leq \frac{M\alpha}{\Gamma(1+\alpha)} \int_0^t (t-s)^{\alpha-1} (1+C) \| F_h(x+m)(s) \| \, ds \]

\[ \leq \frac{M\alpha(1+C)}{\Gamma(1+\alpha)} \int_0^t (t-s)^{\alpha-1} \| f_h(x+m)(s-h) \| + I_f \, ds \]

\[ \leq \frac{Ml\alpha(1+C)}{\Gamma(1+\alpha)} \sqrt{\frac{\tau^{2\alpha-1}}{2\alpha-1}} \| x \|_Z + \frac{M(lr + l_f)(1+C)t^\alpha}{\Gamma(1+\alpha)} \]

\[ \leq \frac{M(1+C)}{\Gamma(1+\alpha)} \left[ l\alpha \sqrt{\frac{\tau^{2\alpha-1}}{2\alpha-1}} \| x \|_Z + lr t^\alpha + l_f t^\alpha \right]. \]

Dividing both side by \( r \) and taking limit as \( r \to \infty \), we get \( \frac{Ml\alpha(1+C)}{\Gamma(1+\alpha)} \geq 1 \), which is a contradiction to (5). Hence \( f_x \) maps \( B_r \) into itself.

Next, we show that \( f_x \) is a compact operator. By assumption (H2) the \( C_0 \)–semigroup is compact. Hence \( T_\alpha(t) \) is also compact (see Lemma 3.4, [26]). This implies that the integral operator \( K \) and hence \( f_x \) are compact.

Then by the Schauder fixed point theorem \( f_x \) has a fixed point \( m_0 \) (say). That is \( f_x(m_0) = Kn = m_0 \). This completes the proof of lemma. \( \square \)

**Theorem 4.2.** The semilinear control system (1) is approximately controllable under the conditions (H1)–(H3).

**Proof.** Let \( x(\cdot) \) be the mild solution of the corresponding linear system (1)* given by

\[
\begin{align*}
x(t) &= \begin{cases} 
S_\alpha(t)\varphi(0) + KBu(t), & t \in [0, \tau]; \\
\psi(t), & t \in [-h, 0]. 
\end{cases} 
\end{align*}
\]

(6)

Now, we have to prove that \( y = x + m_0 \) is the mild solution of the semilinear system given by

\[
\begin{align*}
C D_\alpha^{\alpha} y(t) &= Ay(t) + (Bu - q)(t) + f(t, x(t-h)), & t \in [0, \tau]; \\
y(t) &= \psi(t), & t \in [-h, 0]. 
\end{align*}
\]

(7)

From (4), we have

\[ F_h(x+m_0)(t) = n(t) + q(t). \]

Operating \( K \) on both sides at \( m = m_0 \) (a fixed point of \( f_x \)) and using the definition of \( M_0 \), we get

\[ KF_h(x+m_0)(t) = Kn(t) + Kq(t) = m_0(t) + Kq(t) \]

adding \( x(\cdot) \) on both sides, we get

\[ x(t) + KF_h(x+m_0)(t) = x(t) + m_0(t) + Kq(t). \]
Let \( y(t) = x(t) + m_0(t) \), then
\[
x(t) + K F_h(y)(t) = y(t) + Kq(t).
\]
Using Eq. (6), we get
\[
S_\alpha(t)\varphi(0) + KBu(t) + K F_h(y)(t) = y(t) + Kq(t)
\]
\[
\Rightarrow y(t) = S_\alpha(t)\varphi(0) + K (Bu - q)(t) + K F_h(y)(t).
\]
This is the mild solution of (7) with control \((Bu - q)\).

Moreover, since \( m_0(0) = m_0(\tau) = 0 \) we have
\[
y(0) = x(0) + m_0(0) = x(0) = \varphi(0),
\]
\[
y(\tau) = x(\tau) + m_0(\tau) = x(\tau) \in K_\tau(0).
\]

Further, since \( q \in \overline{R(B)} \) there exists a \( v \in Y \) such that
\[
\|Bv - q\| \leq \epsilon \quad \text{for any given } \epsilon > 0.
\]
Let \( x_w(\cdot) \) be the mild solution of the semilinear control system (1) corresponding to the control \( w = u - v \). Then we can easily prove that
\[
\|y(\tau) - x_w(\tau)\| = \|x(\tau) - x_w(\tau)\| \leq \epsilon.
\]
This implies that \( K_\tau(0) \subseteq K_\tau(f) \). Since \( K_\tau(0) \) is dense in \( V \) (by Theorem 4.1) it follows that \( K_\tau(f) \) is also dense in \( V \). Hence, the semilinear control system (1) is approximately controllable. \( \square \)

**Remark 4.1.** If the system is without delay i.e. \( h = 0 \) and \( \alpha = 1 \), then the main results of [20] are obtained under the condition \( Ml_\tau(1 + C) < 1 \) as a corollary to Theorem 4.2.

**5. Examples**

**Example 5.1.** Let \( V = L_2(0, \pi) \) and \( A = \frac{d^2}{dx^2} \) with \( D(A) \) consisting of all \( y \in V \) with \( \frac{d^2 y}{dx^2} \) and \( y(0) = y(\pi) = 0 \). Put \( \phi_n(x) = (\frac{2}{\pi})^{1/2} \sin(nx) \); \( 0 \leq x \leq \pi \), \( n = 1, 2, \ldots \), then \( \{\phi_n, n = 1, 2, \ldots\} \) is an orthonormal basis for \( V \) and \( \phi_n \) is the eigenfunction corresponding to the eigenvalue \( \lambda_n = -n^2 \) of the operator \( A \). Then the \( C_0 \)-semigroup \( T(t) \) generated by \( A \) has \( \exp(\lambda_n t) \) as the eigenvalues and \( \phi_n \) as their corresponding eigenfunctions [25]. Define an infinite dimensional space \( \hat{V} \) by
\[
\hat{V} = \left\{ u \mid u = \sum_{n=2}^{\infty} u_n \phi_n, \text{ with } \sum_{n=2}^{\infty} u_n^2 < \infty \right\}.
\]
The norm in \( \hat{V} \) is defined by
\[
\|u\|_{\hat{V}} = \left( \sum_{n=2}^{\infty} u_n^2 \right)^{1/2}.
\]
Define a continuous linear map $B$ from $\hat{V}$ to $V$ as

$$ Bu = 2u_{2}\phi_1 + \sum_{n=2}^{\infty} u_n\phi_n \quad \text{for } u = \sum_{n=2}^{\infty} u_n\phi_n \in \hat{V}. $$

Let us consider the following fractional order semilinear control system of the form

$$ C \, D_t^{\alpha} y(t, x) = \frac{\partial^2 y(t, x)}{\partial x^2} + Bu(t, x) + f(t, y(t - h, x)), \quad t \in [0, \tau], \ 0 < x < \pi; $$

$$ y(t, 0) = y(t, \pi) = 0, \quad t > 0; $$

$$ y(t, x) = \varphi(t, x), \quad t \in [-h, 0], $$

where $\varphi(t, x)$ is continuous. The system (8) can be written in the abstract form given by (1). If the conditions (H1)–(H3) are satisfied, then the approximate controllability of the system (8) follows from Theorem 4.2.

**Example 5.2.** Consider the electrical circuit shown in Fig. 1 with given resistances $R_1, R_2, R_3$, inductances $L_1, L_2$ and a nonlinear device $N$ (for example, diode, nonlinear resistor, etc.) connected to a source voltage $u(t)$. Let the nonlinear device produce a voltage $f(i_2(t))$, where $f$ is a nonlinear function of $i_2$ and satisfies Lipschitz condition.

Applying Kirchhoff’s law in closed loop (I) [28], we get

$$ u(t) = i_1(t)R_1 + (i_1(t) - i_2(t))R_3 + L_1 \frac{d^\alpha i_1(t)}{dt^\alpha} \Rightarrow \frac{d^\alpha i_1(t)}{dt^\alpha} = -\frac{(R_1 + R_3)}{L_1} i_1(t) + \frac{R_3}{L_1} i_2(t) + \frac{u(t)}{L_1}. $$

Again, applying Kirchhoff’s law in closed loop (II), we get

$$ 0 = i_2(t)R_2 + L_2 \frac{d^\alpha i_2(t)}{dt^\alpha} + f(i_2(t)) - (i_1(t) - i_2(t))R_3 \Rightarrow \frac{d^\alpha i_2(t)}{dt^\alpha} = \frac{R_3}{L_2} i_1(t) - \frac{(R_2 + R_3)}{L_2} i_2(t) - \frac{f(i_2(t))}{L_2}. $$

Eqs. (9) and (10) can be written in the form

$$ \frac{d^\alpha}{dt^\alpha} \begin{bmatrix} i_1(t) \\ i_2(t) \end{bmatrix} = \begin{bmatrix} \frac{-(R_1 + R_3)}{L_1} & \frac{R_3}{L_2} \\ \frac{R_3}{L_1} & \frac{-(R_2 + R_3)}{L_2} \end{bmatrix} \begin{bmatrix} i_1(t) \\ i_2(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{L_1} \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ -\frac{f(i_2(t))}{L_2} \end{bmatrix}. $$

**Fig. 1.** Electrical control system.
Denoting $x(t) = [i_1(t), i_2(t)]^T$, the above system can be expressed as
\[
\frac{d^\alpha x(t)}{dt^\alpha} = Ax(t) + Bu(t) + f(t, x(t))
\] (11)
where
\[
A = \begin{bmatrix}
-(R_1+R_3) & R_3 \\
L_1 & -L_1 \\
R_2 & -L_2
\end{bmatrix}, \\ B = \begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}
\text{ and } f(t, x(t)) = \begin{bmatrix}
0 \\
-f(i_2(t))
\end{bmatrix}.
\]

It is clear that the linear system corresponding to (11) is controllable as the rank of the matrix $[B, AB]$ is 2, see [3,4]. Since the nonlinear function satisfies the Lipschitz condition, the approximate controllability of the system (11) follows from Theorem 4.2 for $1/2 < \alpha < 1$.

References