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Long-range cohesive interactions of non-local continuum faced by fractional calculus

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ABSTRACT

A non-local continuum model including long-range forces between non-adjacent volume elements has been studied in this paper. The proposed continuum model has been obtained as limit case of two fully equivalent mechanical models: (i) A volume element model including contact forces between adjacent volumes as well as long-range interactions, distance decaying, between non-adjacent elements. (ii) A discrete point-spring model with local springs between adjacent points and non-local springs with distance-decaying stiffness connecting non-adjacent points. Under the assumption of fractional distance-decaying interactions between non-adjacent elements a fractional differential equation involving Marchaud-type fractional derivatives has been obtained for unbounded domains. It is shown that for unbounded domains the two mechanical models revert to Lazopoulos and Eringen model with fractional distance-decaying functions. It has also been shown that for a confined bar, the stress–strain relation is substantially different from that obtained simply using the truncated Marchaud derivatives since a double integral instead of convolution integral appears. Moreover, in the analysis of bounded domains, the governing equations turn out to an integro-differential equation including only the integral part of Marchaud fractional derivatives on finite interval. The mechanical boundary condition for the proposed model has been introduced consistently on the basis of mechanical considerations, and the constitutive law of the proposed continuum model has been reported by mathematical induction. Several numerical applications have been reported to show, verify and assess the concepts listed in this paper.

1. Introduction

Mechanics of generalized continua that accounts for long-range forces in elastomechanics had gained strong interest by the scientists all over the world in the late 1960s as reported in several studies (e.g. Kroner, 1967; Krumhanls, 1967; Eringen, 1972; Eringen and Edelen, 1972). The use of these theories in fields showing the failure of classical continuum mechanics had impressive effects in the explanation of unpredicted phenomenon; for instance, they succeed in considerably smoothing the unrealistic stress-singularities at crack-tips. Anyway, the lack of mechanical grounds in the evaluation of non-local forces led to a progressive indifference to the field orienting researchers to capture non-local effects by description of materials at micro and nano-scale. As in fact any engineering material possesses an internal substructure which may be observed at molecular level. Internal constitutive substructure at molecular or crystalline level may be considered by means of molecular dynamics as shown in some studies conducted in the late fifties.

Despite paradoxes of continuum mechanics observed in the study of some problems, the powerful approach of the well-established mathematical theory of elasticity is extremely attractive to model and solve the engineering problems. Some at-

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tempts to conjugate the accuracy of atomic theory and the simplicity of continuum mechanics have been formulated introducing in the material constitutive equations some terms accounting for long-range interactions between non-adjacent particles. In this context, two wide classes of theories have been well enstablished: The gradient elasticity theory (weak non-locality) and the integral non-local theory (strong non-locality). The first approach consists in the introduction of opportune terms including gradient of strains in the constitutive equations of the considered material (Mindlin and Eshel, 1968; Aifantis, 1994) with opportune coefficients dependent on material microstructure. The main drawback of gradient elasticity model regards fulfilment of the boundary conditions associated to the problem considered. In this context several strategies, which make use of variational formulations, have been recently proposed (Polizzotto, 2001, 2003). In other studies the problem has been framed in thermodynamic setting (Polizzotto and Borino, 1998; Borino et al., 2003). The approach yields results in good agreement with experiments but the mechanical aspects of the boundary conditions and selections of parameters involved in the analysis are still an open problem. For a review of the recent developments in gradient theories see Aifantis (2003) and references cited therein.

As an alternative non-local integral model of elasticity has been introduced as intuitive extensions of interpolation formulas of molecular dynamics accounting for discrete-continuum equivalence (Kroner, 1967; Eringen, 1972). The resulting non-local elastic model includes additional integral term of the strain field with kernel represented by an opportune attenuation function decaying with distance (Gaussian or exponential). In the late 1980s, this approach has been revisited and reconsidered in the fields related to dissipation, damage and plasticity (Baŵzant and Belytschko, 1984; Pijaudier-Cabot and Baŵzant, 1987). For recent advances on the non-local integral theory the readers are referred to several papers (Baŵzant and Jirásek, 2002; Benvenuti et al., 2002; Fuschi and Pisano, 2003). A different framework was proposed lately reconsidering the long-distance forces with the aid of internal state variables accounting for non-local effects (Ganghoffer and de Borst, 2000).

Recently, the problem of non-local continuum has been faced by fractional calculus approach (Lazopoulos, 2006). Fractional calculus has been applied, in the last decade, to several fields of applied mechanics such as the description of damage and fatigue in heterogeneous media (Carpinteri et al., 2001, 2004; Carpinteri and Cornetti, 2002), the representation of viscous forces (Narahari Achar et al., 2004) and in stochastic dynamics setting (Cottone and Di Paola, 2007). The most important feature of fractional derivatives is that they represent an intermediate machinery between differential and integral approach so that non-local mechanics handled with fractional calculus is an intermediate approach between gradient and integral theory of non-local interactions.

On the one hand, the analysis of unbounded non-local continuum with fractional calculus (Lazopoulos, 2006) may be considered an effective procedure to handle generalized continua. On the other hand, bounded media analyzed in the context of fractional calculus provide some inconsistencies that have already been encountered with other approaches involving non-local integral models. In more detail, it has been observed that the local case cannot be obtained as limit case of the Eringen model since some Dirac's delta functions appear at the borders of the bar. Moreover, equivalent non-local formulation for unbounded and bounded analyses is obtained in the latter case, clipping the attenuation function in the neighbourhood of the borders (see, e.g. paper by Fuschi and Pisano, 2003; Benvenuti et al., 2002).

In the authors opinion, these aforementioned inconsistencies are due to the fact that the non-local constitutive law is postulated without underlying mechanical model. In order to formulate properly on the basis of physical interpretation two fully equivalent models are proposed here: (i) The actions on an elementary volume are produced by contact forces arising from surface separation of adjacent volumes and by other central forces, decaying with the distance between non-adjacent volumes. (ii) A point-spring model with springs connecting adjacent points takes into account the local contribution, while the non-local contribution is taken into account with other distance-decaying linear springs connecting the point with all other points. At the limit, when the interdistance between the adjacent points goes to zero, the two aforementioned models give rise to the same differential equation. In the two physical models, one may select any attenuation function (Gaussian, Exponential, Mexican hat, etc.). Here, we select an attenuation function proportional to the interdistance $|x_j - x_h|^{-(1+\alpha)}$. With this choice, the differential equation in terms of displacement involves the Marchaud fractional derivatives. Since the latter operator is an intermediate one between classical derivatives and convolution integrals we may state that the formulation presented in this paper is a unified approach of weak and strong non-locality theory. It is shown that the stress–strain law involves Marchaud fractional integral in such a form that the problem of mechanical boundary conditions is definitively overcome.

The outline of this paper reports on the existent fractional integral model definition in Section 2. In Section 3, the inconsistencies of the non-local integral model with fractional attenuation function are pointed out. Section 4 has been devoted to the analysis of infinite domain with the proposed model of long-range forces with fractional decay. In Section 5, the analysis of finite extension domain has been reported with a proper definition of the mechanical boundary conditions. A point-spring model, totally equivalent to the mechanical model in Section 5 has been proposed in Section 6 and significant numerical applications with closure have been reported in Sections 7 and 8, respectively.

2. Fractional model of integral non-local elasticity

The strong non-local theory of long-range forces has been proposed in the early 1970 (Eringen and Edelen, 1972) to capture some unexpected effects observed in the experimental data and unpredicted by classical mechanics. To this aim, the stress–strain relation for an elastic bar of length L has been taken in the form

$$\sigma(x) = E\varepsilon(x) - \eta \int_0^L \varepsilon(\xi)\tilde{g}(x,\xi)\mathrm{d}\xi, \tag{1}$$

where E is the longitudinal modulus and $\varepsilon(x) = \mathrm{d}u(x)/\mathrm{d}x$ is the local strain, $\sigma(x)$ is the axial stress, η is an opportune constant of proportionality. The kernel $\tilde{g}(x,\xi)$ is the attenuation function, a monotonically decreasing function of the distance $|x-\xi|$, that accounts for the contribution of the strain at abscissa ξ on the stress at location x. Eq. (1) has been proposed in the generalized context of lattice theory of molecular interactions to represent long-distance forces between non-adjacent particles within the context of continuum mechanics (Kroner, 1967). Non-local mechanics modelled as in Eq. (1) requires the specification of the boundary conditions associated to the problem at hand. In the original paper by Kroner the boundary conditions have been specified as in classical "local" fashion (see, e.g. Kroner, 1967 Eq. (25)). In more recent papers, some other comments about the boundary conditions associated in the Eringen model have been addressed (see e.g. Polizzotto, 2001). Anyway, problem involved in the boundary conditions in the non-local integral model in Eq. (1) is still an open problem at the best of the author's knowledge.

The idea to include some fractional integral term in the governing equation of the elastic problem has been proposed by Lazopoulos (2006). He starts by assuming that the strain energy can be assumed as the sum of two contributions: (i) a local part of the kind $E\varepsilon^2(x)/2$ (ii) and a contribution of non-local nature defined as

$$-\eta\varepsilon(x)[(\mathscr{D}_{0+}^{-\beta}\varepsilon)(x) - (\mathscr{D}_{L-}^{-\beta}\varepsilon)(x)]/2, \quad 0<\beta<1, \tag{2}$$

involving the left and right Riemann–Liouville fractional derivatives $(\mathscr{D}_{0+}^{\alpha}\varepsilon)(x)$ and $(\mathscr{D}_{L-}^{\alpha}\varepsilon)(x)$ defined for a generic function $s(x)$ as

$$(\mathscr{D}_{0+}^{\beta}s)(x) \overset{def}{=} \frac{1}{\Gamma(1-\beta)}\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x \frac{s(\xi)\mathrm{d}\xi}{(x-\xi)^{\beta}}, \quad 0<\gamma<1 \tag{3a}$$

$$(\mathscr{D}_{L-}^{\beta}s)(x) \overset{def}{=} \frac{(-1)}{\Gamma(1-\beta)}\frac{\mathrm{d}}{\mathrm{d}x}\int_x^L \frac{s(\xi)\mathrm{d}t}{(\xi-x)^{\beta}}, \quad 0<\gamma<1 \tag{3b}$$

where $\Gamma(\beta)$ is the Euler-gamma function (see Appendix A). Variation of the total stored energy with respect to the state variable $\varepsilon(x)$ yields the stress–strain relation in a form involving fractional derivatives as

$$\sigma(x) = E\varepsilon(x) - \eta[(\mathscr{D}_{0+}^{-\beta}\varepsilon)(x) - (\mathscr{D}_{L-}^{-\beta}\varepsilon)(x)]. \tag{4}$$

Eq. (4) has been formulated (Lazopoulos, 2006) under the assumptions of vanishing boundary condition of the model as $u(0) = u(L) = 0$.

By assuming Eq. (4) as the starting point in the next section we derive the differential equation of equilibrium in order to show some inconsistencies for the bounded domain. To this aim, the fractional non-local model represented in Eq. (4) must be compared with the strong non-local theory of long-range interactions (Eq. (1)) that can be achieved by means of proper manipulations. In more detail accounting for the relations between Riemann–Liouville fractional derivatives and Riemann–Liouville fractional integrals $(\mathscr{D}_{0+}^{-\beta}\varepsilon)(x) = (I_{0+}^{\beta}\varepsilon)(x)$ and $(\mathscr{D}_{L-}^{-\beta}\varepsilon)(x) = -(I_{L-}^{\beta}\varepsilon)(x)$ Eq. (4) may be cast as

$$\sigma(x) = E\varepsilon(x) - \eta[(I_{0+}^{\beta}\varepsilon)(x) + (I_{L-}^{\beta}\varepsilon)(x)], \tag{5}$$

where $(I_{0+}^{\beta}s)(x)$ and $(I_{L-}^{\beta}s)(x)$ are the left and right, respectively, Riemann–Liouville fractional integral is defined as

$$(I_{0+}^{\beta}s)(x) \overset{\mathrm{def}}{=} \frac{1}{\Gamma(\beta)}\int_0^x \frac{s(\xi)}{(x-\xi)^{1-\beta}}\mathrm{d}\xi, \tag{6a}$$

$$(I_{L-}^{\beta}s)(x) \overset{\mathrm{def}}{=} \frac{1}{\Gamma(\beta)}\int_x^L \frac{s(\xi)}{(\xi-x)^{1-\beta}}\mathrm{d}\xi. \tag{6b}$$

The stress–strain relation may be further expanded as

$$\sigma(x) = E\varepsilon(x) - \eta[(I_{0+}^{\beta}\varepsilon)(x) + (I_{L-}^{\beta}\varepsilon)(x)] = E\varepsilon(x) - \eta\int_0^L \varepsilon(\xi)\tilde{g}(x,\xi)\mathrm{d}\xi \tag{7}$$

that is equivalent to Eq. (1) selecting the attenuation function in the form

$$\tilde{g}(x,\xi) = 1/(\Gamma(\beta)\,|\,x-\xi|^{1-\beta}), \quad 0<\beta<1. \tag{8}$$

The particular choice of such an attenuation function is very attractive because the parameter β yields a large variety of distance-decaying interactions and it plays the role of a scale parameter in the strong non-local elasticity theory.

At this stage we may conclude that the strong non-local theory of elasticity (Kroner, 1967) may be framed, under proper assumption about the boundary conditions, in the context of fractional calculus as from the original idea of Lazopoulos (2006). The latter consideration is worthy to be remarked as the strong theory of non-local elasticity showed some inconsistencies in the presence of bounded domains as it will be reported in the following section.

3. Inconsistencies of the eringen model with fractional attenuation function

In this section, some additional comments to the non-local integral model with fractional attenuation function will be introduced to remark that, in the presence of impending boundaries, the non-local integral model yields an inconsistent mechanical formulation. The latter aspect is hereinafter detected by formulating the governing equation of the problem in terms of the displacement function. The equation is obtained, for unbounded domain $x \in \mathbb{R}$, replacing the stress–strain constitutive equation (Eq. (7)) in the equilibrium equation, $\mathrm{d}\sigma(x)/\mathrm{d}x = -f(x)$ with $f(x)$ the axial body force field, yielding the fractional differential equation

$$\frac{\mathrm{d}^2u(x)}{\mathrm{d}x^2} - \frac{\eta}{E}((\mathscr{D}_{+}^{2-\beta}u)(x) + (\mathscr{D}_{-}^{2-\beta}u)(x)) = -\frac{f(x)}{E}, \quad x \in \mathbb{R}, \tag{9}$$

that is a fractional differential equation involving the left and right Riemann–Liouville fractional derivatives. Eq. (9) may be converted in a more appropriate form, by the use of the Marchaud fractional derivatives $(\mathbf{D}_{+}^{\beta}s)(x)$ and $(\mathbf{D}_{-}^{\beta}s)(x)$ related to the Riemann–Liouville operators as (Appendix A)

$$(\mathscr{D}_{+}^{\beta}s)(x) = \frac{1}{\Gamma(1-\beta)}\int_{-\infty}^{x}\frac{s'(\xi)}{(\xi - x)^{\beta}}\mathrm{d}\xi = (\mathbf{D}_{+}^{\beta}s)(x), \tag{10a}$$

$$(\mathscr{D}_{-}^{\beta}s)(x) = \frac{1}{\Gamma(1-\beta)}\int_{x}^{\infty}\frac{s'(\xi)}{(x-\xi)^{\beta}}\mathrm{d}\xi = (\mathbf{D}_{-}^{\beta}s)(x), \tag{10b}$$

and that are defined for unbounded domain as

$$(\mathbf{D}_{+}^{\beta}s)(x) \stackrel{\mathrm{def}}{=} \frac{\beta}{\Gamma(1-\beta)}\int_{-\infty}^{x}\frac{s(x)-s(\xi)}{(x-\xi)^{(1+\beta)}}\mathrm{d}\xi, \tag{11a}$$

$$(\mathbf{D}_{-}^{\beta}s)(x) \stackrel{\mathrm{def}}{=} \frac{\beta}{\Gamma(1-\beta)}\int_{x}^{+\infty}\frac{s(x)-s(\xi)}{(\xi-x)^{(1+\beta)}}\mathrm{d}\xi. \tag{11b}$$

The governing equation (9) may be cast in terms of the Marchaud fractional derivatives as

$$\frac{\mathrm{d}^2u(x)}{\mathrm{d}x^2} - \frac{\eta}{E}((\mathbf{D}_{+}^{2-\beta}u)(x) + (\mathbf{D}_{-}^{2-\beta}u)(x)) = -\frac{f(x)}{E}. \tag{12}$$

Despite the formal identity of Eqs. (9) and (12), the problem formulated in terms of the Marchaud fractional derivatives yields a consistent mechanical representation of the non-local contribution as it will be reported in the following sections. It may be also observed that in the absence of impending boundaries the strong non-local theory, assuming attenuation function reported in Eq. (8), perfectly corresponds to the fractional model of the non-local problem.

A different scenario can be observed solving the elastic problem for a confined bar of length L and replacing the Riemann–Liouville fractional derivatives in the unbounded domain, in (10b) and (10a) with their counterpart defined on finite support. It may be shown, after some straightforward manipulation (see Appendix A) that the following relations hold:

$$(\mathscr{D}_{0+}^{\beta}s)(x) = \frac{\mathrm{d}}{\mathrm{d}x}(I_{0+}^{1-\beta}s(x)) = \frac{s(0)}{\Gamma(1-\beta)x^{\beta}} + \frac{1}{\Gamma(1-\beta)}\int_{0}^{x}\frac{s'(\xi)}{(x-\xi)^{1-\beta}}\mathrm{d}\xi = (\mathbf{D}_{0+}^{\beta}s)(x), \tag{13a}$$

$$(\mathscr{D}_{L-}^{\beta}s)(x) = \frac{\mathrm{d}}{\mathrm{d}x}(I_{L-}^{1-\beta}s(x)) = \frac{s(L)}{\Gamma(1-\beta)(L-x)^{\beta}} + \frac{1}{\Gamma(1-\beta)}\int_{x}^{L}\frac{s'(\xi)}{(\xi-x)^{1-\beta}}\mathrm{d}\xi = (\mathbf{D}_{L-}^{\beta}s)(x) \tag{13b}$$

with the Marchaud fractional derivatives on finite domain $(\mathbf{D}_{0+}^{\beta}s)(x)$ and $(\mathbf{D}_{L-}^{\beta}s)(x)$ defined as

$$(\mathbf{D}_{0+}^{\beta}s)(x) = (\widehat{\mathbf{D}}_{0+}^{\beta}s)(x) + \frac{s(x)}{\Gamma(1-\beta)x^{\beta}}, \tag{14a}$$

$$(\mathbf{D}_{L-}^{\beta}s)(x) = (\widehat{\mathbf{D}}_{L-}^{\beta}s)(x) + \frac{s(x)}{\Gamma(1-\beta)(L-x)^{\beta}}, \tag{14b}$$

where the integral operators $(\widehat{\mathbf{D}}_{0+}^{\beta}s)(x)$ and $(\widehat{\mathbf{D}}_{L-}^{\beta}s)(x)$ are defined as

$$(\widehat{\mathbf{D}}_{0+}^{\beta}s)(x) \stackrel{\mathrm{def}}{=} \frac{\beta}{\Gamma(1-\beta)}\int_{0}^{x}\frac{s(x)-s(\xi)}{(x-\xi)^{(1+\beta)}}\mathrm{d}\xi, \tag{15a}$$

$$(\widehat{\mathbf{D}}_{L-}^{\beta}s)(x) \stackrel{\mathrm{def}}{=} \frac{\beta}{\Gamma(1-\beta)}\int_{x}^{L}\frac{s(x)-s(\xi)}{(\xi-x)^{(1+\beta)}}\mathrm{d}\xi, \tag{15b}$$

yielding the governing equation of the elastic problem in terms of the Marchaud fractional derivatives in the form

$$\frac{\mathrm{d}^2u(x)}{\mathrm{d}x^2} - \frac{\eta}{E}((\mathbf{D}_{0+}^{2-\beta}u)(x) + (\mathbf{D}_{L-}^{2-\beta}u)(x)) = -\frac{f(x)}{E} \tag{16}$$

Observation of Eqs. (16) and (12) shows that they are formally equivalent but not substantially coincident. The main difference is the presence of the algebraic terms in Eqs. (14a) and (14b) that diverge at the borders unless homogeneous boundary conditions in terms of displacements and fixed support strains are imposed $u(0) = u(L) = 0$ and This is automatically accounted in Lazopoulos derivation since derivation has been performed assuming that $u(0) = u(L) = 0$ but in the context of the strong non-local theory with attenuation function in Eq. (8) the elastic problem ruled by Eq. (16) yields mathematical inconsistencies for a bar with free ends.

Anyway, the divergent behaviour in passing from the Marchaud or the Riemann–Liouville derivative on infinite support to the case of bounded bar does not have mechanical explanation at the present time leading to conclude that the requirement of a mathematically and mechanically consistent model is imperative dealing with enriched continuum with cohesive interactions. In the opinion of the authors, the main drawbacks in the strong non-locality model is due to the fact that the governing equation in Eq. (16) is postulated without underlying mechanical model. In this perspective, long-range interactions will be described on mechanical grounds already used in lattice mechanics (Born and Huang, 1954; Lax, 1963). This will be done in the next sections.

4. Elastic bar with long-range interactions: unbounded domain

In this section, the first of the two equivalent physical models will be addressed. Let us consider an elastic bar with infinite length, as depicted in Fig. 1a, loaded with external self-equilibrated volume forces denoted $f(x)$ and let us discretize the bar in volume elements $V_j = A\Delta x (j = -\infty, \ldots, \infty)$ with A the cross-section and Δx the length of the element. Volume element V_j is located at abscissa $x_j = (j-1)\Delta x$ and it is in equilibrium under external loads, contact forces provided by adjacent volume elements, V_{j-1} and V_{j+1}, denoted $N_j$andN_{j+1}, respectively, and the resultant of long-range actions Q_j applied on V_j by the surrounding non-adjacent elements of the bar (Fig. 1b). Under these circumstances the equilibrium equation of volume V_j is provided as

$$\Delta N_j + Q_j = \Delta N_j + \sum_{m=j+1}^{\infty} Q^{(m,j)} - \sum_{m=-\infty}^{j-1} Q^{(m,j)} = -f_j A \Delta x, \tag{17}$$

where $f_j = f(x_j)$, $\Delta N_j = N_{j+1} - N_j$ is the difference between the contact forces N_j and N_{j+1} provided by volume elements V_{j+1}, V_{j-1} and $Q^{(h,j)}$ are the long-range forces that surrounding volume elements V_h $(h = -m, \ldots, -2, -1, 0, 1, 2, \ldots, m\ (m \to \infty), h \neq j)$ apply on element V_j as in Fig. 1c, where only long-range forces have been highlighted. The long-range forces $Q^{(h,j)}$ $(h = -\infty, \ldots, 0, \ldots, h \neq j, \ldots, \infty)$ represent molecular interactions between non-adjacent volume elements, and hence they de-

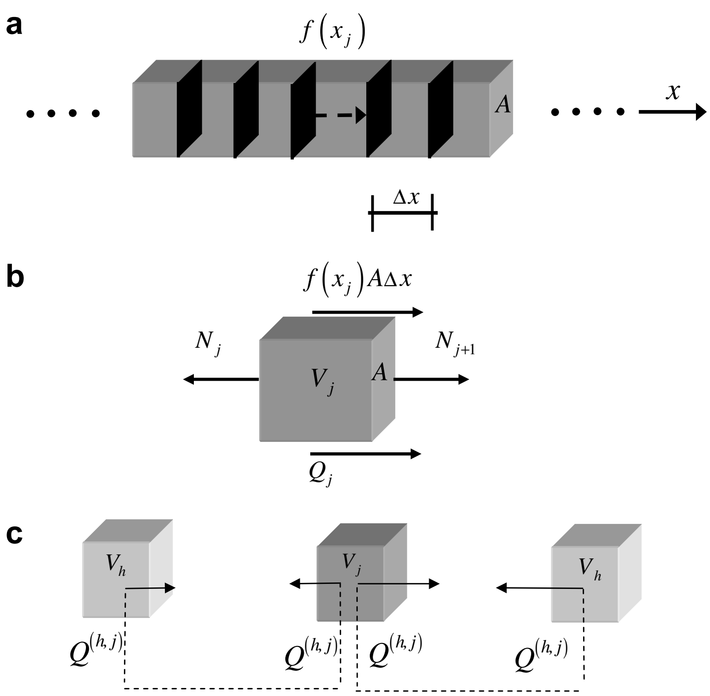

Fig. 1. (a) Discretized elastic bar loaded by an external volume force field $f(x)$. (b) Equilibrium of the volume element V_j. (c) Long-range forces in the equilibrium of volume element V_j.

pend on both volume sizes V_j and V_h of interacting volumes as in applied mechanics problems with interacting axial molecular forces (see, e.g. Krumhanls, 1963, Eq. (18); Kunin, 1963, Eq. 2.4–5). In the following long-distance interactions $Q^{(h,j)}$ will be modelled as forces depending on the products of volume elements V_j and V_h as well as the relative displacement $u(x_h) - u(x_j)$ and on a decaying function $g(|x_h - x_j|)$, that is,

$$Q^{(h,j)} = \text{sgn}(x_h - x_j)(u(x_h) - u(x_j))g(|\, x_j - x_h\,|)V_jV_h, \tag{18}$$

where sgn(·) is the well-known signum function defined as

$$\text{sgn}(x) = \begin{cases} -1, & x < 0, \\ 1, & x \geqslant 0. \end{cases} \tag{19}$$

The decaying function $g(|x_j - x_h|)$ selected is a real-valued, monotonically decreasing function expressed as

$$g(|\, x_j - x_h\,|) = \frac{Ec_\alpha \alpha}{A\Gamma(1-\alpha)\,|\, x_j - x_h|^{1+\alpha}} \quad (0 \leqslant \alpha \leqslant 1). \tag{20}$$

The particular choice of molecular interactions described in Eq. (20) has been selected to capture long-range interactions used in the field of crystal lattices in the presence of central forces. Functional form reported in Eq. (20) for the attenuation function is capable to describe either forces inversely proportional to the square of the distances of unstrained lattice and to the distance lattices (see, e.g. Born and Huang, 1954, Section 11). It will be observed that the classical continuum mechanics, without cohesive forces, may be recovered as $\alpha \to 0$. Direct substitution of Eq. (20) in the equilibrium equation (Eq. (17)), yields the equilibrium equation of volume V_j that, under the assumption $V_j = V_h = V_r = A\Delta x$, may be written as

$$\Delta N_j - \frac{Ec_\alpha \alpha A \Delta x}{\Gamma(1-\alpha)}\left[\sum_{h=-\infty}^{j-1} \frac{u(x_j) - u(x_h)}{(x_j - x_h)^{1+\alpha}}\Delta x + \sum_{r=j+1}^{\infty} \frac{u(x_j) - u(x_r)}{(x_r - x_j)^{1+\alpha}}\Delta x\right] = -f_j A\Delta x. \tag{21}$$

Dividing Eq. (21) by Δx and taking limit for $\Delta x \to 0$ the differential equilibrium equation is obtained as

$$\frac{dN(x)}{dx} - Ec_\alpha A((\mathbf{D}_+^\alpha u)(x) + (\mathbf{D}_-^\alpha u)(x)) = -f(x)A. \tag{22}$$

Eq. (22) may be recast in terms of the local conventional stress $\sigma_l(x) = N(x)/A$ as

$$\frac{d\sigma_l(x)}{dx} - Ec_\alpha((\mathbf{D}_+^\alpha u)(x) + \mathbf{D}_-^\alpha(u)(x)) = -f(x). \tag{23}$$

Eq. (23) is the equilibrium equations of the volume $dV = A\,dx$ located at abscissa x in which long-range interactions between surrounding non-adjacent volumes have been taken into account.

Assuming linear elastic material, the stress–strain relation may be used:

$$\sigma_l(x) = E\varepsilon(x) = E\,du/dx. \tag{24}$$

By using Eq. (24) in Eq. (23), and performing manipulations the equilibrium equation in terms of the displacement field for the infinitesimal volume is written as

$$\frac{d^2u(x)}{dx^2} - c_\alpha((\mathbf{D}_+^\alpha u)(x) + (\mathbf{D}_-^\alpha u)(x)) = -\frac{f(x)}{E}. \tag{25}$$

Direct comparison of Eq. (25) with Eq. (12) shows that the two equations coalesce in the case of unbounded domain since Riemann–Liouville and Marchaud fractional derivatives coincide. That is, by assuming the long-range interactions as in Eq. (18) with the attenuation function reported in Eq. (20), the proposed model coincides formally under condition $\beta = \alpha + 2$ with the Lazopoulos and Eringen model with attenuation function reported in Eq. (3). This is, a remarkable consideration since in the Eringen model a convolution integral of the strain $\varepsilon(x) = du/dx$ is involved, whereas the proposed model of long-range interactions depends on the relative displacements of non-adjacent volumes. In the unbounded domain, these two representation of non-local effects coalesce since fractional operators are involved and 10b and 10a holds. This is not the case of the bounded domain as it will be shown in the following. The mechanical representation of Marchaud fractional derivatives of displacement functions in the fractional integral model of non-local interactions is now highlighted: It represents the resultant of long-range interactions in the equilibrium of volume $dV = A\,dx$.

By summing up, if we select the attenuation function as in Eq. (20) and we assume that long-distance interactions $Q^{(h,j)}$ are reported in Eq. (18), then the differential equation in terms of displacements is an ordinary fractional differential equation formally coalescing with Eq. (12) obtained by manipulation from Lazopoulos model similar to that described in Section 3. On this perspective, the reader could guess that the machinery presented in this section could be avoided by direct introduction of an opportune attenuation function with proper exponent ($\beta = \alpha + 2$) in the Eringen model. At this stage, we may only emphasize that the long-range interactive forces exploited in Eq. (18) now have a clear mechanical interpretation and it allows for two main considerations: (i) The problem of boundary condition in a finite domain will be introduced in a natural way as it will be shown in the following section. (ii) The continuous model proposed here has a correspondence with a mechanical discrete model as will be reported later in the course of this paper. These two main features remain hidden by the direct use of the non-local integral model. It will be stressed that the assumption reported in Lazopulous may now be considered on mechanical basis that the long-range interactive forces have to be expressed as in Eq. (18). This is the crucial point that allow us to formulate the problem in a finite domain.

5. Analysis of finite domain with long-range interactions

In this section, the problem of finite domain with long-range interactions will be treated with the aid of a mechanical interpretation given in the previous section. The problem is introduced considering a bar of finite length L loaded by external axial force field $f(x)$. The same arguments leading to Eq. (17) for the equilibrium equation of volume $V_j = A\Delta x$ with $\Delta x = L/m$ (m the total number of volumes), yield, dividing by Δx, the equation

$$\frac{\Delta N_j}{\Delta x} - \frac{Ec_\alpha \alpha A}{\Gamma(1-\alpha)}\left[\sum_{h=1}^{j-1}\frac{u(x_j)-u(x_h)}{(x_j-x_h)^{1+\alpha}}\Delta x + \sum_{h=j+1}^{m+1}\frac{u(x_j)-u(x_h)}{(x_j-x_h)^{1+\alpha}}\Delta x\right] = -f_j A \tag{26}$$

that represents the analogous equilibrium equation reported in Section 3 but with finite number of terms due to the finite extension of the bar. Discrete equilibrium equation may be converted into integro-differential equation similar to Eq. (23) by the use of Euler–McLaurin interpolation formula assuming that displacement field $u(x) \in C_\infty$ for $x \to 0, L$ with C_∞ the class of infinitely derivable functions. In this case, letting $\Delta x \to 0$ and neglecting higher-order terms in the Euler–McLaurin interpolation formula the integro-differential equilibrium equation may be written, with the same considerations leading to Eq. (23) as

$$\frac{d^2u(x)}{dx^2} - c_\alpha((\widehat{\mathbf{D}}^\alpha_{0^+}u)(x) + (\widehat{\mathbf{D}}^\alpha_{L^-}u)(x)) = -\frac{f(x)}{E}. \tag{27}$$

Such a consideration about the Euler–McLaurin formula is unnecessary in the presence of unbounded domains since no corrective terms are involved in the formula. Direct comparison of terms retained in Eq. (27) with Eq. (16) reveals a substantial difference between the differential equation obtained by direct consideration of the attenuation function in the Eringen model (Eq. (16)) and that derived on the mechanical model of long-range forces proposed here. In Eq. (27), only the integral part of the Marchaud fractional derivative appears instead of the Marchaud fractional derivative on a finite support. The two equations coincide formally only for a bar of infinite length. It is to be stressed that in Eq. (27) the divergent terms at the borders of the bar domain, appearing in the integral non-local model and in the fractional model, are not present. This is a very remarkable result, since it leads us automatically, to a governing integro-differential equation without divergent boundary terms. The latter aspects are a fundamental step in the present derivation since we may now formulate, consistently, the boundary conditions for the non-local continuum.

Boundary conditions associated to Eq. (27) involving kinematic conditions may be imposed for the axial displacements at the restrained locations. If some static boundary condition, say an external force F is applied at the edge then we must define the overall resultant stress σ (x) at cross-section x. To this aim, we observe that the following relation holds:

$$(\widehat{\mathbf{D}}^\alpha_{0^+}u)(x) + (\widehat{\mathbf{D}}^\alpha_{L^-}u)(x) = \frac{\alpha}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{\xi_1:x}^{L}\int_{\xi_2:0}^{x}\frac{u(\xi_1)-u(\xi_2)}{|\,\xi_2-\xi_1|^{1+\alpha}}d\xi_1\,d\xi_2. \tag{28}$$

And Eq. (27) may be recast in terms of the overall stress $\sigma(x)$ as

$$\frac{d}{dx}E\left(\frac{du}{dx} - \frac{c_\alpha\alpha}{\Gamma(1-\alpha)}\int_{\xi_1:x}^{L}\int_{\xi_2:0}^{x}\frac{u(\xi_1)-u(\xi_2)}{|\,\xi_2-\xi_1|^{1+\alpha}}d\xi_1\,d\xi_2\right) = \frac{d\sigma(x)}{dx} = -f(x) \tag{29}$$

with the stress $\sigma(x)$ defined by

$$\sigma(x) = E\left(\frac{du}{dx} - \frac{c_\alpha\alpha}{\Gamma(1-\alpha)}\int_{\xi:x}^{L}\int_{\xi_2:0}^{x}\frac{u(\xi_1)-u(\xi_2)}{|\,\xi_2-\xi_1|^{1+\alpha}}d\xi_1\,d\xi_2\right). \tag{30}$$

From Eq. (30) we may observe that the overall stress $\sigma(x)$ is the sum of the local stress $\sigma_l(x) = E\,du/dx$ and a non-local contribution $\sigma_{nl}(x)$ represented by the second term on the right-hand side of Eq. (30). To derive the mechanical boundary conditions this fundamental equation does not show any mathematical inconsistence, and static boundary conditions may now be applied requiring, as in classical local mechanics, that the applied force at the edges F is equal to σA. Moreover, at the boundary of the bar ($x = 0, L$), the contribution to the overall stress due to the non-local term in Eq. (30) vanishes, that is, the mechanical boundary conditions are simply $\sigma_l(0)A = -F_0$ and $\sigma_l(L)A = F_L$. Eq. (30) may be also drawn from mechanical basis, but this cannot be assessed at the present stage, and it will be addressed after the introduction of the mechanical model of the non-local interactions reported in Section 6. Summing up, if we postulate that the long-range forces descend by the Eringen model we may use the usual rule of the fractional calculus in the presence of homogeneous boundary condition (Eq. (16)) to formulate a governing equation similar to Eq. (12). Since engineering problems may involve also non-homogeneous boundary conditions, we propose to assume long-range forces on physical grounds. In this context, the analysis of finite bar does not involve divergent terms anymore allowing for non-homogeneous boundary conditions but integral operators

$(\widehat{\mathbf{D}}^{\alpha}_{0^+}\cdot)(x)and(\widehat{\mathbf{D}}^{\alpha}_{L^-}\cdot)(x)$ are now fractional operators of different natures (and the usual rules of fractional calculus do not hold). The mechanical boundary conditions for a finite-extension bar may be easily imposed by means of Eq. (30).

Now we suppose that the discrete form of Eq. (27) (that is expressed in Eq. (26)) is not known and we want to use the tools of discretization of fractional calculus. This may be provided resorting to fractional finite differences (Shkanukov, 1996). In this context, introducing a proper discretization of the bar in m intervals of amplitude $\Delta x = L/m$ and representing the fractional differential operator $(\widehat{D}^{\alpha}\cdot)(x) = (\widehat{\mathbf{D}}^{\alpha}_{0+}\cdot)(x) + (\widehat{\mathbf{D}}^{\alpha}_{L^-}\cdot)(x)$ at the material point $x_j = (j-1)\Delta x \; j = 1,2,\dots,m+1$ by the difference operator $\Delta^{\alpha}_{x_i}$ given by

$$\widehat{D}^{\alpha}[s](x_j) = \Delta^{\alpha}_{x_j}s(x) + \mathrm{O}(\Delta x), \tag{31}$$

where $\mathrm{O}(\Delta x)$ means a quantity of order Δx and the fractional difference operator $\Delta^{\alpha}_{x_i}$ is represented as

$$\Delta^{\alpha}_{x_j}s(x) = \frac{\alpha^{-1}}{\Gamma(1-\alpha)}\left\{\sum_{h=1}^{j-1}[(x_{j-h+1})^{-\alpha} - (x_{j-h})^{-\alpha}]s(x_h) + \sum_{r=j+1}^{m}[(x_{j-r})^{-\alpha} - (x_{j+1-r})^{-\alpha}]s(x_r)\right\}. \tag{32}$$

Discretizing Eq. (27) by operator in Eq. (32) and neglecting terms of order Δx an algebraic, fractional difference, system in the unknown displacement field $u(x_j)$ is obtained as

$$\begin{aligned}&\tfrac{EA}{\Delta x}\Delta^2 u(x_j) - \tfrac{Ec_{\alpha}A\Delta x}{\Gamma(1-\alpha)}\left[\sum_{h=1}^{j-1} u(x_j)((x_{j-h})^{-\alpha} - (x_{j+1-h})^{-\alpha}) - u(x_h)((x_{j-h})^{-\alpha} - (x_{j+1-h})^{-\alpha})\right]\\ &+ - \tfrac{Ec_{\alpha}A\Delta x}{\Gamma(1-\alpha)}\left[\sum_{r=j}^{m} u(x_j)((x_{r-j})^{-\alpha} - (x_{r+1-j})^{-\alpha}) - u(x_{r+1})((x_{r-j})^{-\alpha} - (x_{r+1-j})^{-\alpha})\right] = -F(x_j)\Delta x,\end{aligned} \tag{33}$$

holding for $j = 1,2,\dots,m+1$ and with the finite differences $\Delta^2 u(x_j) = u(x_{j+1}) - 2u(x_j) + u(x_{j-1})$ with $F(x_j) = f(x_j)A$. In passing, we remark that the approximation scheme involved by fractional finite differences may be applied also at the boundary-value problem in Eq. (16), since it requires homogeneous boundary condition for the unknown function. This behaviour is due to the presence of the divergent boundary terms at the boundaries that is overcome for homogeneous boundary conditions, as it has been proposed in scientific literature with Riemann–Liouville fractional derivative with bounded intervals (see, e.g. Kilbas et al., 2006, pp. 272).

System of m algebraic equations reported in Eq. (33) in the unknown displacements $u(x_j)$ of the grid points used to discretize fractional differential equation may be reported in compact form as

$$\mathbf{Ku} = \mathbf{f}, \tag{34}$$

where nodal displacement and force vectors $\mathbf{u}$ and $\mathbf{f}$, respectively, are given as

$$\mathbf{u}^{\mathrm{T}} = [\, u_1 \quad u_2 \quad \dots \quad u_m \,], \tag{35a}$$

$$\mathbf{f}^{\mathrm{T}} = [\, f_1 \quad \dots \quad \dots \quad f_m \,]A\Delta x, \tag{35b}$$

and the non-local coefficient matrix $\mathbf{K} = \mathbf{K}^{\mathrm{l}} + \mathbf{K}^{\mathrm{nl}}$ has been introduced in which contact contributions due to adjacent elements have been considered in the tri-diagonal matrix $\mathbf{K}^{\mathrm{l}}$, collecting elements $K^{\mathrm{l}} = EA/\Delta x$ as

$$\mathbf{K}^{\mathrm{l}} = \begin{bmatrix} K^{\mathrm{l}} & -K^{\mathrm{l}} & \dots & \dots & 0\\ -K^{\mathrm{l}} & 2K^{\mathrm{l}} & -K^{\mathrm{l}} & \dots & 0\\ \dots & \dots & \dots & \dots & \dots\\ \dots & \dots & -K^{\mathrm{l}} & 2K^{\mathrm{l}} & -K^{\mathrm{l}}\\ 0 & \dots & \dots & -K^{\mathrm{l}} & K^{\mathrm{l}} \end{bmatrix} \tag{36}$$

and non-local interactions have been considered in the symmetric, fully populated, matrix

$$\begin{aligned}&\mathbf{K}^{\mathrm{nl}} = \tfrac{c_{\alpha}A\Delta x}{\Gamma(1-\alpha)}\\ &\begin{bmatrix} K^{\mathrm{nl}}_{11} & -\Delta x^{-\alpha} & [\Delta x^{-\alpha} - (2\Delta x)^{-\alpha}] & \dots & [((m-1)\Delta x)^{-\alpha} - (m\Delta x)^{-\alpha}]\\ -\Delta x^{-\alpha} & K^{\mathrm{nl}}_{22} & -\Delta x^{-\alpha} & \dots & [((m-2)\Delta x)^{-\alpha} - ((m-1)\Delta x)^{-\alpha}]\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ [((m-2)\Delta x)^{-\alpha} - ((m-1)\Delta x)^{-\alpha}] & \dots & \dots & \dots & \dots\\ [((m-1)\Delta x)^{-\alpha} - (m\Delta x)^{-\alpha}] & \dots & \dots & -\Delta x^{-\alpha} & K^{\mathrm{nl}}_{mm} \end{bmatrix},\end{aligned} \tag{37}$$

where elements jh of the matrix $\mathbf{K}^{\mathrm{nl}}$, $K^{\mathrm{nl}}_{jh} = \int_{x_j}^{x_h} g(x_j,\xi)\mathrm{d}\xi (j \neq h)$ and $K^{\mathrm{nl}}_{jj} = -\sum_{h=1,\, h\neq j}^{m} K^{\mathrm{nl}}_{jh}$, with function $g(x_j,\xi)$ defined in Eq. (20). Displacements at the grid points used to discretize the model are provided by inversion of the stiffness matrix $\mathbf{K}$

accounting for the appropriate kinematic and static boundary conditions at the borders of the solid. Formal equivalence of Eq. (34) with the solving equations of elastic problems suggests that an elastic mechanical model may be used to represent mechanics of long-range enriched continuum as will be reported in the following section.

6. The mechanical equivalent model of non-local bar

At this point, some new insights on the mechanics of the non-local problem may be introduced with the aid of the discrete spring-point model reported in Fig. 2 with few nodes for clarity. A similar idea to capture non-local effects has also been proposed in the context of finite element method (Liu et al., 2004) connecting two adjacent nodes by a spring to account for local effects and nodes $i-2, i, i+2$ with other springs to obtain non-local effects.

In the following, a more refined non-local model that reflects the mechanical framework proposed in this paper will be reported representing local forces between adjacent particles have been considered by springs with elastic stiffness $K^{\rm l} = EA/\Delta x$. Long-distance interactions have been introduced by mechanical connections of all non-adjacent particles with linear springs with distance-decaying stiffness as $K^{\rm nl}_{jh} = g(|\, x_h - x_j\,|)$. Under these circumstances, the model in Fig. 2 may be studied by the classical displacement approach, observing that the equilibrium equations of the elastic model including long-range effects may be formulated as

$$K^{\rm l}u_1 - K^{\rm l}u_2 - A^2\Delta x\sum_{h=2}^{m} g(|\, x_1 - x_h\,|)(u_h - u_1) = F_1, \tag{38a}$$

$$-K^{\rm l}u_{j-1} + 2K^{\rm l}u_j - K^{\rm l}u_{j+1} - A^2\Delta x\sum_{\substack{h=1\\ h\neq j}}^{m+1} g(|\, x_j - x_h\,|)(u_h - u_j) = F_j, \quad j = 2,\ldots,m-1, \tag{38b}$$

$$K^{\rm l}u_m - K^{\rm l}u_{m-1} - A^2\Delta x\sum_{h=2}^{m} g(|\, x_m - x_h\,|)(u_m - u_h) = F_m. \tag{38c}$$

First terms in Eqs. (38a)–(38c) correspond to contact forces and the sums represent non-local forces applied at material particle located at abscissa x_j by the surrounding particles located at abscissas x_h. The right-hand side of Eqs. (38a) and (38c) is related to the body forces applied at material particles. Nodal forces reported on the right-hand side of Eqs. (38a)–(38c) are expressed as $F_j = f(x_j)A$.

Equilibrium equations reported in Eqs. (38a) and (38c) may be rewritten in matrix form similar to Eq. (34) introducing the non-local stiffness matrix $\mathbf{K} = \mathbf{K}^{\rm l} + \mathbf{K}^{\rm nl}$ in which we denoted $\mathbf{K}^{\rm l}$ the local stiffness matrix (Eq. (36)). In the discrete point-spring model, it may be easily shown that the non-local interactions are described in the symmetric, fully populated, non-local stiffness matrix as

$$\mathbf{K}^{\rm nl} = \begin{bmatrix} K^{\rm nl}_{11} & -A^2\Delta x g(|\, x_2 - x_1\,|) & -A^2\Delta x g(|\, x_3 - x_1\,|) & \ldots & -A^2\Delta x g(|\, x_m - x_1\,|) \\ -A^2\Delta x g(|\, x_2 - x_1\,|) & K^{\rm nl}_{22} & -A^2\Delta x g(|\, x_3 - x_2\,|) & \ldots & -A^2\Delta x g(|\, x_m - x_2\,|) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -A^2\Delta x g(|\, x_m - x_1\,|) & \ldots & \ldots & -A^2\Delta x g(|\, x_2 - x_1\,|) & K^{\rm nl}_{mm} \end{bmatrix}, \tag{39}$$

where $K^{\rm nl}_{jj} = \sum_{h=1, h\neq j}^{m} k_{jk}$ and $k_{jk} = A^2\Delta x g(|\, x_j - x_h\,|)$. The non-local stiffness matrix reported in Eq. (39) has been obtained by induction evaluating the equilibrium equations for an increasing number of interconnected points. Moreover, close observation of Eq. (39) contrasted with coefficient matrix in Eq. (37) shows that by selecting the spatially decaying function

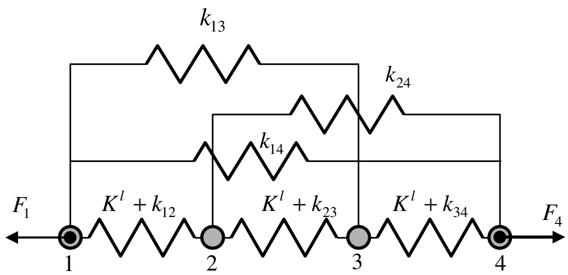

Fig. 2. Discrete point-spring non-local model with spatially decaying stiffness.

$g(|x_j - x_h|)$ with expression in Eq. (20) as soon as $\Delta x \to 0$ matrix in Eq. (39) reverts to the non-local coefficient matrix defined in Eq. (37) and obtained with fractional finite differences. This is a very remarkable result enabling us to validate the proposed non-local model of long-range interactions and gives a new perspective in the analysis of enriched continuum.

At this stage, we may use the point-spring model to gain some additional insights about the boundary conditions associated to the mechanics of the non-local continuum. First of all, we observe that if some node of the model is restrained, then it involves the cancellation of the corresponding row and column in the stiffness matrix $\mathbf{K} = \mathbf{K}^{l} + \mathbf{K}^{nl}$. If some static boundary condition is involved in the considered problem, then this is automatically accounted introducing the known load at the corresponding joint in the point-spring model. Now, we want to derive the constitutive law of the non-local model by the analysis of the point-spring model. The overall stress $\sigma(x)$ in the discrete model is the resultant stress provided by the springs by a fictitious cut of the model in two parts, for instance at location $0 < x < \Delta x$. Thus, the overall stress is furnished as

$$\sigma(x) = \frac{1}{A}\left(\sum_{j=2}^{m} Q^{(1,j)} + k_l(u_2 - u_1)\right), \quad 0 < x < \Delta x, \tag{40}$$

where the long-distance forces $Q^{(1,j)}$ are represented with the proposed point-spring model as $Q^{(1,j)} = g(|x_j - x_1|)(u_j - u_1)V_jV_1$. If the overall stress $\sigma(x)$ is evaluated in the interval $\Delta x < x < 2\Delta x$ or $2\Delta x < x < 3\Delta x$, then it reads respectively,

$$\sigma(x) = \frac{1}{A}\left(\sum_{j=3}^{m} Q^{(1,j)} + \sum_{j=3}^{m} Q^{(2,j)} + k_l(u_3 - u_2)\right), \quad \Delta x < x < 2\Delta x, \tag{41a}$$

$$\sigma(x) = \frac{1}{A}\left(\sum_{j=4}^{m} Q^{(1,j)} + \sum_{j=4}^{m} Q^{(2,j)} + \sum_{j=4}^{m} Q^{(3,j)} + k_l(u_4 - u_3)\right), \quad \Delta x < x < 3\Delta x \tag{41b}$$

yielding, by mathematical induction, the overall stress for $r\Delta x < x < (r+1)\Delta x$ as

$$\sigma(x) = \frac{1}{A}\left(\sum_{j=r+1}^{m} Q^{(1,j)} + \sum_{j=r+1}^{m} Q^{(2,j)} + \cdots + \sum_{j=r+1}^{m} Q^{(r-1)j} + k_l(u_{r+1} - u_r)\right) = \frac{1}{A}\left(\sum_{j=r+1}^{m}\sum_{h=1}^{r} Q^{(h,j)} + k_l(u_{r+1} - u_r)\right). \tag{42}$$

The continuous model with long-range forces is derived from Eq. (42), substituting for the long-range interactions $Q^{(h,j)}$, the expressions reported in Eq. (18), accounting for Eq. (20), that reads

$$\sigma(x) = \left(\frac{\alpha c_\alpha A}{\Gamma(1-\alpha)}\sum_{j=r+1}^{m}\sum_{h=1}^{r}\frac{(u_j - u_h)}{|x_j - x_h|^{1+\alpha}}(\Delta x)^2 + E\frac{(u_r - u_{r-1})}{\Delta x}\right) \tag{43}$$

and letting $\Delta x \to 0$, as in Section 5, Eq. (30) is fully recovered and now has a remarkable mechanical equivalence since it represents the non-local Cauchy stress at cross-section x obtained as the sum of two contributions: (i) The local Cauchy stress represented by the first term in Eq. (43) and (ii) a non-local stress represented by the latter contribution in Eq. (43). Static boundary conditions may then be applied to the continuous model with long-range interactions requiring that the overall stress, coalescing at the borders with the local stress σ_l, is equivalent to the applied load as $\sigma A = F$.

Summing up the elastic equilibrium problem for 1D solid, in the presence of fractional form of long-range interactions involves equilibrium, compatibility and constitutive relations, respectively, given as

$$\frac{d\sigma(x)}{dx} = \frac{d}{dx}(\sigma_l(x) + \sigma_{nl}(x)) = -f(x), \tag{44a}$$

$$\frac{du}{dx} = \varepsilon(x), \quad 0 \leqslant x \leqslant L, \tag{44b}$$

$$\sigma(x) = E\left(\frac{du}{dx} - \frac{\alpha c_\alpha}{\Gamma(1-\alpha)}\int_{\xi_1:x}^{L}\int_{\xi_2:0}^{x}\frac{u(\xi_1) - u(\xi_2)}{|\xi_2 - \xi_1|^{1+\alpha}}d\xi_2 d\xi_1\right), \tag{44c}$$

with associated boundary conditions, respectively, kinematic or static, in the form

$$u(0) = u_0, \quad u(L) = u_L, \tag{45a}$$

$$\sigma(0) = \sigma_l(0) = -F_0/A, \quad \sigma(L) = \sigma_l(L) = F_L/A, \tag{45b}$$

since the non-local contribution provided by the double integral in Eq. (44c) vanishes at the edges (for $x \to 0$or $x \to L$) or their combination in the case of mixed boundary conditions.

It is to be remarked that all the considerations provided about the mechanical models presented in this paper hold true also for different classes (exponential-type, Gaussian, Mexican hat, etc.) of the distance-decaying stiffness of the springs representing long-range forces as it will be reported in a forthcoming paper dedicated to this topic. Moreover, we must claim that all the proposed governing equations and associated boundary conditions may be also derived by the application of variational calculus, once appropriate form of the strain energy function has been deduced from the mechanical model presented here. This aspect has already been investigated and derived by the authors and it cannot be reported in this paper for brevity. Anyway, the fundamental relations of the proposed continuum with long-range forces such as the virtual work principles, the elastic potentials and all the variational formulations, already obtained by the authors, will be presented in a study totally devoted to this fundamental matter.

In the presence of unbounded domains (44a) and (44b) hold true while (44c) must be replaced with

$$\sigma(x) = E\left(\frac{du}{dx} - \frac{\alpha c_\alpha}{\Gamma(1-\alpha)}\int_x^\infty \int_{-\infty}^x \frac{u(\xi_1)-u(\xi_2)}{|\xi_2-\xi_1|^{1+\alpha}} d\xi_2\, d\xi_1\right), \tag{46}$$

that has been derived from the relation

$$(\mathbf{D}_+^\alpha u)(x) + (\mathbf{D}_-^\alpha u)(x) = \frac{\alpha}{\Gamma(1-\alpha)}\frac{d}{dx}\int_x^\infty \int_{-\infty}^x \frac{u(\xi_1)-u(\xi_2)}{|\xi_2-\xi_1|^{1+\alpha}} d\xi_1\, d\xi_2 \tag{47}$$

A close inspection of Eq. (47) reveals that double integral at the right-hand side times $\alpha/\Gamma(1-\alpha)$ is the Marchaud fractional integral that coincides in unbounded domains with the difference of Riemann–Liouville fractional integrals as (see Appendix A for details)

$$(I_+^\alpha u)(x) - (I_-^\alpha u)(x) = (\mathbf{I}_{-\infty,\infty}^\alpha u)(x) = \frac{\alpha}{\Gamma(1-\alpha)}\int_x^\infty \int_{-\infty}^x \frac{u(\xi_1)-u(\xi_2)}{|\xi_2-\xi_1|^{1+\alpha}} d\xi_1\, d\xi_2. \tag{48}$$

A different scenario happens for the double integral in Eq. (44c) that does not coalesce with the fractional Riemann–Liouville fractional integral on a finite support, since it remains just the integral counterpart of the sum of $(\widehat{\mathbf{D}}_{0^+}^\alpha u)(x)and(\widehat{\mathbf{D}}_{L^-}^\alpha u)(x)$ as

$$(\widehat{\mathbf{I}}_{0,L}^\alpha u)(x) = \frac{\alpha}{\Gamma(1-\alpha)}\int_x^L \int_0^x \frac{u(\xi_1)-u(\xi_2)}{|\xi_2-\xi_1|^{1+\alpha}} d\xi_1\, d\xi_2. \tag{49}$$

The stress–strain relation, for unbounded and bounded domain may then be reported as

$$\sigma(x) = E\varepsilon(x) - Ec_\alpha(\mathbf{I}_{-\infty,\infty}^\alpha u)(x), \quad -\infty < x < \infty, \tag{50a}$$

$$\sigma(x) = E\varepsilon(x) - Ec_\alpha(\widehat{\mathbf{I}}_{0,L}^\alpha u)(x), \quad 0 \leqslant x \leqslant L. \tag{50b}$$

Such a stress–strain relation for a bounded domain is very different from the constitutive relationship proposed in Eq. (1), without underlying the mechanical model or with that exploited in Eq. (7) in the fractional model, because they depend of the relative displacements at different locations instead of the strain field as in Eqs. (1) and (7). Moreover, a double integral is involved in (50b) and (50a) instead of single convolution integral involved in the non-local integral theories.

At this stage, we now have the machinery to represent the mechanical equivalence of the non-integral terms retained in the non-local integral model obtained by the direct use of Eringen model (Eq. (16)). The divergent boundary terms in Eqs. (14a) and (14b) are, in the point-spring non-local model, elastic springs connecting the point to the ground with location-dependent stiffness (as shown in Fig. 3) that reads, at location $x_j = (j-1)\Delta x$:

$$k_j = \frac{Ec_\alpha \Delta x}{A\Gamma(1-\alpha)}\left(\frac{1}{x_j^\alpha} + \frac{1}{(L-x_j)^\alpha}\right) \tag{51}$$

Thus for the model directly derived from the Eringen model the stiffness matrix of the non-local model is provided as $\mathbf{K} = \mathbf{K}^{l} + \mathbf{K}^{nl} + \mathbf{K}^{r}$ with the additional, diagonal matrix $\mathbf{K}^{r}$ of the form

$$\mathbf{K}^r = \begin{bmatrix} k_1 & 0 & \dots & 0 \\ 0 & k_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & k_m \end{bmatrix} \tag{52}$$

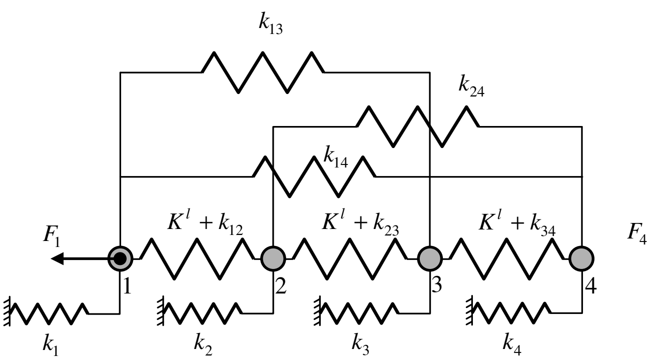

Fig. 3. Mechanical equivalence of the Eringen model with point-spring model with additional elastic restraints.

As soon as $\Delta x \to 0$ matrix in Eq. (39) tends to the fractional difference matrix and the non-integral terms in the integral non-local model are provided by the additional terms reported in Eq. (51), yielding the governing Eq. (16). The specific form of the additional stiffness in Eq. (51) led us to the conclusion that (i) the stiffness of the spring located at the border of the bar is infinitely large corresponding to a fixed support; (ii) the presence of the additional springs is not consistent with the studied bar that is not connected to the ground. These considerations appearing for the finite bar are not involved in the analysis of bar with unbounded domain, since in that latter case the stiffness of the additional springs is vanishing everywhere.

As a conclusion, Eq. (16) directly derived from the Eringen model is inconsistent from a mechanical point of view, because it corresponds to some additional restraints and spring connections that are not present in the mechanical model. It follows that the only way to define the non-local model of a finite bar is that provided in Eq. (27) that contains integral parts $\widehat{\mathbf{D}}^{\alpha}_{0^+}[u(x)]and\widehat{\mathbf{D}}^{\alpha}_{L^-}[u(x)]$ instead of the Marchaud fractional derivatives on finite supports and it has formulated on mechanical grounds. Henceforth, we conclude that the mechanical model introduced in this paper does not correspond, for a bounded bar, neither to the Eringen model nor to the fractional model presented by Lazopulos but that it represents a novel model to account for long-range forces and appropriate investigations will be reported elsewhere.

As soon as the linear point-spring model of long-range interactions has been introduced and established, we may inquire about the influence of parameters α, c_α and E on the non-local response of a system. The couple of parameters α and c_α are involved in the non-local contribution to the overall Cauchy stress (see 44c). Increasing value of parameter c_α corresponds to larger long-range interactions for specified distance of the particles, whereas the real parameter α controls the influence of the edge-effect on the structural response. Values of these parameters strongly depend on the inner structure of the material and they must be set to describe the measured data field. Parameter E influences the amount of strain energy stored in the specimen and it may be estimated once a standard experimental set-up has been established.

Several numerical investigations have shown that the concepts expressed in this paper about the mechanical representation of long-range interactions may be applied also for different classes of attenuation functions (Gaussian, Mexican hat, exponential) as it will be reported in forthcoming papers.

7. Numerical applications

Numerical investigations reported in this paper have been devoted to highlight concepts and discussions of the previous sections. To this aim in Fig. 4a–c, a critical comparison between the integral non-local model and the proposed representation of cohesive forces has been reported for different cases of bar length. The bar has been loaded by self-equilibrated forces applied at fixed distance d = 5 mm from the central cross-section of the bar allowing to represent edge effect. In Fig. 4a, the axial displacements of the bar obtained with the proposed model (continuous line) have been contrasted with the discrete point-spring model of non-local interactions (dots) and with integral fractional model of Section 3 (dashed line). It may be observed with the length of the bar L = 200 mm the Eringen model yields displacement function almost similar to the axial displacement field obtained with the proposed interpretation of long-range forces. These effects may be discussed with the arguments of Section 4, where we reported the case of bar of infinite length remarking that in this context the integral non-local model yields the same governing equation of the proposed model of non-local interactions. This may be explained observing that the axial displacements of the free end of the bar are nearly vanishing for the mechanical model in Fig. 2, so that the fixed support at the end of the Eringen integral model in Fig. 3 is ineffective.

A different scenario is provided with other lengths of the bar, namely L = 50 mm and L = 10 mm, as it may be observed in Fig. 4b and c, respectively, where the marked differences between the non-local integral model and the proposed representation of long-range interactions may be detected.

This behaviour is explained by the considerations reported in Section 5 for the mechanical model of long-range interactions proposed. As soon as the distance between the borders of the bar and the external loads decreases, stronger edge effects due to the elastically restrained supports may be observed as in Fig. 4b and c, showing significant differences with the proposed non-local model. This is because as soon as the external forces are closer to the end of the bar, the proposed non-local model yields non-vanishing axial displacements at the borders that are no more coincident with the Eringen fractional integral model of non-local interactions (Fig. 4). This effect is still more highlighted for the bar in Fig. 4), where axial displacements corresponding to the mechanical model of non-local interactions are very different from the non-local fractional integral model.

The displacement field obtained for assigned boundary forces has been reported in Fig. 5a for different values of the size of the grid scheme used. It has been observed that the equivalent point-spring model do provide the same numerical values and it has not been reported for clarity. Number of points used to discretize the model has been assumed equal to m = 400 (dotted line), m = 800 (dot-dashed line), m = 1600 (dashed line) and m = 2400 (continuous line). In particular, it may be observed that as soon as the grid used become more and more refined, the displacement field tends toward the continuous line that may be assumed coinciding with the correct solution of Eq. (27) (obtained for m = 4800). Moreover, the presence of long-range forces is significant at the border of the domain, where the displacement field shows significant deviations from the linear behaviour predicted by classical continuum mechanics.

In the central part of the specimen, almost linear behaviour of the displacement field may be withdrawn from Fig. 5b leading to conclude that the presence of long-range forces does not alter the characters of the displacement field in the core domain. This behaviour is enhanced in Fig. 5b reporting the strain field of the specimen for different values of the grid scheme used and it may be observed that, independently of the discretization step, non-uniform strains are present at

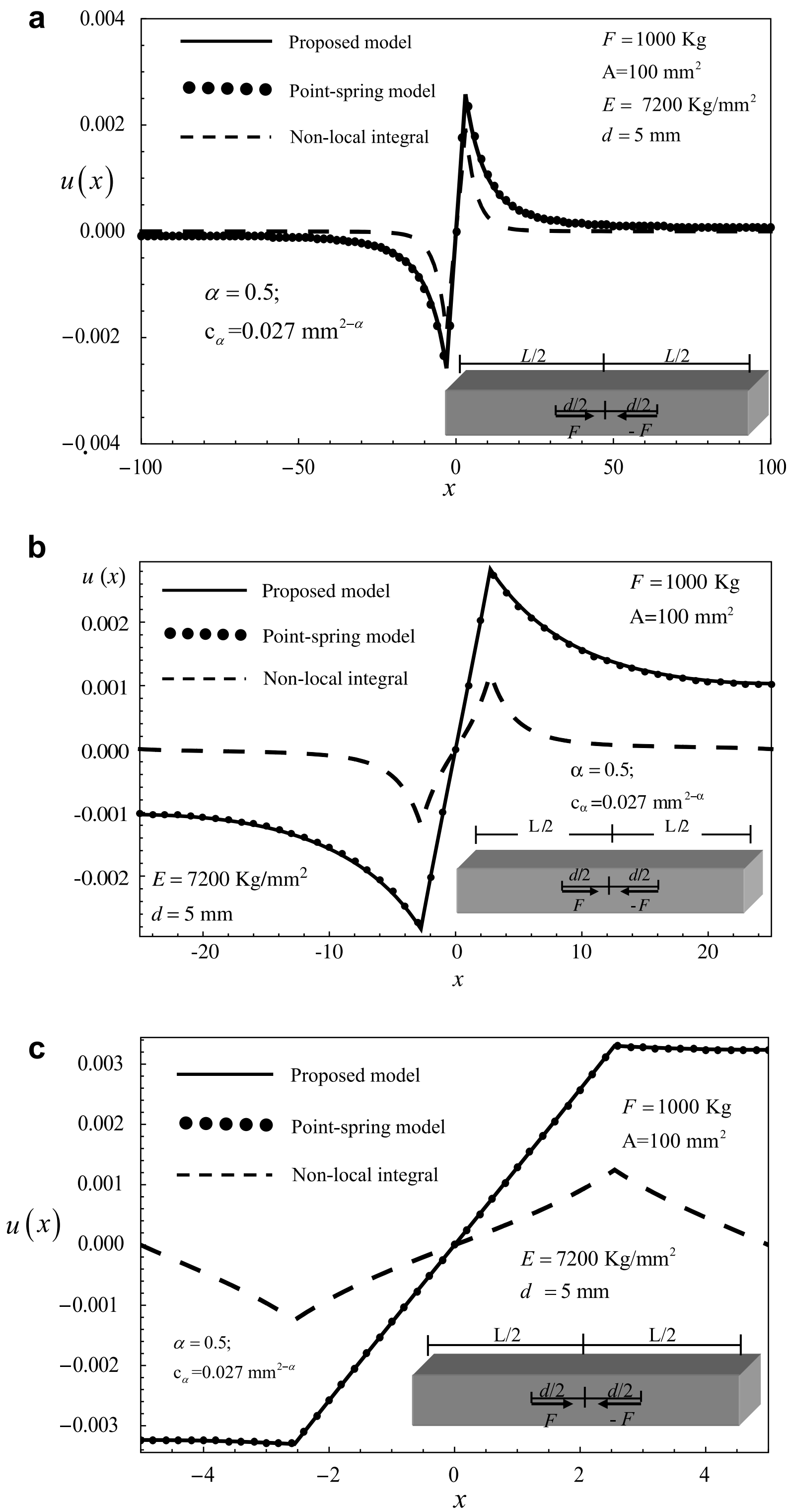

Fig. 4. (a) Axial displacements of the proposed fractional model vs the equivalent point-springs model (dots) and the Eringen model (dashed) for a self-equilibrated bar with $L = 200$ mm. (b) Axial displacements of the proposed fractional model vs the equivalent point-springs model (dots) and the Eringen model (dashed) for a self-equilibrated bar with $L = 50$ mm. (c) Axial displacements of the proposed fractional model vs the equivalent point-springs model (dots) and the Eringen model (dashed) for a self-equilibrated bar with $L = 10$ mm.

the border of the specimen. In more detail, the non-local character of the model is always respected and as soon as the gird step decreases the strain field tends toward the continuous line representing the solution of Eq. (27) (obtained for $m = 4800$).

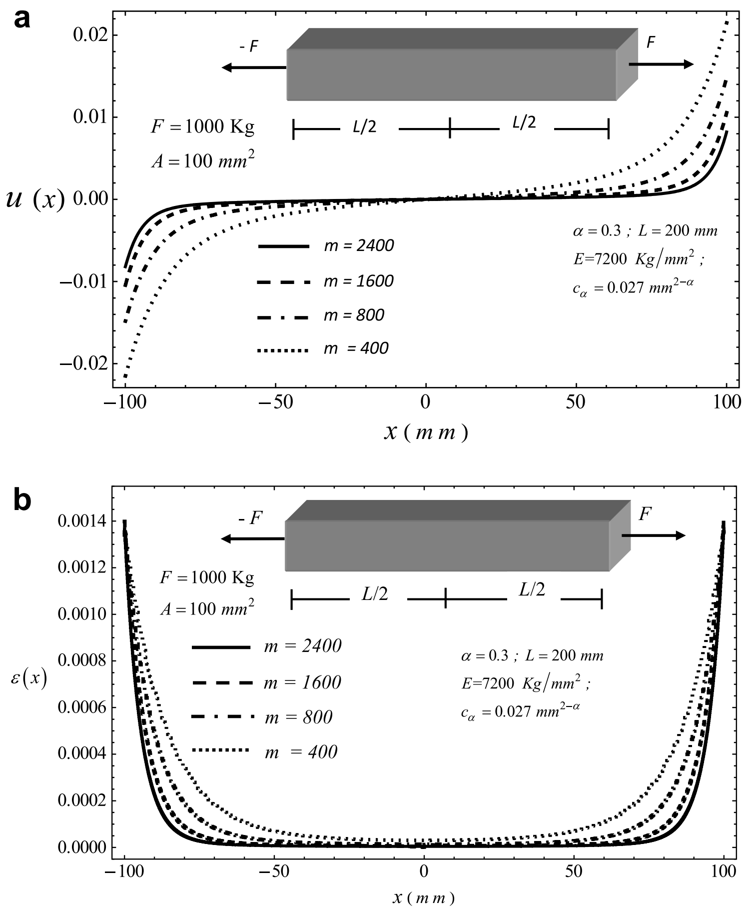

Fig. 5. (a) Axial displacement field of fractional continuum of a free-free bar for different finite difference discretization grids. (b) Axial strain of fractional continuum of a free-free bar for different finite difference discretization grids.

The effects of the parameters involved in the proposed model of long-range interactions have been reported in Fig. 6a and b illustrating the behaviour of the conventional strain $\varepsilon(x)$ of a free-free bar with applied self-equilibrated axial loads for different values of the coefficients α (Fig. 6a) and c_α (Fig. 6b). It may be observed that, for the limiting case of $\alpha \to 0$ or $c_\alpha \to 0$, the proposed non-local model yields the well-known local case without distance-decaying long-range forces.

8. Conclusions and discussion

This paper aims to overcome the problem of mechanical boundary conditions for a non-local bar with finite length as well as to unify the gradient and the integral theory of non-local continua.

Initially, it has been shown that proper selection of the attenuation function in the Eringen model yields stress–strain relation involving fractional derivatives instead of classical derivatives or convolution integrals. Fractional derivatives or fractional integrals are neither else than differentials of convolution integrals and then by introducing the non-local contribution in the stress–strain relation we obtain an intermediate operator between weak and strong non-local theories. It has been shown that the direct substitution of the fractional attenuation function in the Eringen integral model yields some inconsistencies at the border of the finite bar that do not allow to satisfy the mechanic boundary conditions. This happens, in the author's opinion, because the direct introduction of the non-local contribution in the constitutive law of the material does not have any mechanical equivalence and only in the presence of unbounded domain these inconsistencies disappear.

Non-local effects have then been proposed, consequently, introducing two different, but totally equivalent, mechanical models. The first one is obtained by considering that in each volume element we have contact forces arising from the surface of separation of adjacent volumes and other central forces provided by the surrounding, non-adjacent, volumes. The latter forces depend on the product of non-adjacent volumes, on the relative displacements of the non-adjacent volumes and on a decaying function of the distance between the interacting masses. It has been shown that for unbounded domain, by selecting the decaying function as proportional to $|x_j - x_h|^{-(1+\alpha)}$ then the Marchaud fractional derivative of the displacement field appears in the equilibrium equation. Moreover, for a bar of finite extension, only the integral part of the Marchaud fractional

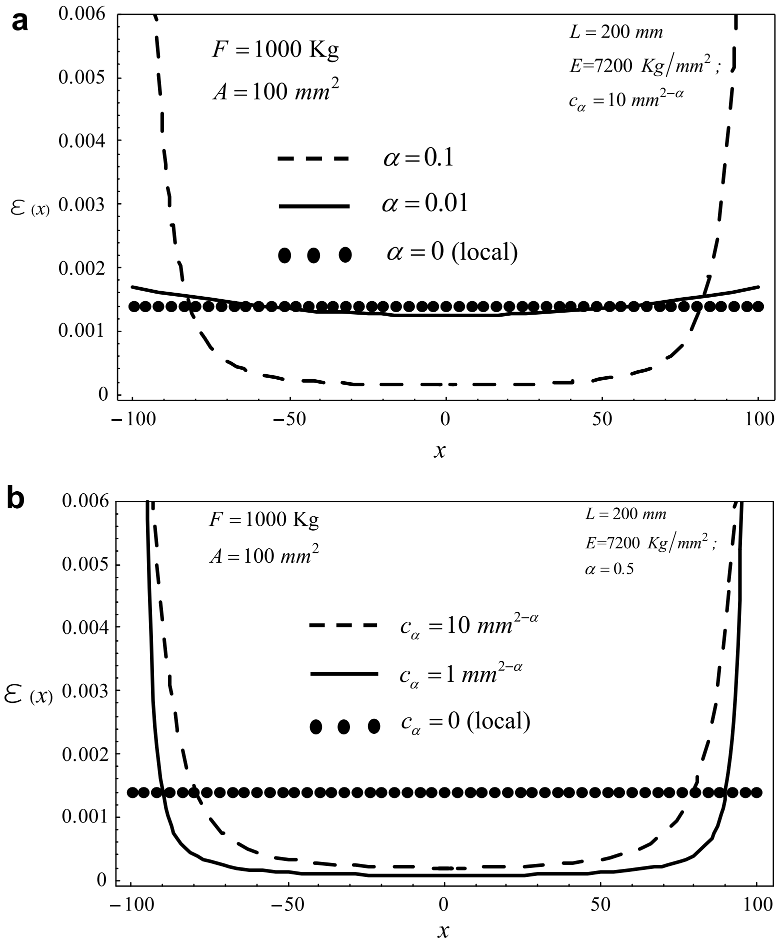

Fig. 6. (a) Axial strain field of clamped-free for different values of real parameter α. (b) Axial strain field of clamped-free 1 for different values of real parameter c_α.

derivative appears. It follows that the divergent term of Marchaud fractional derivative on finite support is not involved in the analysis. This is a very remarkable aspect since for an unbounded bar the non-local contribution is represented by the Marchaud fractional derivative, while for a finite support, only the integral term of the Marchaud fractional derivative in finite support is present and then the governing differential equation in terms of displacements is no more a fractional differential equation. This feature leads us to conclude that Eringen model with fractional attenuation function is equivalent to the proposed mechanical model of non-local interaction only in the case of unbounded domain. This is due to the equivalence of the Riemann–Liouville fractional derivatives present in the Eringen model with Marchaud fractional derivatives involved in the present analysis for unbounded domains. This is no more the case for bounded bar where the proposed non-local interaction model involve only the integral part of the Marchaud fractional derivative and this part is no more equivalent to the Riemann–Lioville fractional derivative on bounded intervals involved in Eringen model with a proper selection of the decaying function.

The second equivalent mechanical model accounting for non-local contribution is a point-spring model. The local contribution is taken into account by springs connecting adjacent points while the non-local contributions are considered by springs with distance-decaying stiffness. It is shown that at the limit, for a bar of finite length, the governing equations exactly coalesce with the previously proposed model. This second mechanical model of long-range interactions allows to define the overall Cauchy stress in a given location of the model simply by the sum of two contributions: The first is the local stress due to the contact of the points (local spring); the second is the resultant, in terms of the stress of the springs connecting non-adjacent points (non-local springs). The overall stress leads us to formulate, consistently, the boundary condition of the 1D continuum with long-range forces.

The proposed models of long-range forces lead us to derive a stress–strain relation for the non-local 1D bar for both unbounded and bounded domains that is composed by the local contribution and the non-local one represented by a Marchaud fractional integral. Since the latter is a double convolution integral we may state that the proposed non-local interaction model with mechanical equivalence does not correspond to the Eringen model in bounded domains.

The concepts presented in this paper are not only related to the choice of the attenuation function in order to obtain a fractional differential equation. As in fact the readers may verify that with different attenuation functions describing long-range forces the non-local stress is still defined by a double convolution integral (no more Marchaud-type).

Several numerical applications have been reported in this paper to highlight the effect of long-range forces and impending boundaries on the proposed mechanical model of the solid.

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Appendix A. Details about fractional calculus

In this appendix, we give some details on fractional calculus and we consider functions defined in a finite interval. Given a Lebesgue measurable function $w(x)$ on the closed interval $[a,b]$, briefly $w(x) \in Leb_1([a,b])$, it is possible to define the *left–handed RL fractional derivative* $(\mathscr{D}^{\gamma}_{a+}w)(x)$ with $\gamma \in \mathbb{R}$, given by

$$(\mathscr{D}^{\gamma}_{a+}w)(x) \stackrel{\text{def}}{=} \frac{1}{\Gamma(1-\gamma)}\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x \frac{w(\xi)\mathrm{d}\xi}{(x-\xi)^{\gamma}}, \quad 0<\gamma<1 \tag{A1}$$

and the *right–handed RL fractional derivative* $(\mathscr{D}^{\gamma}_{b-}w)(x)$ in the form

$$(\mathscr{D}^{\gamma}_{b-}w)(x) = \frac{(-1)}{\Gamma(1-\gamma)}\frac{\mathrm{d}}{\mathrm{d}x}\int_x^b \frac{w(\xi)\mathrm{d}t}{(\xi-x)^{\gamma}}, \quad 0<\gamma<1, \tag{A2}$$

Useful representations of Eq. (A1) and (A2) are

$$(\mathscr{D}^{\gamma}_{a+}w)(x) = \frac{1}{\Gamma(1-\gamma)}\left[\frac{w(a)}{(x-a)^{\gamma}} + \int_a^x \frac{w\prime(\xi)\mathrm{d}\xi}{(x-\xi)^{\gamma}}\right], \quad 0<\gamma<1, \tag{A3}$$

$$(\mathscr{D}^{\gamma}_{b-}w)(x) = \frac{1}{\Gamma(1-\gamma)}\left[\frac{w(b)}{(b-x)^{\gamma}} - \int_x^b \frac{w'(\xi)\mathrm{d}\xi}{(\xi-x)^{\gamma}}\right], \quad 0<\gamma<1, \tag{A4}$$

as reported in the book by Samko et al. (1988). In order to extend the definition of fractional derivative of order greater than 1, first, we recall a standard notation, indicating with $[\gamma]$ the integer part of a real number and with $\{\gamma\}$ the fractional part, that is $\gamma = [\gamma] + \{\gamma\}$. Then, for every positive real number γ the Riemann–Liouville fractional derivatives are defined as

$$(\mathscr{D}^{\gamma}_{a+}w)(x) = \frac{1}{\Gamma(n-\gamma)}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\int_a^x \frac{w(\xi)\mathrm{d}t}{(x-\xi)^{\gamma-n+1}}, \quad n=[\gamma]+1, \tag{A5}$$

$$(\mathscr{D}^{\gamma}_{b-}w)(x) = \frac{(-1)^n}{\Gamma(n-\gamma)}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\int_x^b \frac{w(\xi)\mathrm{d}\xi}{(\xi-x)^{\gamma-n+1}}, \quad n=[\gamma]+1, \tag{A6}$$

Comparing the definitions, it follows that the fractional derivatives and fractional integrals are related by the simple relations

$$(\mathscr{D}^{\gamma}_{a+}w)(x) = \frac{\mathrm{d}^n}{\mathrm{d}x^n}(I^{n-\gamma}_{a+}w)(x), \quad n=[\gamma]+1, \tag{A7}$$

$$(\mathscr{D}^{\gamma}_{b-}w)(x) = (-1)^n\frac{\mathrm{d}^n}{\mathrm{d}x^n}(I^{n-\gamma}_{b-}w)(x), \quad n=[\gamma]+1. \tag{A8}$$

The presence of the derivatives of order n in the fractional derivatives definitions involves more strict conditions to the existence of the fractional derivative. A sufficient condition is the function having continuous derivatives up to the order $[\alpha]-1$.

In the presence of unbounded domain, the RL fractional derivatives reads

$$(\mathscr{D}^{\gamma}_{\pm}w)(x) = \frac{1}{\Gamma(1-\gamma)}\frac{\mathrm{d}}{\mathrm{d}x}\int_0^{\infty} \frac{w(x\mp\xi)\mathrm{d}\xi}{\xi^{\gamma}} \tag{A9}$$

for $0 < \gamma < 1$ or, for $\gamma > 0$, it is expressed as

$$(\mathscr{D}^{\gamma}_{\pm}w)(x) = \frac{(\pm 1)^n}{\Gamma(n-\gamma)}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\int_0^{\infty} \xi^{n-\gamma-1}w(x\mp\xi)\mathrm{d}\xi, \quad n=[\gamma]+1. \tag{A10}$$

On the real axis, Eq. (A9) can be written in more convenient form, working out a little on definition. In fact, suppose first that the function $w(x)$ is continuously differentiable and with its first derivative $w'(x)$, vanishes at infinity as $|x|^{\gamma-1-\varepsilon}, \varepsilon > 0$, and consider $0 < \gamma < 1$. Under these assumptions the chain of equalities is true:

$$
\begin{aligned}
(\mathscr{D}^{\gamma}_{\pm}w)(x) &= \tfrac{1}{\Gamma(1-\gamma)}\tfrac{\mathrm{d}}{\mathrm{d}x}\int_0^\infty \tfrac{w(x\mp\xi)\mathrm{d}\xi}{\xi^\gamma} = \tfrac{1}{\Gamma(1-\gamma)}\int_0^\infty \tfrac{w'(x\mp\xi)\mathrm{d}\xi}{\xi^\gamma} = \\
&= \tfrac{\gamma}{\Gamma(1-\gamma)}\int_0^\infty w'(x\mp\xi)\mathrm{d}\xi\int_t^\infty \tfrac{\mathrm{d}\xi}{\xi^{1+\gamma}} = \\
&= \tfrac{\gamma}{\Gamma(1-\gamma)}\int_0^\infty \tfrac{w(x)-w(x\mp\xi)\mathrm{d}\xi}{\xi^{1+\gamma}} \overset{\text{def}}{=} (\mathbf{D}^{\gamma}_{\pm}w)(x)
\end{aligned} \tag{A11}
$$

The operators $(\mathbf{D}^{\gamma}_{+}w)(x)$ and $(\mathbf{D}^{\gamma}_{-}w)(x)$ in Eq. (A11) are the *Marchaud fractional derivatives* for an unbounded domain. The advantage of this definition is that the integral converges under more general assumptions for the function, not requiring a good behavior at infinity, i.e., a function growing at infinity as $|x|^{\gamma-\varepsilon}$, with $\varepsilon > 0$ has a Marchaud fractional derivative. Therefore, the RL derivative and the Marchaud derivative coincide only for a class of functions. Conditions on the equivalence are reported in (Samko, pp 224–229). The Marchaud fractional derivatives in a finite interval is obtained from Eq. (A11) by continuing the function $f(x)$ by zero beyond the interval $[a,b]$, that is

$$
w^*(x) = \begin{cases} w(x), & x\in[a,b], \\ 0, & x\notin[a,b], \end{cases} \tag{A12}
$$

obtaining the useful relations

$$
(\mathbf{D}^{\gamma}_{a+}w)(x) \overset{\text{def}}{=} (\widehat{\mathbf{D}}^{\gamma}_{a+}w)(x) + \frac{w(x)}{\Gamma(1-\gamma)(x-a)^\gamma}, \quad x\in[a,b], \tag{A13}
$$

$$
(\mathbf{D}^{\gamma}_{b-}w)(x) \overset{\text{def}}{=} (\widehat{\mathbf{D}}^{\gamma}_{b-}w)(x) + \frac{w(x)}{\Gamma(1-\gamma)(b-x)^\gamma}, \quad x\in[a,b], \tag{A14}
$$

where $(\widehat{\mathbf{D}}^{\gamma}_{a+}w)(x)$ and $(\widehat{\mathbf{D}}^{\gamma}_{b-}w)(x)$ represent the defined integrals

$$
(\widehat{\mathbf{D}}^{\gamma}_{a+}w)(x) = \frac{\gamma}{\Gamma(1-\gamma)}\int_a^x \frac{w(x)-w(\xi)}{(x-\xi)^{(1+\gamma)}}\mathrm{d}\xi \tag{A15}
$$

$$
(\widehat{\mathbf{D}}^{\gamma}_{b-}w)(x) = \frac{\gamma}{\Gamma(1-\gamma)}\int_x^b \frac{w(x)-w(\xi)}{(\xi-x)^{(1+\gamma)}}\mathrm{d}\xi \tag{A16}
$$

Equivalence between Riemann–Liouville fractional derivative and Marchaud fractional derivatives in the presence of bounded intervals may be proved by integrating in Eqs. (A3) and (A4) by parts obtaining

$$
\begin{aligned}
(\mathscr{D}^{\gamma}_{a+}w)(x) &= \tfrac{1}{\Gamma(1-\gamma)}\left[\tfrac{w(a)}{(x-a)^\gamma} + \int_a^x (x-\xi)^{-\gamma}\mathrm{d}[w(\xi)-w(x)]\right] \\
&= \tfrac{1}{\Gamma(1-\gamma)}\left[\tfrac{w(x)}{(x-a)^\gamma} + \lim_{\xi\to x}\tfrac{w(\xi)-w(x)}{(x-\xi)^\gamma} + \gamma\int_a^x \tfrac{w(x)-w(\xi)}{(x-\xi)^{(\gamma+1)}}\mathrm{d}\xi\right] \\
&\overset{\text{def}}{=} \tfrac{1}{\Gamma(1-\gamma)}\left[\tfrac{w(x)}{(x-a)^\gamma} + \gamma\int_a^x \tfrac{w(x)-w(\xi)}{(x-\xi)^{(\gamma+1)}}\mathrm{d}\xi\right] \\
&= (\mathbf{D}^{\gamma}_{a+}w)(x),
\end{aligned} \tag{A17}
$$

$$
\begin{aligned}
(\mathscr{D}^{\gamma}_{b+}w)(x) &= \tfrac{1}{\Gamma(1-\gamma)}\left[\tfrac{w(b)}{(b-x)^\gamma} + \int_x^b (x-\xi)^{-\gamma}\mathrm{d}[w(\xi)-w(x)]\right] \\
&= \tfrac{1}{\Gamma(1-\gamma)}\left[\tfrac{w(x)}{(b-x)^\gamma} + \lim_{\xi\to x}\tfrac{w(\xi)-w(x)}{x-\xi)^\gamma} + \gamma\int_x^b \tfrac{w(x)-w(\xi)}{(\xi-x)^{(\gamma+1)}}\mathrm{d}\xi\right] \\
&\overset{\text{def}}{=} \tfrac{1}{\Gamma(1-\gamma)}\left[\tfrac{w(x)}{(b-x)^\gamma} + \gamma\int_a^x \tfrac{w(x)-w(\xi)}{(\xi-x)^{(\gamma+1)}}\mathrm{d}\xi\right] \\
&= (\mathbf{D}^{\gamma}_{b+}w)(x).
\end{aligned} \tag{A18}
$$

The equivalence between the Riemann–Liouville fractional integrals and the proposed Marchaud fractional integral in the unbounded domain reported in Eq. (50) may be withdrawn from Eq. (A11) casting the sum $(\mathbf{D}^{\alpha}_{+}w)(x) + (\mathbf{D}^{\alpha}_{-}w)(x)$ in the form

$$
(\mathbf{D}^{\gamma}_{+}w)(x) + (\mathbf{D}^{\gamma}_{-}w)(x) = (\mathscr{D}^{\gamma}_{+}w)(x) + (\mathscr{D}^{\gamma}_{-}w)(x) = \frac{1}{\Gamma(1-\gamma)}\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_{-\infty}^x \frac{w(\xi)}{(x-\xi)^\gamma}\mathrm{d}\xi - \int_x^\infty \frac{w(\xi)}{(x-\xi)^\gamma}\mathrm{d}\xi\right). \tag{A19}
$$

and the left-hand side of Eq. (A19) is written, accounting for Eq. (49), in the equivalent form

$$
(\mathscr{D}^{\gamma}_{+}w)(x) + (\mathscr{D}^{\gamma}_{-}w)(x) = \frac{\gamma}{\Gamma(1-\gamma)}\frac{\mathrm{d}}{\mathrm{d}x}\int_x^\infty\int_{-\infty}^x \frac{w(\xi_1)-w(\xi_2)}{|\,\xi_2-\xi_1|^{1+\gamma}}\mathrm{d}\xi_1\,\mathrm{d}\xi_2 \tag{A20}
$$

And by equating the right-hand side of Eqs. (A19) and (A20) the following equality holds:

$$
\frac{\gamma}{\Gamma(1-\gamma)}\frac{\mathrm{d}}{\mathrm{d}x}\int_x^\infty\int_{-\infty}^x \frac{w(\xi_1)-w(\xi_2)}{|\,\xi_2-\xi_1|^{1+\gamma}}\mathrm{d}\xi_1\,\mathrm{d}\xi_2 = \frac{1}{\Gamma(1-\gamma)}\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_{-\infty}^x \frac{w(\xi)}{(x-\xi)^\gamma}\mathrm{d}\xi - \int_x^\infty \frac{w(\xi)}{(x-\xi)^\gamma}\mathrm{d}\xi\right). \tag{A21}
$$

Thus, conducting to the conclusion that apart an inessential constant their primitives must coincide

$$\frac{\gamma}{\Gamma(1-\gamma)}\int_x^\infty\int_{-\infty}^x\frac{w(\xi_1)-w(\xi_2)}{\mid\xi_2-\xi_1\mid^{1+\gamma}}\mathrm{d}\xi_1\,\mathrm{d}\xi_2=(\mathbf{I}^{1-\gamma}_{-\infty,\infty}w)(x)=(I^{1-\gamma}_{+}w)(x)-(I^{1-\gamma}_{-}w)(x)\tag{A22}$$

as reported in Eq. (50).

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