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Analytical inversion of general periodic tridiagonal matrices

M.A. El-Shehawey *, Gh.A. El-Shreef, A.Sh. Al-Henawy

Department of Mathematics, Damietta Faculty of Science, PO Box 6, New Damietta, Egypt

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ABSTRACT

In this paper we present an analytical forms for the inversion of general periodic tridiagonal matrices, and provide some very simple analytical forms which immediately lead to closed formulae for some special cases such as symmetric or perturbed Toeplitz for both periodic and non-periodic tridiagonal matrices. An efficient computational algorithm for finding the inverse of any general periodic tridiagonal matrices from the analytical form is given, it is suited for implementation using Computer Algebra systems such as MAPLE, MATLAB, MACSYMA, and MATHEMATICA. An example is also given to illustrate the algorithm.

1. Introduction

The solution of a variety of problems in many areas of physics and mathematics as well as in electrical engineering requires finding analytical formulae for the inversion of the general periodic tridiagonal matrices:

$$J(d_1, c_n; d_k, a_k, c_k; a_1, a_n) = \begin{bmatrix} a_1 & c_1 & 0 & \cdots & 0 & -d_1 \\ -d_2 & a_2 & c_2 & & & 0 \\ 0 & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ 0 & & & -d_{n-1} & a_{n-1} & c_{n-1} \\ c_n & 0 & \cdots & 0 & -d_n & a_n \end{bmatrix}_{(n\times n)}. \tag{1.1}$$

In the simplest cases the matrices $J(0,0;d,a,c;a,a)$, $J(0,0;d,a,c;a_1,a_n)$, $J(d,c;d,a,c;a,a)$, and $J(d_1,c_n;d,a,c;a_1,a_n)$ are respectively, called Toeplitz, perturbed Toeplitz, Toeplitz periodic, and perturbed Toeplitz periodic tridiagonal matrices.

The inversion of $J(d_1,c_n;d_k,a_k,c_k;a_1,a_n)$ in (1.1) has been studied extensively with an attempt to find a simple and analytic expression for the inverse. However, most of the efforts ended up with formulae for some special cases like $J(0,0;d_k,a_k,c_k;a_1,a_n)$, $J(0,0;-c_k,a_k,c_k;a_1,a_n)$, $J(0,0;d,a,c;a,a),\ldots,$ see for example [3,6,7,9,11,12,14–23] in addition to several others.

This comment is motivated by the work of Huang and McColl [15] who solved for the inverse of the general tridiagonal matrices $J(0,0;d_k,a_k,c_k;a_1,a_n)$. This paper is concerned with generalizing their work, to provide analytical formulae for the inverse of $J(d_1,c_n;d_k,a_k,c_k;a_1,a_n)$ as in (1.1). These formulae are expressed in term of determinants of specific submatrices of $J(d_1,c_n;d_k,a_k,c_k;a_1,a_n)$. The values of the determinants are related to the solutions of second order linear recurrences, which can explicitly be expressed in term of the elements of the matrices. The present formulae can immediately lead to explicitly closed forms for certain matrices such as perturbed Toeplitz for both periodic and non-periodic tridiagonal cases.

* Corresponding author.
E-mail address: el_shehawy@mans.edu.eg (M.A. El-Shehawey).

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In 1979, Yamamoto and Ikebe [21] obtained formulae for the inverses of banded matrices. In 1988, Chakraborty [2] gave an efficient algorithms for solving general periodic Toeplitz systems. A best known algorithm designed for serial implementation, for the solution of periodic tridiagonal linear system is given by Chawla [3]. In 2006, El-Mikkawy and Karawia [9] presented an efficient algorithms to find the inverse of a general tridiagonal matrix. An efficient computational algorithm for finding the inverse of any general periodic tridiagonal matrices $J(d_1, c_n; d_k, a_k, c_k; a_1, a_n)$ as in (1.1) is given, it is suited for implementation using Computer Algebra systems such as MAPLE, MATLAB, MACSYMA, and MATHEMATICA. An example is also given to illustrate the algorithm.

The paper is organized as follows. In Section 2, we discuss properties of some tridiagonal matrices. Section 3 is devoted to the main analytical results, in which we state and prove some relationships between two sequences of determinants. The proof of the main theorem is also obtained. Illustrative examples and computational algorithm for general periodic tridiagonal matrix inversion are given in Section 4.

2. Properties of some tridiagonal matrices

Let $J(d_k, a_k, c_k) \equiv J(0, 0; d_k, a_k, c_k; a_1, a_n)$ be an invertible tridiagonal matrix of order n such that

$$J(d_k, a_k, c_k) = \begin{bmatrix} a_1 & c_1 & 0 & \cdots & 0 & 0 \\ -d_2 & a_2 & c_2 & & & 0 \\ 0 & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ 0 & & & -d_{n-1} & a_{n-1} & c_{n-1} \\ 0 & 0 \cdots & 0 & & -d_n & a_n \end{bmatrix}_{(n\times n)}. \tag{2.1}$$

The determinant Δ_1^n of $J(d_k, a_k, c_k)$ is easily evaluated from three-term recurrence relations

$$\Delta_1^i = a_i \Delta_1^{i-1} + d_i c_{i-1} \Delta_1^{i-2}, \quad i = 2, 3, \ldots, n, \tag{2.2}$$

where $\Delta_1^{-1} = 0$, $\Delta_1^0 = 1$, $\Delta_1^1 = a_1$, and

$$\Delta_j^n = a_j \Delta_{j+1}^n + d_{j+1} c_j \Delta_{j+2}^n, \quad j = n-1, n-2, \ldots, 1, \tag{2.3}$$

where $\Delta_{n+2}^n = 0$, $\Delta_{n+1}^n = 1$, $\Delta_n^n = a_n$.

The elements of the vectors $(\Delta_1^i)_i$ and $(\Delta_j^n)_j$ in (2.2) and (2.3) are precisely the determinants of specific submatrices of the matrix $J(d_k, a_k, c_k)$.

The elements of the inverse matrix $J^{-1}(d_k, a_k, c_k) = (L_{ij})$, $1 \leqslant i, j \leqslant n$ can be computed according to the next lemma, which has been derived in [19], see also [4,11,15,20] and [23]:

Lemma 2.1. *The elements of $J^{-1}(d_k, a_k, c_k) = (L_{ij})$, $1 \leqslant i, j \leqslant n$, can be expressed as*

$$L_{ij} = \begin{cases} (-1)^{i+j} \frac{\Delta_1^{i-1} \Delta_{j+1}^n}{\Delta_1^n} \prod_{k=i}^{j-1} c_k & \text{for } i \leqslant j, \\ \frac{\Delta_1^{j-1} \Delta_{i+1}^n}{\Delta_1^n} \prod_{k=j+1}^{i} d_k & \text{for } i \geqslant j, \end{cases} \tag{2.4}$$

where $j = 1, 2, \ldots, n$, $d_1 = 0$, $c_n = 0$.

Notice that the problem of the determination of the elements L_{ij}, $1 \leqslant i, j \leqslant n$ reduces to solving the recurrent relations (2.2) and (2.3). Those are homogeneous linear recurrence relations of second order with variable coefficients. In 2001, Mallik [16] obtained closed form expression for the solutions of such recurrence relations. In fact, based on these results in recurrence relations in [16], explicit expressions for the elements L_{ij}, $1 \leqslant i, j \leqslant n$ were given in terms of the matrix elements. These results can be summarized in the following two propositions (cf. formulae (16), (18), (63a), (63b) and (72) in [16]):

Proposition 2.1. *The solution of the recurrence relation*

$$E_m(l) = \frac{-a_m}{c_m} E_{m-1}(l) + \frac{d_m}{c_m} E_{m-2}(l), \quad 1 \leqslant m \leqslant n, \ 1 \leqslant l \leqslant m \tag{2.5}$$

is given by

$$E_m(l) := \begin{cases} \frac{(-1)^{m-l+1}}{\prod_{j=l}^m c_j} \left(\prod_{j=l}^m a_j + \sum_{q=1}^{\lfloor \frac{m-l+1}{2} \rfloor} \sum_{(k_1,\ldots,k_q) \in S_q(l+1,m)} \sigma_{l,m}(k_1, \ldots, k_q) \right) & \text{for } l = 1, \ldots, m-1, \ m = 2, \ldots, n, \\ -\frac{a_m}{c_m} & \text{for } l = m, \ m = 1, \ldots, n, \\ 1 & \text{for } l = m+1, \ m = 0, \ldots, n, \\ 0 & \text{otherwise}, \end{cases} \tag{2.6}$$

where

$$\sigma_{l,m}(k_1,\dots,k_q)=\prod_{j=l}^{m}a_j\prod_{i=1}^{q}\frac{c_{k_i-1}d_{k_i}}{a_{k_i-1}a_{k_i}}, \tag{2.7}$$

and the definition of the set $S_q(l+1,m)$ has been introduced in [16] *as follows: Let $\mathbb{N}$ denote the set of natural numbers. For $q, L, U\in\mathbb{N}$, $S_q(L,U)$ is the set of all q-tuples with elements from $\{L,L+1,\dots,U\}$ arranged in ascending order so that no two consecutive elements are present, that is*:

$$S_q(L,U)=\begin{cases}\{L,L+1,\dots,U\} & \text{for } U\geqslant L \text{ and } q=1,\\ \{(k_1,\dots,k_q)\colon k_1,\dots,k_q\in\{L,L+1,\dots,U\};\ k_l-k_{l-1}\geqslant 2 \text{ for } l=2,\dots,q\} & \\ & \text{for } U\geqslant L+2 \text{ and } 2\leqslant q\leqslant\lfloor\frac{U-L+2}{2}\rfloor,\\ \emptyset, & \text{otherwise}.\end{cases} \tag{2.8}$$

Proposition 2.2. *Explicit formula for the elements $L_{i,j}$, $1\leqslant i,j\leqslant n$ is given by*

$$L_{ij}=\begin{cases}-\frac{E_{i-1}(1)E_n(j+1)}{c_jE_N(1)} & \text{for } i\leqslant j,\\ \frac{F_{j-1}(1)F_n(i+1)}{d_{i+1}F_n(1)} & \text{for } i>j,\end{cases} \tag{2.9}$$

where $E_n(1)\neq 0$, $c_1,c_2,\dots,c_{n-1}\neq 0$, $d_2,d_3,\dots,d_n\neq 0$ and

$$F_m(l):=(-1)^{m-l}\prod_{j=l}^{m}\left(\frac{c_j}{d_{j+1}}\right)E_m(l),\quad \text{for } l\leqslant m. \tag{2.10}$$

Remark 2.1. The expression for $\sigma_{l,m}(k_1,\dots,k_q)$ is well defined even if some of the a's are zero, since all the denominators $a_{k_i-1}a_{k_i}$ $(i=1,\dots,q)$ cancel out with some factors in $\prod_{j=l}^{m}a_j$.

Formulae (2.9) and (2.10) follow from (71) and (72) in [16], and the invertibility condition is a consequence of formula (77) in [16], which reads as

$$\det J(d_k,a_k,c_k)=(-1)^n\left(\prod_{j=1}^{n}c_j\right)E_n(1). \tag{2.11}$$

In fact, Mallik [16] obtained the explicit expressions (2.9) and (2.10) for the elements L_{ij}, $1\leqslant i,j\leqslant n$ by solving recurrence relation (2.5) with (2.7), as well as

$$E_m(l)=\frac{-a_l}{c_l}E_m(l+1)+\frac{d_{l+1}}{c_{l+1}}E_m(l+2),\quad 1\leqslant l\leqslant m,\ 1\leqslant m\leqslant n \tag{2.12}$$

(cf. (25), (63a) and (63b) in [16]). Now, using (2.2), (2.5) and (2.7) one observes that

$$\Delta_1^m=(-1)^m\left(\prod_{j=1}^{m}c_j\right)E_m(1),\quad -1\leqslant m\leqslant n. \tag{2.13}$$

Similarly, using (2.3), (2.7) and (2.12), one finds that

$$\Delta_m^n=(-1)^{n-m+1}\left(\prod_{j=m}^{n}c_j\right)E_n(m),\quad 1\leqslant m\leqslant n+2. \tag{2.14}$$

The relations (2.13) and (2.14) show that Lemma 2.1 and Proposition 2.2 are consistent. In fact, Lemma 2.1 can be derived from Proposition 2.2.

Notice that:

1. In our results, there are no conditions on the elements of the matrix, but in Mallik [16], the elements in the off-diagonals are not allowed to equal zero.
2. It is not easy to directly work with the explicit expression for $E_m(l)$ as in Proposition 2.1.

Lemma 2.2. *Let A be invertible matrix of order n, partitioned in the form*

$$A=\left[\begin{array}{c|c}A_{11} & A_{12}\\ \hline A_{21} & A_{22}\end{array}\right], \tag{2.15}$$

in which A_{11} and A_{22} are of order m and $n-m$, respectively, $1\leqslant m<n$, and will be assumed invertible. A_{12}' and A_{21} are of $n-m$ rows and m columns, where A_{12}' refers to the transpose of A_{12}. Then

(i) $B = A_{11} - A_{12}A_{22}^{-1}A_{21}$ *is invertible,*
(ii) $C = A_{22} - A_{21}A_{11}^{-1}A_{12}$ *is invertible.*

In this case two different representations for the inverse of A are given by

$$A^{-1} = \left[\begin{array}{c|c} B^{-1} & -B^{-1}A_{12}A_{22}^{-1} \\ \hline -A_{22}^{-1}A_{21}B^{-1} & A_{22}^{-1} + A_{22}^{-1}A_{21}B^{-1}A_{12}A_{22}^{-1} \end{array}\right] \tag{2.16}$$

and

$$A^{-1} = \left[\begin{array}{c|c} A_{11}^{-1} + A_{11}^{-1}A_{12}C^{-1}A_{21}A_{11}^{-1} & -A_{11}^{-1}A_{12}C^{-1} \\ \hline -C^{-1}A_{21}A_{11}^{-1} & C^{-1} \end{array}\right]. \tag{2.17}$$

A direct verification that the identity (2.16) (or (2.17)) holds can be obtained by multiplying the right member in either direction by the right member of (2.15), yielding the unit matrix.

A matrix with the form of B (or C) is called a Schur complement. An excellent review of Schur complement and their applications is given by Cottle [5], see also [8,10] and [13, Section 07.3, p. 18].

Remark 2.2. The off-diagonal blocks of A^{-1} can alternatively be expressed using the identities

$$B^{-1}A_{12}A_{22}^{-1} = A_{11}^{-1}A_{12}C^{-1} \quad \text{and} \quad A_{22}^{-1}A_{21}B^{-1} = C^{-1}A_{21}A_{11}^{-1}. \tag{2.18}$$

Therefore

$$\det A = \det A_{11}.\det C = \det A_{22}.\det B. \tag{2.19}$$

3. Main analytical results

The main purpose of this contribution is to find analytical formulae for the inverse of a general periodic tridiagonal matrices (1.1). It is formulated in the next theorem:

Theorem 3.1. *The elements of* $J^{-1}(d_1, c_n; d_k, a_k, c_k; a_1, a_n) = (q_{ij})$, $1 \leqslant i, j \leqslant n$ *can be expressed as*

$$q_{ij} = \frac{1}{\Delta}\begin{cases} (-1)^{i+j}[\Delta_1^{i-1}\Delta_{j+1}^{n} + d_1c_n\Delta_2^{i-1}\Delta_{j+1}^{n-1}]\prod_{k=i}^{j-1} c_k + \Delta_{i+1}^{j-1}\prod_{k=j+1}^{n} d_k \prod_{k=1}^{i} d_k & \text{for } i \leqslant j, \\ [\Delta_1^{j-1}\Delta_{i+1}^{n} + d_1c_n\Delta_2^{j-1}\Delta_{i+1}^{n-1}]\prod_{k=j+1}^{i} d_k + (-1)^{i+j+n}\Delta_{j+1}^{i-1}\prod_{k=1}^{j-1} c_k \prod_{k=i}^{n} c_k & \text{for } i \geqslant j, \end{cases} \tag{3.1}$$

where

$$\Delta = a_n\Delta_1^{n-1} + d_1c_n\Delta_2^{n-1} + d_nc_{n-1}\Delta_1^{n-2} - (-1)^n\prod_{k=1}^{n} c_k - \prod_{k=1}^{n} d_k \tag{3.2}$$

with $\prod_{l+1}^{l}(.) = 1$, *and the sequence of determinants* $\{\Delta_i^j\}$ *is computed by the three-term recurrence relations* (3.4), (3.5) *and* (3.6).

Before giving the proof of Theorem 3.1, we first state and prove two auxiliary lemmas which establish relationships between the elements of the sequence $\{\Delta_i^j\}$.

3.1. Relationships between two sequences of determinants

The determinant Δ_i^j can be written as

$$\Delta_i^j = \begin{array}{c} \\ i \\ i+1 \\ i+2 \\ \vdots \\ k \\ \vdots \\ j-1 \\ j \end{array} \begin{array}{c} \begin{array}{cccccccc} i & i+1 & i+2 & \cdots & k & \cdots & j-1 & j \end{array} \\ \left|\begin{array}{cccccccc} a_i & c_i & 0 & \cdots & & & & 0 \\ -d_{i+1} & a_{i+1} & c_{i+1} & & & & & \vdots \\ 0 & \ddots & \ddots & \ddots & & & & \\ \vdots & & & & & & & \\ & & & -d_k & a_k & c_k & & \\ & & & & \ddots & \ddots & \ddots & 0 \\ & & & & & -d_{j-1} & a_{j-1} & c_{j-1} \\ 0 & \cdots & & & & 0 & -d_j & a_j \end{array}\right| \end{array}, \quad j > i \tag{3.3}$$

which has the following recurrence relations, $i < j$:

$$\Delta_i^j = a_j \Delta_i^{j-1} + d_j c_{j-1} \Delta_i^{j-2}, \tag{3.4}$$

and

$$\Delta_i^j = a_i \Delta_{i+1}^j + d_{i+1} c_i \Delta_{i+2}^j, \tag{3.5}$$

with

$$\Delta_i^j = \begin{cases} a_i, & i = j, \\ 1, & i = j+1, \\ 0, & i \geqslant j+2. \end{cases} \tag{3.6}$$

Following, for example, [6] (see also [18,20]), it is easy to verify that

$$\Delta_i^j = \prod_{k=j}^{j} \gamma_k \quad \text{and} \quad \Delta_i^j = \prod_{k=i}^{j} \delta_k, \tag{3.7}$$

where

$$\gamma_k = \begin{cases} a_i, & k = i, \\ a_k + \frac{c_{k-1} d_k}{\gamma_{k-1}}, & k = i+1, i+2, \ldots, j, \end{cases} \tag{3.8}$$

$$\delta_k = \begin{cases} a_j, & k = j, \\ a_k + \frac{c_k d_{k+1}}{\delta_{k+1}}, & k = j-1, j-2, \ldots, i. \end{cases} \tag{3.9}$$

Lemma 3.1.

$$\Delta_i^j = \Delta_i^k \Delta_{k+1}^j + c_k d_{k+1} \Delta_i^{k-1} \Delta_{k+2}^j, \quad i \leqslant k \leqslant j. \tag{3.10}$$

Proof. Expanding the determinant Δ_i^j in terms of elements of the kth row, $i \leqslant k \leqslant j$ leads to the relation

$$\Delta_i^j = \Delta_{k+1}^j \left(a_k \Delta_i^{k-1} + c_{k-1} d_k \Delta_i^{k-2}\right) + c_k d_{k+1} \Delta_i^{k-1} \Delta_{k+2}^j.$$

Using formula (3.4), formula (3.10) immediately follows. □

Formula (3.10) agrees with that of [1,11].

Note that formula (3.10) can be expressed as the following alternating form

$$\Delta_1^j = \Delta_1^{i-1} \Delta_i^j + c_{i-1} d_i \Delta_1^{i-2} \Delta_{i+1}^j. \tag{3.11}$$

Lemma 3.2.

$$\Delta_1^k \Delta_i^n = \Delta_i^k \Delta_1^n - (-1)^{k-i-1} \Delta_1^{i-2} \Delta_{k+2}^n \prod_{l=i-1}^{k} c_l \prod_{l=i}^{k+1} d_l, \quad 1 \leqslant i, j \leqslant n. \tag{3.12}$$

Proof. For $i = 1$, $j = n$, formula (3.10) becomes

$$\Delta_1^n = \Delta_1^k \Delta_{k+1}^n + c_k d_{k+1} \Delta_1^{k-1} \Delta_{k+2}^n. \tag{3.13}$$

On multiplying the both sides of formula (3.15) with Δ_i^k, we get

$$\Delta_1^n \Delta_i^k = \Delta_i^k \Delta_1^k \Delta_{k+1}^n + c_k d_{k+1} \Delta_i^k \Delta_1^{k-1} \Delta_{k+2}^n. \tag{3.14}$$

It is easy to see, after adding and subtracting the term $c_k d_{k+1} \Delta_i^{k-1} \Delta_{k+2}^n \Delta_1^k$ to the right-hand side of (3.14) and using formula (3.13), that

$$\Delta_1^n \Delta_i^k = \Delta_i^n \Delta_1^k + c_k d_{k+1} \Delta_{k+2}^n \left(\Delta_i^k \Delta_1^{k-1} - \Delta_i^{k-1} \Delta_1^k\right). \tag{3.15}$$

Using formula (3.11), we get

$$\Delta_i^k \Delta_1^{k-1} - \Delta_i^{k-1} \Delta_1^k = c_{i-1} d_i \Delta_1^{i-2} \left(\Delta_i^k \Delta_{i+1}^{k-1} - \Delta_i^{k-1} \Delta_{i+1}^k\right). \tag{3.16}$$

Using formula (3.5). Formula (3.16) becomes

$$\Delta_i^k \Delta_1^{k-1} - \Delta_i^{k-1} \Delta_1^k = -c_{i-1} c_i d_i d_{i+1} \Delta_1^{i-2} \left(\Delta_{i+1}^k \Delta_{i+2}^{k-1} - \Delta_{i+1}^{k-1} \Delta_{i+2}^k\right). \tag{3.17}$$

Using formula (3.5) repeatedly, we obtain

$$\begin{aligned}\Delta_i^k\Delta_1^{k-1}-\Delta_i^{k-1}\Delta_1^k&=\prod_{l=i-1}^{i+1}c_l\prod_{l=i}^{i+2}d_l\Delta_1^{i-2}\left[\Delta_{i+3}^{k-1}\Delta_{i+2}^k-\Delta_{i+3}^k\Delta_{i+2}^{k-1}\right]=\cdots\\&=(-1)^{k-i-1}\prod_{l=i-1}^{k-2}c_l\prod_{l=i}^{k-1}d_l\Delta_1^{i-2}\left(\Delta_k^{k-1}\Delta_{k-1}^k-\Delta_k^k\Delta_{k-1}^{k-1}\right)\\&=(-1)^{k-i-1}\prod_{l=i-1}^{k-1}c_l\prod_{l=i}^{k}d_l\Delta_1^{i-2}.\end{aligned}\tag{3.18}$$

Insertion of formula (3.18) into formula (3.15) leads to the formula (3.12). □

Some other relationships between the sequence of determinants of specific submatrices are collected in the following corollary.

Corollary 3.1.

$$\frac{\Delta_1^{n-2}\Delta_{l+1}^{n-1}}{\Delta_1^{n-1}}=\Delta_{l+1}^{n-2}-(-1)^{n-l-4}\frac{\Delta_1^{l-1}}{\Delta_1^{n-1}}\prod_{r=l}^{n-2}c_r\prod_{r=l+1}^{n-1}d_r,\tag{3.19}$$

$$\frac{\Delta_2^{n-1}\Delta_1^{l-1}}{\Delta_1^{n-1}}=\Delta_2^{l-1}-(-1)^{l-4}\frac{\Delta_{l+1}^{n-1}}{\Delta_1^{n-1}}\prod_{r=1}^{l-1}c_r\prod_{r=2}^{l}d_r,\tag{3.20}$$

$$\frac{\Delta_1^{l-1}\Delta_{k+1}^{n-1}}{\Delta_1^{n-1}}=\Delta_{k+1}^{l-1}-(-1)^{l-k-3}\frac{\Delta_1^{k-1}\Delta_{l+1}^{n-1}}{\Delta_1^{n-1}}\prod_{r=k}^{l-1}c_r\prod_{r=k+1}^{l}d_r.\tag{3.21}$$

Proof of Theorem 3.1. Let $J(d_1,c_n;d_k,a_k,c_k;a_1,a_n)$ be partitioned in the form

$$J=\left[\begin{array}{cccccc|c}a_1&c_1&0&\cdots&&0&-d_1\\-d_2&a_2&c_2&&&&0\\0&\ddots&\ddots&\ddots&&&\vdots\\\vdots&&\ddots&\ddots&\ddots&&0\\0&&&-d_{n-1}&a_{n-1}&&c_{n-1}\\\hline c_n&0&\cdots&0&-d_n&&a_n\end{array}\right]\equiv\left[\begin{array}{c|c}J_{11}&J_{12}\\\hline J_{21}&J_{22}\end{array}\right]\tag{3.22}$$

with $J_{22}\equiv$ the single element a_n, $J_{11}\equiv(n-1)\times(n-1)$ matrix, $J_{12}\equiv(n-1)\times1$ column vector, $J_{21}\equiv1\times(n-1)$ row vector, where

$$J_{11}=\begin{bmatrix}a_1&c_1&0&\cdots&0&0\\-d_2&a_2&c_2&&&0\\0&\ddots&\ddots&\ddots&&\vdots\\\vdots&&\ddots&\ddots&\ddots&0\\0&&&-d_{n-2}&a_{n-2}&c_{n-2}\\0&0&\cdots&0&-d_{n-1}&a_{n-1}\end{bmatrix}_{(n-1\times n-1)},$$

$$J_{12}'=[\,-d_1\quad0\quad\cdots\quad0\quad c_{n-1}\,],\quad\text{and}\quad J_{21}=[\,c_n\quad0\quad\cdots\quad0\quad-d_n\,].$$

We then have that the elements of $J_{11}^{-1}=(L_{ij})_{n-1\times n-1}$, are the same as that given as in Lemma 2.1 with the indices i and j range from 1 to $n-1$. If

$$J^{-1}(d_1,c_n;d_k,a_k,c_k;a_1,a_n)=\left[\begin{array}{c|c}C_{11}&C_{12}\\\hline C_{21}&C_{22}\end{array}\right],\tag{3.23}$$

is partitioned conformally, we then have that, from Lemma 2.2, C_{22} is invertible, and the Schur complement of J_{11} is given by

$$J_{22} - J_{21} J_{11}^{-1} J_{12}$$
$$= a_n - \left\{ c_n \left[c_{n-1} \frac{\Delta_1^0 \Delta_n^{n-1}}{\Delta_1^{n-1}} (-1)^n \prod_{l=1}^{n-2} c_l - d_1 \frac{\Delta_1^0 \Delta_2^{n-1}}{\Delta_1^{n-1}} \prod_{l=2}^{1} d_l \right] - d_n \left[c_{n-1} \frac{\Delta_1^{n-2} \Delta_n^{n-1}}{\Delta_1^{n-1}} \prod_{l=n-1}^{n-2} c_l - d_1 \frac{\Delta_1^0 \Delta_n^{n-1}}{\Delta_1^{n-1}} \prod_{l=2}^{n-1} d_l \right] \right\}$$
$$= \frac{\Delta}{\Delta_1^{n-1}} \equiv C_{22}^{-1}, \tag{3.24}$$

where Δ is the same as that given in formula (3.2).

To find the elements of the matrix $C_{11} = (c_{ij})$, $1 \leqslant i, j \leqslant n-1$, we first calculate the elements of the column vector $J_{11}^{-1} J_{12}$. It is easy to verify that

$$J_{11}^{-1} J_{12} = \frac{1}{\Delta_1^{n-1}} \left((-1)^{i+n-1} \Delta_1^{i-1} \prod_{k=i}^{n-1} c_k - \Delta_{i+1}^{n-1} \prod_{k=1}^{i} d_k \right)_{1 \leqslant i \leqslant n-1}. \tag{3.25}$$

Secondary, we calculate the elements of the row vector $J_{21} J_{11}^{-1}$, it is easy to verify that

$$J_{21} J_{11}^{-1} = \frac{1}{\Delta_1^{n-1}} \left((-1)^{1+j} \Delta_{j+1}^{n-1} c_n \prod_{k=1}^{j-1} c_k - \Delta_1^{j-1} \prod_{k=j+1}^{n} d_k \right)_{1 \leqslant j \leqslant n-1}. \tag{3.26}$$

By formula (2.17), and using formulae (3.24), (3.25), (3.26) and Lemma 2.1 with the indices i and j range from 1 to $n-1$, we have

$$c_{ij} = \frac{1}{\Delta_1^{n-1} \Delta} \begin{cases} \{[a_n \Delta_1^{n-1} \Delta_1^{i-1} \Delta_{j+1}^{n-1} + d_1 c_n \Delta_2^{n-1} \Delta_1^{i-1} \Delta_{j+1}^{n-1} + d_n c_{n-1} \Delta_1^{n-2} \Delta_1^{i-1} \Delta_{j+1}^{n-1}](-1)^{i+j} \prod_{k=i}^{j-1} c_k \\ \quad + (-1)^{i+j+n} \Delta_1^{i-1} \Delta_{j+1}^{n-1} \prod_{k=1}^{n} c_k \prod_{k=i}^{j-1} c_k - (-1)^{1+j} \Delta_{i+1}^{n-1} \Delta_{j+1}^{n-1} c_n \prod_{k=1}^{j-1} c_k \prod_{k=1}^{i} d_k \\ \quad + (-1)^{i+j+1} \Delta_1^{i-1} \Delta_{j+1}^{n} \prod_{k=i}^{j-1} c_k \prod_{k=1}^{n} d_k + (-1)^{i+j+n} \Delta_1^{i-1} \Delta_{j+1}^{n-1} c_n \prod_{k=1}^{j-1} c_k \prod_{k=i}^{n-1} d_k \\ \quad - (-1)^{i+n-1} \Delta_1^{i-1} \Delta_1^{j-1} \prod_{k=i}^{n-1} c_k \prod_{k=j+1}^{n} d_k + \Delta_{i+1}^{n-1} \Delta_1^{j-1} \prod_{k=1}^{i} d_k \prod_{k=j+1}^{n} d_k\}, \quad i \leqslant j, \\ \{[a_n \Delta_1^{n-1} \Delta_1^{j-1} \Delta_{i+1}^{n-1} + d_1 c_n \Delta_2^{n-1} \Delta_1^{j-1} \Delta_{i+1}^{n-1} + d_n c_{n-1} \Delta_1^{n-2} \Delta_1^{j-1} \Delta_{i+1}^{n-1}] \prod_{k=j+1}^{i} d_k \\ \quad - (-1)^n \Delta_1^{j-1} \Delta_{i+1}^{n-1} \prod_{k=1}^{n} c_k \prod_{k=j+1}^{i} d_k + (-1)^{i+j+n} \Delta_1^{i-1} \Delta_{j+1}^{n-1} \prod_{k=1}^{j-1} c_k \prod_{k=i}^{n} c_k \\ \quad - (-1)^{1+j} \Delta_{i+1}^{n-1} \Delta_{j+1}^{n-1} c_n \prod_{k=1}^{j-1} c_k \prod_{k=1}^{i} d_k - (-1)^{i+n-1} \Delta_1^{i-1} \Delta_1^{j-1} \prod_{k=i}^{n-1} c_k \prod_{k=j+1}^{n} d_k\}, \quad i \geqslant j. \end{cases} \tag{3.27}$$

Using Corollary 3.1, after simplification, formula (3.27) becomes

$$c_{ij} = \frac{1}{\Delta} \begin{cases} (-1)^{i+j} [\Delta_1^{i-1} (a_n \Delta_{j+1}^{n-1} + d_n c_{n-1} \Delta_{j+1}^{n-2}) + d_1 c_n \Delta_2^{i-1} \Delta_{j+1}^{n-1}] \prod_{k=i}^{j-1} c_k + \Delta_{i+1}^{j-1} \prod_{k=j+1}^{n} d_k \prod_{k=1}^{i} d_k \\ \quad \text{for } i \leqslant j, \\ [\Delta_1^{j-1} (a_n \Delta_{i+1}^{n-1} + d_n c_{n-1} \Delta_{i+1}^{n-2}) + d_1 c_n \Delta_2^{j-1} \Delta_{i+1}^{n-1}] \prod_{k=j+1}^{i} d_k + (-1)^{i+j+n} \Delta_{j+1}^{i-1} \prod_{k=1}^{j-1} c_k \prod_{k=i}^{n} c_k \\ \quad \text{for } i \geqslant j, \end{cases} \tag{3.28}$$

where Δ is the same as that given in formula (3.2).

Observe that the elements of the vectors $C_{12} = (\alpha_j)_j$, $C_{21} = (\beta_j)_j$, and C_{22} in (3.23), are obtained from the elements of the matrix $C_{11} = (c_{ij})$, by replacing, j by n in the case $i \leqslant j$; i by n in the case $i \geqslant j$, and i, j by n in the case $i \leqslant j$ or $i \geqslant j$, respectively. Finally, we get

$$\alpha_j = \frac{1}{\Delta} \left((-1)^{1+j} \Delta_{j+1}^{n-1} c_n \prod_{k=1}^{j-1} c_k - \Delta_1^{j-1} \prod_{k=j+1}^{n} d_k \right), \tag{3.29}$$

$$\beta_j = \frac{1}{\Delta} \left((-1)^{1+j} \Delta_{j+1}^{n-1} c_n \prod_{k=1}^{j-1} c_k - \Delta_1^{j-1} \prod_{k=j+1}^{n} d_k \right), \tag{3.30}$$

$$C_{22} = \frac{\Delta_1^{n-1}}{\Delta}, \tag{3.31}$$

where Δ is the same as that given in formula (3.2). Hence the theorem is proved. □

This result agrees with that obtained by Cichocki and Unbehauen [4] using different approaches based on the coats flow graph technique of a linear system.

As a simple consequence of Theorem 3.1, we consider a perturbed Toeplitz periodic tridiagonal matrices $J(d_1, c_n; d, a, c; a_1, a_n)$, in which the boundary elements a_1, d_1, c_n and a_n are different from the remaining elements, i.e., $a_i = a$ for $i = 2, 3, \ldots, n-1$, $a_1, a_n \neq a$; $d_i = d$ for $i = 2, 3, \ldots, n$; $d_1 \neq d$ and $c_i = c$ for $i = 1, 3, \ldots, n-1$, $c_n \neq c$

$$J(d_1, c_n; d, a, c; a_1, a_n) = \begin{bmatrix} a_1 & c & 0 & \cdots & 0 & -d_1 \\ -d & a & c & \ddots & & 0 \\ 0 & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \ddots & & & & 0 \\ 0 & & & -d & a & c \\ c_n & 0 & \cdots & 0 & -d & a_n \end{bmatrix}. \tag{3.32}$$

For this matrix, we have the following corollary.

Corollary 3.2. *The elements of $J^{-1}(d_1, c_n; d, a, c; a_1, a_n) = (u_{ij})$, $1 \leqslant i, j \leqslant n$, can be expressed in the following form*

$$u_{ij} = \frac{1}{\Delta_1} \begin{cases} (-c)^{j-i}[\Delta_1^{i-1}\Delta_{j+1}^{n} + d_1 c_n \Delta_2^{i-1}\Delta_{j+1}^{n-1}] + d_1 \Delta_{i+1}^{j-1} d^{n-j+i-1} & \text{for } i \leqslant j, \\ [\Delta_1^{j-1}\Delta_{i+1}^{n} + d_1 c_n \Delta_2^{j-1}\Delta_{i+1}^{n-1}] d^{i-j} - c_n \Delta_{j+1}^{i-1} c^{n+j-i-1} & \text{for } i \geqslant j, \end{cases} \tag{3.33}$$

where

$$\Delta_1 = d_1 c_n \Delta_2^{n-1} - d_1 d^{n-1} + (-1)^{n+1} c_n c^{n-1} + \bigl[h_{n+1} + (a_1 + a_n - 2a)h_n + (a_n - a)(a_1 - a)h_{n-1}\bigr]. \tag{3.34}$$

For $\lambda_1 \neq \lambda_2$, $\lambda_1^k - \lambda_2^k = h_k(\lambda_1 - \lambda_2)$, $\lambda_{1,2} = \frac{1}{2}[a \pm \sqrt{a^2 + 4dc}\,]$,

$$\Delta_k^l = h_{l-k+2} + (a_k + a_l - 2a)h_{l-k+1} + (a_n - a_l)(a_1 - a_k)h_{l-k}, \quad 1 \leqslant k, l \leqslant n. \tag{3.35}$$

For $\lambda_1 = \lambda_2 = \frac{a}{2}$, $1 \leqslant k, l \leqslant n$

$$\Delta_k^l = \left(\frac{a}{2}\right)^{l-k-1} \left\{(l-k)(a_1 - a_k)(a_n - a_l) + (l-k+1)(a_k + a_l - 2a)\left(\frac{a}{2}\right) + (l-k+2)\left(\frac{a}{2}\right)^2\right\}. \tag{3.36}$$

In the case when $d_1 = 0$, $c_n = 0$, the elements of $J^{-1}(0, 0; d, a, c; a_1, a_n) = (u_{ij})$, $1 \leqslant i, j \leqslant n$, become

$$u_{ij} = \frac{1}{\Delta_1} \begin{cases} (-c)^{j-i}\Delta_1^{i-1}\Delta_{j+1}^{n} & \text{for } i \leqslant j, \\ \Delta_1^{j-1}\Delta_{i+1}^{n} d^{i-j} & \text{for } i \geqslant j, \end{cases} \tag{3.37}$$

where

$$\Delta_1 = h_{n+1} + (a_1 + a_n - 2a)h_n + (a_n - a)(a_1 - a)h_{n-1} \tag{3.38}$$

and Δ_k^l defined as in (3.35) *and* (3.36)*.*

It is straightforward to show that the present results are generalized some well known results, see for example [14,15,17,20–22] and [23].

4. Illustrative examples

To illustrate the advantages of the proposed approach and the usefulness of the formulae (3.1), (3.2) we present a number of special cases (examples) and discuss some numerical results.

Example 4.1. In the numerical solution of some partial differential equations which arise in electromagnetic field theory there occurs the need to invert the following matrix

$$J(pr, 1+qr, r) = \begin{bmatrix} 1+qr & r & 0 & \cdots & 0 & 0 \\ -pr & 1+qr & r & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & & \\ \vdots & \ddots & & & & 0 \\ & & & -pr & 1+qr & r \\ 0 & 0 & \cdots & 0 & -pr & 1+qr \end{bmatrix}. \tag{4.1}$$

Putting $a_1 = a_n = a = 1 + qr$, $d_1 = c_n = 0$, $c = r$ and $d = pr$ in Corollary 3.2, we obtain immediately the elements of $J^{-1}(pr, 1 + qr, r) = (v_{ij})$, $1 \leqslant i, j \leqslant n$, given by

$$v_{ij} = \frac{1}{h_{n+1}} \begin{cases} h_i h_{n-j+1}(-r)^{j-i} & \text{for } i \leqslant j, \\ h_j h_{n-i+1}(pr)^{i-j} & \text{for } i \geqslant j, \end{cases} \tag{4.2}$$

for $\lambda_1 \neq \lambda_2$, $\lambda_1^k - \lambda_2^k = h_k(\lambda_1 - \lambda_2)$, $\lambda_{1,2} = \frac{1+qr}{2} \pm \sqrt{(\frac{1+qr}{2})^2 + pr^2}$, and

$$v_{ij} = \frac{1}{n+1} \begin{cases} i(n-j+1)(-\frac{2}{1+qr})^{j-i} & \text{for } i \leqslant j, \\ j(n-i+1)(\frac{2pr}{1+qr})^{i-j} & \text{for } i \geqslant j, \end{cases} \tag{4.3}$$

for $\lambda_1 = \lambda_2 = \frac{1}{2}(1 + qr)$.

Note that, in the special case when $p < 0$ the inverse can be expressed by, $U_n(x)$, the nth Chebyshev polynomials of the second kind, given by

$$v_{ij} = \frac{|p|^{(j-i-1)/2}}{rU_n(x)} \begin{cases} U_{i-1}(x)U_{n-j}(x) & \text{for } i \leqslant j, \\ U_{j-1}(x)U_{n-i}(x) & \text{for } i \geqslant j, \end{cases} \tag{4.4}$$

where $x = \frac{1+qr}{2r\sqrt{|p|}}$.

Example 4.2. Consider the following periodic matrix $J(1, 1; 1, a, 1; 1, 1)$

$$J(1,1;1,a,1;1,1) = \begin{bmatrix} a & -1 & 0 & \cdots & 0 & -1 \\ -1 & a & -1 & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & & \\ \vdots & \ddots & & & & 0 \\ & & & -1 & a & -1 \\ -1 & 0 & \cdots & 0 & -1 & a \end{bmatrix}. \tag{4.5}$$

It is easy verify that the elements of $J^{-1}(1, 1; 1, a, 1; 1, 1) = (\theta_{ij})$, $1 \leqslant i, j \leqslant n$, by putting $a_1 = a_n = a$, $d_1 = d = 1$ and $c_n = c = -1$ in Corollary 3.2, are:

$$\theta_{ij} = \frac{1}{h_{n+1} + h_{n-1} - 2} \begin{cases} h_{n-j+1}h_i - h_{i-1}h_{n-j} + h_{j-i} & \text{for } i \leqslant j, \\ h_{n-i+1}h_j + h_{j-1}h_{n-i} - (-1)^{n+j-i}h_{i-j} & \text{for } i \geqslant j, \end{cases} \tag{4.6}$$

for $\lambda_1 \neq \lambda_2$, where $\lambda_{1,2} = \frac{1}{2}[a \pm \sqrt{a^2 - 4}]$, $\lambda_1^k - \lambda_2^k = h_k(\lambda_1 - \lambda_2)$, and

$$\theta_{ij} = \frac{1}{\delta} \begin{cases} i(n-j+1)(\frac{a}{2})^{n+i-j-1} - (i-1)(n-j)(\frac{a}{2})^{n+i-j-3} + (j-i)(\frac{a}{2})^{j-i-1} & \text{for } i \leqslant j, \\ j(n-i+1)(\frac{a}{2})^{n+j-i-1} + (j-1)(n-i)(\frac{a}{2})^{n+j-i-3} - (-1)^{n+j-i}(j-i)(\frac{a}{2})^{j-i-1} & \text{for } i \geqslant j, \end{cases} \tag{4.7}$$

for $\lambda_1 = \lambda_2 = \frac{a}{2}$, where $\delta = (n+1)(\frac{a}{2})^n - (n-1)(\frac{a}{2})^{n-2} - 2$.

It should be noted that this example has been previously studied by several authors; see, for instance, [11,18] and [22].

4.1. Algorithm for general periodic tridiagonal matrix

In this subsection, we will develop an efficient computational algorithm for inverting the general periodic tridiagonal matrices (1.1):

Algorithm 4.1. To find the $n \times n$ inverse matrix of a general periodic tridiagonal matrices $A \equiv J(d_1, c_n; d_k, a_k, c_k; a_1, a_n)$ of the form (1.1), we may proceed as follows:

INPUT Order of the matrix n and the components d_k, a_k, c_k, $k = 1, 2, \ldots, n$.
OUTPUT The entries q_{ij}, $1 \leqslant i, j \leqslant n$ of the matrix inversion $A^{-1} \equiv J^{-1}(d_1, c_n; d_k, a_k, c_k; a_1, a_n)$.
Step 1 Designing a function for computing the product of any series S from i to j and name it $P(S, i, j)$ as follows:
Step 1-1 if $i < j$, then $P(S, i, j) = 1$,
Step 1-2 set $f = 1$,
Step 1-3 for k from i to j set $f = f * S(k)$,
Step 1-4 set $P(C, i, j) = f$.

Step 2 Designing a function for computing the continuants Δ_i^j and name it $C(A,i,j)$ as follows:
Step 2-1 for k from 1 to n, set $a(k)=A(k,k)$, $c(k)=A(k,k+1)$, $d(k+1)=-A(k+1,k)$,
Step 2-2 set $G(1)=a(i)$,
Step 2-3 set $G(1)=x$ whenever $G(1)=0$,
Step 2-4 for k from 2 to $j-i+1$, set $G(k)=a(i+k-1)+c(i+k-2)*d(i+k-1)/G(k-1)$,
Step 2-5 set $G(k)=x$ whenever $G(k)=0$ for any $k=2,3,\dots,j-i$,
Step 2-6 set $C(A,i,j)=P(G,1,j-i+1)$.
Step 3 Set $D1=C(A,1,n)+d(1)*c(n)*C(A,2,n-1)-P(c,1,n)*(-1)\hat{}n-P(d,1,n)$.
Step 4 If the $D1=0$, then OUTPUT ('no inverse exist'); STOP.
Step 5 For i from 1 to n, for j from 1 to n:
Step 5-1 if $i\leqslant j$, then set $q(i,j)=((C(A,j+1,n)*C(A,1,i-1)+d(1)*c(n)*C(A,2,i-1)*C(A,j+1,n-1))*P(c,i,j-1)*(-1)\hat{}(i+j)+C(A,i+1,j-1)*P(d,j+1,n)*P(d,1,i))/D1$,
Step 5-2 else, set $q(i,j)=((C(A,i+1,n)*C(A,1,j-1)+d(1)*c(n)*C(A,2,j-1)*C(A,i+1,n-1))*P(d,j+1,i)+((-1)\hat{}(n+i+j))*C(A,j+1,i-1)*P(c,1,j-1)*P(c,i,n))/D1$.
Step 6 OUTPUT the inverse matrix $A^{-1}=(q_{ij})$, $1\leqslant i,j\leqslant n$.

Appendix A. A Maple procedure for inverting the general periodic tridiagonal matrices

```
> # A Maple procedure for inverting the periodic general tridiagonal matrices
> restart:
> with(LinearAlgebra);
> # A function for computing the product of any Series C from i to j
> pro:=proc(C,i,j)
> local M,k:
> if j<i then M:=1
> else
> M:=1:
> for k from i to j do
> M:=M*C[k]:
> end do
> end if:
> eval(M);
> end proc:
> # A function for computing the contiuant of the matrix from i to j
> continuant:=proc(A,i,j)
> local k,a,c,d,G,M,n:n:=RowDimension(A):
> a:=Array(1..n):c:=Array(1..n-1): d:=Array(2..n):
> if i=j+1 then M:=1
> elif i=j+2 then M:=0
> else
> G:=Array(1..j-i+1):
> for k from 1 to n do
> a[k]:=A[k,k]:
> end do:
> for k from 1 to n-1 do
> c[k]:=A[k,k+1]:
> d[k+1]:=-A[k+1,i]:
> end do:
> G[1]:=a[i]:
> if G[1]=0 then G[1]:=x end if:
> for k from 2 to j-i+1 do
> G[k]:=a[i+k-1]+c[i+k-2]*d[i+k-1]/G[k-1]:
> if G[k]=0 then G[k]:=x end if:
> end do:
> M:=1:
> for k from 1 to j-i+1 do
> M:=M*G[k]:
> end do:
> end if:
> eval(M);
> end proc:
```

```
> # The main program
> ptdmi:=proc(A)
> local k, l,a, c, d, F, O, s, n:
> n:=RowDimension(A):
> a:=Array(1..n):c:=Array(1..n):d:=Array(1..n): s:=Matrix(1..n,1..n):
> for k from 1 to n do
> a[k]:=A[k,k]:
> end do:
> for k from 1 to n-1 do
> c[k]:=A[k,k+1]:
> end do:
> for k from 2 to n do
> d[k]:=-A[k,k-1]:
> end do:
> c[n]:=A[n,1]:
> d[1]:=-A[1,n]:
> D1:=continuant(A,1,n)+d[1]*c[n]*continuant(A,2,n-1)
    -pro(c,1,n)*(-1)^n-pro(d,1,n);
> if D1=0 then
> print ("singular matrix, no inverse");
> break:
> end if:
> for k from 1 to n do
> for l from 1 to n do
> if k<= l then
> s[k,l]:=(continuant(A,l+1,n)*continuant(A,1,k-1)+d[1]*c[n]
   *continuant(A,2,k-1)*continuant(A,l+1,n-1))*pro(c,k,l-1)*(-1)^(k+l)
   +continuant(A,k+1,l-1)*pro(d,l+1,n)*pro(d,1,k):
> else
> s[k,l]:=(continuant(A,k+1,n)*continuant(A,1,l-1)+d[1]*c[n]
   *continuant(A,2,l-1)*continuant(A,k+1,n-1))*pro(d,l+1,k)+((-1)^(n+k+l))
   *continuant(A,l+1,k-1)*pro(c,1,l-1)*pro(c,k,n):
> end if
> end do
> end do:
> print("A=",A);print("Inverse of A=",'1/D'(subs(x=0,simplify(s))));
> print("D="(subs(x=0,simplify(D1))));
> end proc:
```

Consider the following matrix

$$A=\begin{bmatrix} a & 1 & 0 & -1 \\ -1 & a & 1 & 0 \\ 0 & -1 & a & 1 \\ 1 & 0 & -1 & a \end{bmatrix}$$

to use the last procedure, just define our matrix as follows

```
> A := Matrix(1..4,1..4,[[a,1,0,-1],[-1,a,1,0],[0,-1,a,1],[1,0,-1,a]]);
> ptdmi(A);
```

Then we can get the following result

$$\text{Inverse of } A=\frac{1}{D}\begin{bmatrix} a(a^2+2) & -a^2 & 2a & a^2 \\ a^2 & a(a^2+2) & -a^2 & 2a \\ 2a & a^2 & a(a^2+2) & -a^2 \\ -a^2 & 2a & a^2 & a(a^2+2) \end{bmatrix}, \qquad D=a^2\left(a^2+a\right).$$

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