PERGAMON

Applied Mathematics Letters 16 (2003) 653–655

Applied Mathematics Letters

www.elsevier.com/locate/aml

A Certain Connection between Starlike and Convex Functions

N. TAKAHASHI AND M. NUNOKAWA
Department of Mathematics, Gunma University
Aramaki, Maebashi Gunma 371-8510, Japan
norihiro@ed.edu.gunma-u.ac.jp
nunokawa@edu.gunma-u.ac.jp

(Received July 2001; revised and accepted September 2002)

Abstract—We define two certain classes of functions $S^*(\alpha, \beta)$ and $C(\alpha, \beta)$ and obtain a certain connection between these classes.

Keywords—Starlike, Convex.

1. INTRODUCTION

Let $\mathcal{A}$ be the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$. A function $f(z)$ in $\mathcal{A}$ is said to be starlike if it satisfies

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > 0, \qquad z \in U.$$

We denote by S^* the subclass of $\mathcal{A}$ consisting of all starlike functions in U. A function $f(z)$ in $\mathcal{A}$ is said to be convex if it satisfies

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, \qquad z \in U.$$

We denote by C the subclass of $\mathcal{A}$ consisting of all convex functions in U.

Nunokawa, Owa, Saitoh, Cho and Takahashi [1] obtained the following result.

LEMMA. *Let $p(z)$ be analytic in U with $p(0) = 1$ and $p(z) \neq 0$. If there exist two points $z_1 \in U$ and $z_2 \in U$ such that*

$$-\frac{\pi\beta}{2} = \arg p(z_1) < \arg p(z) < \arg p(z_2) = \frac{\pi\alpha}{2}$$

The authors sincerely appreciate the helpful comments made by the referee.

for $\alpha > 0$, $\beta > 0$, *and for* $|z| < |z_1| = |z_2|$, *then we have*

$$\frac{z_1 p'(z_1)}{p(z_1)} = -i\,\frac{\alpha+\beta}{2}\,m$$

and

$$\frac{z_2 p'(z_2)}{p(z_2)} = i\,\frac{\alpha+\beta}{2}\,m,$$

where

$$m \geq \frac{1-|a|}{1+|a|}$$

and

$$a = i\tan\frac{\pi}{4}\left(\frac{\alpha-\beta}{\alpha+\beta}\right).$$

From the lemma, we define two classes of functions. Let $S^*(\alpha,\beta)$ be the subclass of $\mathcal{A}$ which satisfies

$$-\frac{\pi\beta}{2} < \arg\frac{zf'(z)}{f(z)} < \frac{\pi\alpha}{2}, \qquad z \in U,$$

for $0 \leq \alpha < 1$, $0 \leq \beta < 1$, and let $C(\alpha,\beta)$ be the subclass of $\mathcal{A}$ which satisfies

$$-\frac{\pi\beta}{2} < \arg\left(1+\frac{zf''(z)}{f'(z)}\right) < \frac{\pi\alpha}{2}, \qquad z \in U,$$

for $0 \leq \alpha < 1$, $0 \leq \beta < 1$. We can see that $S^*(\alpha,\beta) \subset S^*$ and $C(\alpha,\beta) \subset C$. In this paper, applying the lemma, we obtain a certain connection between $S^*(\alpha,\beta)$ and $C(\alpha,\beta)$.

2. MAIN THEOREM

THEOREM. *Let* $f(z) \in C(\gamma(\alpha,\beta),\, \delta(\alpha,\beta))$. *Then* $f(z) \in S^*(\alpha,\beta)$ *where*

$$\gamma(\alpha,\beta) = \alpha + \frac{2}{\pi}\operatorname{Tan}^{-1}\frac{(1-|a|)\sigma(\alpha,\beta)\sin(\pi/2)(1-\alpha)}{1+|a|+(1-|a|)\sigma(\alpha,\beta)\cos(\pi/2)(1-\alpha)},$$

$$\delta(\alpha,\beta) = \beta + \frac{2}{\pi}\operatorname{Tan}^{-1}\frac{(1-|a|)\sigma(\alpha,\beta)\sin(\pi/2)(1-\beta)}{1+|a|+(1-|a|)\sigma(\alpha,\beta)\cos(\pi/2)(1-\beta)},$$

$$\sigma(\alpha,\beta) = \frac{\alpha+\beta}{2-\alpha-\beta}\left(\frac{2-\alpha-\beta}{2+\alpha+\beta}\right)^{(2+\alpha+\beta)/4},$$

and

$$a = i\tan\frac{\pi}{4}\left(\frac{\alpha-\beta}{\alpha+\beta}\right).$$

PROOF. Let us put $p(z) = zf'(z)/f(z)$ and $f(z) \in C(\gamma(\alpha,\beta),\, \delta(\alpha,\beta))$. If there exist two points $z_1 \in U$ and $z_2 \in U$ such that

$$-\frac{\pi\beta}{2} = \arg p(z_1) < \arg p(z) < \arg p(z_2) = \frac{\pi\alpha}{2},$$

for $|z| < |z_1| = |z_2|$, then from the proof of the lemma [1], we have

$$\frac{z_1 p'(z_1)}{p(z_1)} = -i\cdot\frac{\alpha+\beta}{4}\cdot\frac{1+{t_1}^2}{t_1}\cdot m \qquad \text{and} \qquad \frac{z_2 p'(z_2)}{p(z_2)} = i\cdot\frac{\alpha+\beta}{4}\cdot\frac{1+{t_2}^2}{t_2}\cdot m, \tag{2.1}$$

where

$$e^{-i(\pi/2)((\alpha-\beta)/(\alpha+\beta))}p(z_1)^{2/(\alpha+\beta)} = -it_1, \qquad e^{-i(\pi/2)((\alpha-\beta)/(\alpha+\beta))}p(z_2)^{2/(\alpha+\beta)} = it_2, \tag{2.2}$$

$t_1 > 0$, $t_2 > 0$, and

$$m \geq \frac{1-|a|}{1+|a|}.$$

By logarithmic differentiation of $p(z) = zf'(z)/f(z)$, we have

$$1 + \frac{zf''(z)}{f'(z)} = p(z) + \frac{zp'(z)}{p(z)} = p(z)\left(1 + \frac{1}{p(z)} \cdot \frac{zp'(z)}{p(z)}\right). \tag{2.3}$$

Let us put $z = z_2$. Then from (2.1)–(2.3), we have

$$\begin{aligned} 1 + \frac{z_2 f''(z_2)}{f'(z_2)} &= p(z_2)\left(1 + \frac{1}{p(z_2)} \cdot \frac{z_2 p'(z_2)}{p(z_2)}\right) \\ &= t_2^{(\alpha+\beta)/2} e^{i(\pi/2)\alpha}\left(1 + t_2^{-(\alpha+\beta)/2} e^{-i(\pi/2)\alpha}\left(i \cdot \frac{\alpha+\beta}{4} \cdot \frac{1+t_2^2}{t_2} \cdot m\right)\right) \\ &= t_2^{(\alpha+\beta)/2} e^{i(\pi/2)\alpha}\left(1 + m \cdot \frac{\alpha+\beta}{4}\left(t_2^{1-(\alpha+\beta)/2} + t_2^{-1-(\alpha+\beta)/2}\right) e^{i(\pi/2)(1-\alpha)}\right). \end{aligned}$$

Let us put $g(x) = x^{1-(\alpha+\beta)/2} + x^{-1-(\alpha+\beta)/2}$, $x > 0$. Then $g(x)$ takes the minimum value at $x = \sqrt{(2+\alpha+\beta)/(2-\alpha-\beta)}$. Therefore, we have

$$\begin{aligned} \arg\left(1 + \frac{z_2 f''(z_2)}{f'(z_2)}\right) &= \arg p(z_2) + \arg\left(1 + \frac{1}{p(z_2)} \cdot \frac{z_2 p'(z_2)}{p(z_2)}\right) \\ &= \frac{\pi\alpha}{2} + \arg\left(1 + m \cdot \frac{\alpha+\beta}{4}\left(t_2^{1-(\alpha+\beta)/2} + t_2^{-1-(\alpha+\beta)/2}\right) e^{i(\pi/2)(1-\alpha)}\right) \\ &\geq \frac{\pi\alpha}{2} + \arg\left(1 + \frac{1-|a|}{1+|a|} \cdot \frac{\alpha+\beta}{4}\left(\left(\frac{2+\alpha+\beta}{2-\alpha-\beta}\right)^{(2-\alpha-\beta)/4}\right.\right. \\ &\quad \left.\left. + \left(\frac{2+\alpha+\beta}{2-\alpha-\beta}\right)^{(-2-\alpha-\beta)/4}\right) e^{i(\pi/2)(1-\alpha)}\right) \\ &= \frac{\pi\alpha}{2} + \arg\left(1 + \frac{1-|a|}{1+|a|} \cdot \frac{\alpha+\beta}{2-\alpha-\beta}\left(\frac{2+\alpha+\beta}{2-\alpha-\beta}\right)^{(2+\alpha+\beta)/4} e^{i(\pi/2)(1-\alpha)}\right) \\ &= \frac{\pi\alpha}{2} + \operatorname{Tan}^{-1} \frac{(1-|a|)\,\sigma(\alpha,\beta)\,\sin(\pi/2)(1-\alpha)}{1+|a|+(1-|a|)\,\sigma(\alpha,\beta)\,\cos(\pi/2)(1-\alpha)}, \end{aligned}$$

where

$$\sigma(\alpha,\beta) = \frac{\alpha+\beta}{2-\alpha-\beta}\left(\frac{2-\alpha-\beta}{2+\alpha+\beta}\right)^{(2+\alpha+\beta)/4}.$$

This contradicts the assumption of the theorem.

For the case $z = z_1$, applying the same method as the above, we have

$$\arg\left(1 + \frac{z_2 f''(z_2)}{f'(z_2)}\right) \leq -\frac{\pi\beta}{2} - \operatorname{Tan}^{-1} \frac{(1-|a|)\,\sigma(\alpha,\beta)\,\sin(\pi/2)(1-\beta)}{1+|a|+(1-|a|)\,\sigma(\alpha,\beta)\,\cos(\pi/2)(1-\beta)}.$$

This contradicts the assumption of the theorem. We complete the proof of the theorem. ∎

REMARK. Let us put $\alpha = \beta$ in the theorem. Then we have the result in [2].

REFERENCES

1. M. Nunokawa, S. Owa, H. Saitoh, N.E. Cho and N. Takahashi, Some properties of analytic functions at extremal points for arguments, (submitted).
2. M. Nunokawa, On the order of strongly starlikeness of strongly convex functions, *Proc. Japan. Acad., Ser. A* **69** (7), 234–237, (1993).