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# Existence and uniqueness of periodic solutions for a kind of Liénard equation with a deviating argument[☆]

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**Abstract**

In this work, we use the coincidence degree theory to establish new results on the existence and uniqueness of $T$-periodic solutions for a kind of Liénard equation with a deviating argument of the form

$$x''(t) + f(x(t))x'(t) + g(t, x(t - \tau(t))) = p(t).$$



*Keywords:* Liénard equation; Deviating argument; Periodic solution; Existence; Uniqueness; Coincidence degree

## 1. Introduction

Consider the Liénard equation with a deviating argument of the form

$$x''(t) + f(x(t))x'(t) + g(t, x(t - \tau(t))) = p(t), \tag{1.1}$$

where $f, \tau, p : R \to R$ and $g : R \times R \to R$ are continuous functions, $\tau$ and $p$ are $T$-periodic, $g$ is $T$-periodic in its first argument, and $T > 0$. In recent years, the problem of the periodic solutions of Eq. (1.1) has been extensively studied in the literature. We refer the reader to [1,3–8] and the references cited therein. However, to the best of our knowledge, most authors of the bibliographies listed above only consider the existence of periodic solutions of Eq. (1.1), and there exist few results for the *existence and uniqueness* of periodic solutions of Eq. (1.1). Thus, it is worthwhile to study the existence and uniqueness of the periodic solutions of Eq. (1.1).

The main purpose of this work is to establish sufficient conditions for the existence and uniqueness of $T$-periodic solutions of Eq. (1.1). The results of this work are new and they complement previously known results.

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For ease of exposition, throughout this work we will adopt the following notation:

$$|x|_k = \left(\int_0^T |x(t)|^k \mathrm{d}t\right)^{1/k}, \qquad |x|_\infty = \max_{t\in[0,T]} |x(t)|.$$

Let

$$X = \{x | x \in C^1(R, R),\ x(t+T) = x(t),\ \text{for all } t \in R\}$$

and

$$Y = \{x | x \in C(R, R),\ x(t+T) = x(t),\ \text{for all } t \in R\}$$

be two Banach spaces with the norms

$$\|x\|_X = \max\{|x|_\infty,\ |x'|_\infty\}, \quad \text{and} \quad \|x\|_Y = |x|_\infty.$$

Define a linear operator $L : D(L) \subset X \longrightarrow Y$ by setting

$$D(L) = \{x | x \in X, x'' \in C(R, R)\}$$

and for $x \in D(L)$,

$$Lx = x''. \tag{1.2}$$

We also define a nonlinear operator $N : X \longrightarrow Y$ by setting

$$Nx = -f(x(t))x'(t) - g(t, x(t-\tau(t))) + p(t). \tag{1.3}$$

It is easy to see that

$$\operatorname{Ker} L = R, \quad \text{and} \quad \operatorname{Im} L = \left\{x | x \in Y, \int_0^T x(s)\mathrm{d}s = 0\right\}.$$

Thus the operator $L$ is a Fredholm operator with index zero.

Define the continuous projector $P : X \longrightarrow \operatorname{Ker} L$ and the averaging projector $Q : Y \longrightarrow Y$ by setting

$$Px(t) = x(0) = x(T)$$

and

$$Qx(t) = \frac{1}{T}\int_0^T x(s)\mathrm{d}s.$$

Hence, $\operatorname{Im} P = \operatorname{Ker} L$ and $\operatorname{Ker} Q = \operatorname{Im} L$. Denoting by $L_P^{-1} : \operatorname{Im} L \longrightarrow D(L) \cap \operatorname{Ker} P$ the inverse of $L|_{D(L)\cap \operatorname{Ker} P}$, we have

$$L_P^{-1} y(t) = -\frac{t}{T}\int_0^T (t-s)y(s)\mathrm{d}s + \int_0^t (t-s)y(s)\mathrm{d}s. \tag{1.4}$$

It is convenient to introduce the following assumptions.

($A_0$) Assume that there exist nonnegative constants $C_1$ and $C_2$ such that

$$|f(x_1) - f(x_2)| \le C_1|x_1 - x_2|, \qquad |f(x)| \le C_2, \quad \text{for all } x_1, x_2,\ x \in R.$$

## 2. Preliminary results

In view of (1.2) and (1.3), the operator equation $Lx = \lambda Nx$ is equivalent to the following equation:

$$x'' + \lambda[f(x(t))x'(t) + g(t, x(t-\tau(t)))] = \lambda p(t), \tag{$2.1_\lambda$}$$

where $\lambda \in (0, 1)$.

For convenience of use, we introduce the Continuation Theorem [4] as follows.

**Lemma 2.1.** *Let $X$ and $Y$ be two Banach spaces. Suppose that $L : D(L) \subset X \longrightarrow Y$ is a Fredholm operator with index zero and $N : X \longrightarrow Y$ is $L$-compact on $\overline{\Omega}$, where $\Omega$ is an open bounded subset of $X$. Moreover, assume that all the following conditions are satisfied:*

(1) $Lx \neq \lambda Nx$, *for all* $x \in \partial\Omega \cap D(L)$, $\lambda \in (0, 1)$;
(2) $Nx \notin \operatorname{Im} L$, *for all* $x \in \partial\Omega \cap \operatorname{Ker} L$;
(3) *the Brouwer degree*

$$\deg\{QN, \ \Omega \cap \operatorname{Ker} L, 0\} \neq 0.$$

*Then equation $Lx = Nx$ has at least one solution on $\overline{\Omega}$.*

The following lemmas will be useful for proving our main results in Section 3.

**Lemma 2.2.** *If $x \in C^2(R, R)$ with $x(t+T) = x(t)$, then*

$$|x'(t)|_2^2 \leq \left(\frac{T}{2\pi}\right)^2 |x''(t)|_2^2. \tag{2.2}$$

**Proof.** Lemma 2.2 is a direct consequence of the Wirtinger inequality, and see [2,3] for its proof. □

**Lemma 2.3.** *Suppose that there exists a constant $d > 0$ such that one of the following conditions holds:*

($A_1$) $x(g(t, x) - p(t)) < 0$, *for all* $t \in R$, $|x| \geq d$;
($A_2$) $x(g(t, x) - p(t)) > 0$, *for all* $t \in R$, $|x| \geq d$.

*If $x(t)$ is a $T$-periodic solution of ( $2.1_\lambda$), then*

$$|x|_\infty \leq d + \sqrt{T}|x'|_2. \tag{2.3}$$

**Proof.** Let $x(t)$ be a $T$-periodic solution of ($2.1_\lambda$). Then, integrating ($2.1_\lambda$) from 0 to $T$, we have

$$\int_0^T [g(t, x(t - \tau(t))) - p(t)]\mathrm{d}t = 0.$$

This implies that there exists $\xi \in [0, T]$ such that

$$g(\xi, x(\xi - \tau(\xi))) - p(\xi) = 0.$$

Thus, taking this together with ($A_1$) (or ($A_2$)), we have

$$|x(\xi - \tau(\xi))| < d.$$

Let $\xi - \tau(\xi) = mT + t_0$, where $t_0 \in [0, T]$ and $m$ is an integer. Then, using the Schwarz inequality and the following relation:

$$|x(t)| = \left| x(t_0) + \int_{t_0}^t x'(s)\mathrm{d}s \right| \leq d + \int_0^T |x'(s)|\mathrm{d}s, \quad t \in [0, T],$$

we obtain

$$|x|_\infty = \max_{t \in [0,T]} |x(t)| \leq d + \sqrt{T}|x'|_2.$$

This completes the proof of Lemma 2.3. □

**Lemma 2.4.** *Let ($A_0$) and ($A_1$) (or ($A_2$)) hold. Assume that the following condition is satisfied:*

($A_3$) *there exists a nonnegative constant $b$ such that*

$$C_2\frac{T}{2\pi} + b\frac{T^2}{2\pi} < 1, \quad \textit{and} \quad |g(t, x_1) - g(t, x_2)| \leq b|x_1 - x_2|, \quad \textit{for all } t, x_1, x_2 \in R;$$

*If $x(t)$ is a $T$-periodic solution of Eq. (1.1), then*

$$|x'|_2 \leq \frac{[bd + \max\{|g(t,0)| : 0 \leq t \leq T\} + |p|_\infty]T}{1 - \left(C_2\frac{T}{2\pi} + b\frac{T^2}{2\pi}\right)} := D. \tag{2.4}$$

**Proof.** Let $x(t)$ be a $T$-periodic solution of Eq. (1.1). From ($A_1$) (or ($A_2$)), we can easily show that (2.3) also holds. Multiplying $x''(t)$ and Eq. (1.1) and then integrating it from 0 to $T$, in view of (2.2), (2.3), ($A_3$) and the inequality of Schwarz, we have

$$\begin{aligned} |x''|_2^2 &= -\int_0^T f(x(t))x'(t)x''(t)\mathrm{d}t - \int_0^T g(t,x(t-\tau(t)))x''(t)\mathrm{d}t + \int_0^T p(t)x''(t)\mathrm{d}t \\ &\leq C_2\frac{T}{2\pi}|x''|_2^2 + \int_0^T [|g(t,x(t-\tau(t))) - g(t,0)| + |g(t,0)|]\cdot|x''(t)|\mathrm{d}t + \int_0^T |p(t)|\cdot|x''(t)|\mathrm{d}t \\ &\leq C_2\frac{T}{2\pi}|x''|_2^2 + b\int_0^T |x(t-\tau(t))|\cdot|x''(t)|\mathrm{d}t + \int_0^T |g(t,0)|\cdot|x''(t)|\mathrm{d}t + \int_0^T |p(t)|\cdot|x''(t)|\mathrm{d}t \\ &\leq C_2\frac{T}{2\pi}|x''|_2^2 + b|x|_\infty\sqrt{T}|x''|_2 + [\max\{|g(t,0)| : 0 \leq t \leq T\} + |p|_\infty]\sqrt{T}|x''|_2 \\ &\leq \left(C_2\frac{T}{2\pi} + b\frac{T^2}{2\pi}\right)|x''|_2^2 + [bd + \max\{|g(t,0)| : 0 \leq t \leq T\} + |p|_\infty]\sqrt{T}|x''|_2, \end{aligned} \tag{2.5}$$

which, together with ($A_3$), implies that

$$|x''|_2 \leq \frac{[bd + \max\{|g(t,0)| : 0 \leq t \leq T\} + |p|_\infty]\sqrt{T}}{1 - \left(C_2\frac{T}{2\pi} + b\frac{T^2}{2\pi}\right)}. \tag{2.6}$$

Since $x(0) = x(T)$, there exists a constant $\zeta \in [0, T]$ such that

$$x'(\zeta) = 0,$$

and

$$|x'(t)| = |x'(\zeta) + \int_\zeta^t x''(s)\mathrm{d}s| \leq \sqrt{T}|x''|_2, \quad \text{for all } t \in [0, T]. \tag{2.7}$$

Thus, in view of (2.6) and (2.7), we have

$$|x'|_\infty \leq \frac{[bd + \max\{|g(t,0)| : 0 \leq t \leq T\} + |p|_\infty]T}{1 - \left(C_2\frac{T}{2\pi} + b\frac{T^2}{2\pi}\right)} := D.$$

This completes the proof of Lemma 2.4. □

**Lemma 2.5.** *Let ($A_1$) (or ($A_2$)) hold. Assume that the following condition is satisfied:*

($A_4$) *Suppose that ($A_0$) holds, $g(t,x)$ is a strictly monotone function in $x$, and there exists a nonnegative constant $b$ such that*

$$C_1 D\frac{T^2}{2\pi} + C_2\frac{T}{2\pi} + b\frac{T^2}{2\pi} < 1, \quad \textit{and} \quad |g(t,x_1) - g(t,x_2)| \leq b|x_1 - x_2|, \quad \textit{for all } t, x_1, x_2 \in R.$$

*Then Eq. (1.1) has at most one $T$-periodic solution.*

**Proof.** Suppose that $x_1(t)$ and $x_2(t)$ are two $T$-periodic solutions of Eq. (1.1). Then, we have

$$(x_1(t) - x_2(t))'' + (f(x_1(t))x_1'(t) - f(x_2(t))x_2'(t)) + (g(t,x_1(t-\tau(t))) - g(t,x_2(t-\tau(t)))) = 0. \tag{2.8}$$

Set $Z(t) = x_1(t) - x_2(t)$. Then, from (2.8), we obtain

$$Z''(t) + (f(x_1(t))x_1'(t) - f(x_2(t))x_2'(t)) + (g(t,x_1(t-\tau(t))) - g(t,x_2(t-\tau(t)))) = 0. \tag{2.9}$$

Since $x_1(t)$ and $x_2(t)$ are $T$-periodic, integrating (2.9) from 0 to $T$, we obtain

$$\int_0^T (g(t, x_1(t-\tau(t))) - g(t, x_2(t-\tau(t))))\mathrm{d}t = 0.$$

Thus, in view of the integral mean value theorem, it follows that there exists a constant $\gamma \in [0, T]$ such that

$$g(\gamma, x_1(\gamma-\tau(\gamma))) - g(\gamma, x_2(\gamma-\tau(\gamma))) = 0. \tag{2.10}$$

Let $\gamma - \tau(\gamma) = nT + \widetilde{\gamma}$, where $\widetilde{\gamma} \in [0, T]$ and $n$ is an integer. Then, (2.10), together with $(A_4)$, implies that there exists a constant $\widetilde{\gamma} \in [0, T]$ such that

$$Z(\widetilde{\gamma}) = x_1(\widetilde{\gamma}) - x_2(\widetilde{\gamma}) = x_1(\gamma-\tau(\gamma)) - x_2(\gamma-\tau(\gamma)) = 0. \tag{2.11}$$

Hence,

$$|Z(t)| = |Z(\widetilde{\gamma}) + \int_{\widetilde{\gamma}}^t Z'(s)\mathrm{d}s| \le \int_0^T |Z'(s)|\mathrm{d}s, \quad \text{for all } t \in [0, T],$$

and

$$|Z|_\infty \le \sqrt{T}|Z'|_2. \tag{2.12}$$

Multiplying $Z''(t)$ and (2.9) and then integrating it from 0 to $T$, from (2.2) and (2.12) and Schwarz inequality, we get

$$\begin{aligned}
|Z''|_2^2 &= -\int_0^T (f(x_1(t))x_1'(t) - f(x_2(t))x_2'(t))Z''(t)\mathrm{d}t - \int_0^T (g(t, x_1(t-\tau(t))) \\
&\quad - g(t, x_2(t-\tau(t))))Z''(t)\mathrm{d}t \\
&\le \int_0^T |f(x_1(t))||x_1'(t) - x_2'(t)||Z''(t)|\mathrm{d}t + \int_0^T |f(x_1(t)) - f(x_2(t))||x_2'(t)||Z''(t)|\mathrm{d}t \\
&\quad + b\int_0^T |x_1(t-\tau(t)) - x_2(t-\tau(t))||Z''(t)|\mathrm{d}t \\
&\le \int_0^T C_2|x_1'(t) - x_2'(t)||Z''(t)|\mathrm{d}t + \int_0^T C_1|x_1(t) - x_2(t)|D|Z''(t)|\mathrm{d}t \\
&\quad + b\int_0^T |x_1(t-\tau(t)) - x_2(t-\tau(t))||Z''(t)|\mathrm{d}t \\
&\le C_2|Z'|_2|Z''|_2 + C_1D|Z|\infty\sqrt{T}|Z''|_2 + b|Z|_\infty\sqrt{T}|Z''|_2,
\end{aligned}$$

which implies that

$$|Z''|_2^2 \le C_2\frac{T}{2\pi}|Z''|_2^2 + (C_1D + b)\sqrt{T}|Z'|_2\sqrt{T}|Z''|_2 \le \left(C_1D\frac{T^2}{2\pi} + C_2\frac{T}{2\pi} + b\frac{T^2}{2\pi}\right)|Z''|_2^2. \tag{2.13}$$

Since $Z(t)$, $Z'(t)$ and $Z''(t)$ are $T$-periodic and continuous functions, in view of $(A_4)$, (2.11) and (2.13), we have

$$Z(t) \equiv Z'(t) \equiv Z''(t) \equiv 0, \quad \text{for all } t \in R.$$

Thus, $x_1(t) \equiv x_2(t)$, for all $t \in R$. Therefore, Eq. (1.1) has at most one $T$-periodic solution. The proof of Lemma 2.5 is now complete. □

## 3. Main results

**Theorem 3.1.** *Let* $(A_1)$ *(or* $(A_2)$*) hold. Assume that the condition* $(A_4)$ *is satisfied. Then Eq.* (1.1) *has a unique* $T$*-periodic solution.*

**Proof.** By Lemma 2.5, it is easy to see that Eq. (1.1) has at most one $T$-periodic solution. Thus, to prove Theorem 3.1, it suffices to show that Eq. (1.1) has at least one $T$-periodic solution. To do this, we shall apply Lemma 2.1. Firstly, we will claim that the set of all possible $T$-periodic solutions of Eq. ($2.1_\lambda$) is bounded.

Let $x(t)$ be a $T$-periodic solution of Eq. ($2.1_\lambda$). Multiplying $x''(t)$ and Eq. ($2.1_\lambda$) and then integrating from 0 to $T$, in view of (2.2) and (2.3), ($A_4$) and the inequality of Schwarz, we have

$$\begin{aligned}|x''|_2^2 &= -\lambda\int_0^T f(x(t))x'(t)x''(t)\mathrm{d}t - \lambda\int_0^T g(t,x(t-\tau(t)))x''(t)\mathrm{d}t + \lambda\int_0^T p(t)x''(t)\mathrm{d}t\\ &\le \left(C_2\frac{T}{2\pi} + b\frac{T^2}{2\pi}\right)|x''|_2^2 + [bd + \max\{|g(t,0)| : 0\le t\le T\} + |p|_\infty]\sqrt{T}|x''|_2, \end{aligned} \tag{3.1}$$

which, together with ($A_4$), implies that there exist positive constants $D_1$ and $D_2$ such that

$$|x''|_2 < D_1, \tag{3.2}$$

and

$$|x'|_2 < D_2, \qquad |x|_\infty < D_2. \tag{3.3}$$

Since $x(0) = x(T)$, there exists a constant $\bar{\zeta} \in [0, T]$ such that

$$x'(\bar{\zeta}) = 0,$$

and

$$|x'(t)| = \left|x'(\bar{\zeta}) + \int_{\bar{\zeta}}^t x''(s)\mathrm{d}s\right| \le \sqrt{T}|x''|_2 < \sqrt{T}D_1, \quad \text{for all } t \in [0, T]. \tag{3.4}$$

Therefore, in view of (3.3) and (3.4), there exists a positive constant $M_1 > \sqrt{T}D_1 + D_2$ such that

$$\|x\|_X \le |x|_\infty + |x'|_\infty < M_1.$$

If $x \in \Omega_1 = \{x | x \in \operatorname{Ker} L \cap X, \text{ and } Nx \in \operatorname{Im} L\}$, then there exists a constant $M_2$ such that

$$x(t) \equiv M_2, \quad \text{and} \quad \int_0^T [g(t, M_2) - p(t)]\mathrm{d}t = 0. \tag{3.5}$$

Thus,

$$|x(t)| \equiv |M_2| < d, \quad \text{for all } x(t) \in \Omega_1. \tag{3.6}$$

Let $M = M_1 + d + 1$. Set

$$\Omega = \{x | x \in X, |x|_\infty < M, |x'|_\infty < M\}.$$

It is easy to see from (1.3) and (1.4) that $N$ is $L$-compact on $\overline{\Omega}$. We have from (3.5) and (3.6) and the fact that $M > \max\{M_1, d\}$ that the conditions (1) and (2) in Lemma 2.1 hold.

Furthermore, define continuous functions $H_1(x,\mu)$ and $H_2(x,\mu)$ by setting

$$H_1(x,\mu) = (1-\mu)x - \mu\cdot\frac{1}{T}\int_0^T [g(t,x) - p(t)]\mathrm{d}t; \quad \mu \in [0\ 1],$$

$$H_2(x,\mu) = -(1-\mu)x - \mu\cdot\frac{1}{T}\int_0^T [g(t,x) - p(t)]\mathrm{d}t; \quad \mu \in [0\ 1].$$

If ($A_1$) holds, then

$$xH_1(x,\mu) \ne 0, \quad \text{for all } x \in \partial\Omega \cap \operatorname{Ker} L.$$

Hence, using the homotopy invariance theorem, we have

$$\begin{aligned}\deg\{QN, \Omega\cap\operatorname{Ker} L, 0\} &= \deg\left\{-\frac{1}{T}\int_0^T [g(t,x) - p(t)]\mathrm{d}t, \Omega\cap\operatorname{Ker} L, 0\right\}\\ &= \deg\{x, \Omega\cap\operatorname{Ker} L, 0\} \ne 0.\end{aligned}$$

If $(A_2)$ holds, then

$$xH_2(x,\mu) \neq 0, \quad \text{for all } x \in \partial\Omega \cap \operatorname{Ker} L.$$

Hence, using the homotopy invariance theorem, we obtain

$$\begin{aligned}\deg\{QN, \Omega \cap \operatorname{Ker} L, 0\} &= \deg\left\{-\frac{1}{T}\int_0^T [g(t,x) - p(t)]\mathrm{d}t, \Omega \cap \operatorname{Ker} L, 0\right\} \\ &= \deg\{-x, \Omega \cap \operatorname{Ker} L, 0\} \neq 0.\end{aligned}$$

In view of all the discussions above, we conclude from Lemma 2.1 that Theorem 3.1 is proved. □

## 4. Example and remark

**Example 4.1.** Let $g(t,x) = \frac{1}{6\pi}x$, for all $t, x \in R$. Then the Liénard equation

$$x''(t) + \frac{1}{80}(\sin x(t))x'(t) + g(t, x(t - \sin^2 t)) = \frac{1}{6\pi}\mathrm{e}^{\cos t - 1} \tag{4.1}$$

has a unique $2\pi$-periodic solution.

**Proof.** By (4.1), we have $d = 1$, $b = \frac{1}{6\pi}$, $C_1 = C_2 = \frac{1}{80}$, $\tau(t) = \sin^2 t$, $T = 2\pi$ and $p(t) = \frac{1}{6\pi}\mathrm{e}^{\cos t - 1}$; then

$$\frac{[bd + \max\{|g(t,0)| : 0 \le t \le T\} + |p|_\infty]T}{1 - \left(C_2\frac{T}{2\pi} + b\frac{T^2}{2\pi}\right)} := D = \frac{\left[\frac{1}{6\pi} + \frac{1}{6\pi}\right] \times 2\pi}{1 - \frac{1}{8} - \frac{1}{3}} = \frac{16}{13},$$

$$C_1 D\frac{T^2}{2\pi} + C_2\frac{T}{2\pi} + b\frac{T^2}{2\pi} = \frac{4\pi}{157} + \frac{83}{240} < 1.$$

It is obvious that the assumptions $(A_2)$ and $(A_4)$ hold. Hence, by Theorem 3.1, Eq. (4.1) has a unique $2\pi$-periodic solution. □

**Remark 4.1.** Eq. (4.1) is a very simple version of a Liénard equation. Since $f(x) = \frac{1}{8}\sin x$ and $\tau(t) = \sin^2 t$, all the results in [1,3–8] and the references therein cannot be applicable to Eq. (4.1) for obtaining the existence and uniqueness of $2\pi$-periodic solutions. This implies that the results of this work are essentially new.

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