On module variety dimension for coverings and degenerations of algebras

Piotr Dowbor *, Adam Hajduk (Toruń)
Faculty of Mathematics and Computer Science, Nicolaus Copernicus University, Chopina 12/18, 87–100 Toruń, Poland

A R T I C L E  I N F O

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A B S T R A C T

Given a pair of \( G \)-covering functors \( F^1 : R \to R_1 \) and \( F^0 : R \to R_0 \) such that \( F^0 \) is a Galois covering, the inequality \( \dim \text{mod}^1_{R_0}(z, t) \leq \dim \text{mod}^1_{R_1}(z, t) \) for all \( z, t \), of the dimensions of the first kind module sets under some assumptions is proved (Theorem 2.2). The result is applied to show the equality of the module variety dimensions for some special degenerations of algebras. Certain consequences for preserving wild and tame representation types by \( G \)-covering functors are also presented (Theorems 2.4 and 3.1).

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0. Introduction

One of the most important and classical problems of modern representation theory of finite dimensional algebras is to classify all indecomposable representations for an algebra \( A \). Generally this problem is difficult and often it is replaced by a simpler one, namely, that of determining the representation type for \( A \). The natural way of studying the representation type for an individual algebra \( A \) requires usually a detailed knowledge of the structure of the category \( \text{mod} A \) formed by all finite dimensional \( A \)-modules, in particular, a description of the indecomposable objects in \( \text{mod} A \). However, there also exists an alternative approach based on the geometric characterization due to Geiss and de la Peña [23,24,28], which is phrased in terms of certain dimension estimations for module varieties \( \text{mod} A(z, t) \), for \( z, t \in \mathbb{N} \) (cf. Proposition 1.3).

This approach was applied by Geiss and Crawley-Boevey for proving that if an algebra \( A \) admits a (geometric) degeneration \( \tilde{A} \), which is tame, then \( A \) is tame itself (see [24,6]; cf. [21]). More precisely, they showed, that for an algebra \( \tilde{A} \) being a degeneration of \( A \) the inequalities \( \dim \text{mod} A(z, t) \leq \dim \text{mod} \tilde{A}(z, t) \) hold for all \( z, t \in \mathbb{N} \). In this paper we discuss when these dimensions are equal, or in other words, when \( A \) and \( \tilde{A} \) can be of the same representation type.

We study this question in the context of coverings — a powerful and efficient tool, which allowed to solve many important problems of contemporary representation theory (see e.g. [19,20,17,29,32,33]). Connections between these two notions are rather natural, especially, for the representation finite case (see [3, 5.2]). It happens quite often that a degeneration of algebras, \( A \) to \( \tilde{A} \) (indicated symbolically by \( A \leadsto \tilde{A} \) below) can be “build into the triangle” of the form

\[
\begin{array}{ccc}
A & \sim & \tilde{A} = \tilde{A}/G \\
\downarrow & & \downarrow \\
\tilde{A} & \sim & \tilde{A}/G \\
\end{array}
\]

(\( \Delta \))

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* Corresponding author.

E-mail addresses: dowbor@mat.uni.torun.pl (P. Dowbor), ahajduk@mat.uni.torun.pl (A. Hajduk).

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where $\tilde{A}$ is a locally bounded $k$-category equipped with a free action of the group $G \subseteq \text{Aut}_k(\tilde{A})$ on $\text{ob} \tilde{A}$ such that $\tilde{A}$ is isomorphic to the (quotient) orbit category $\tilde{A}/G$ of $\tilde{A}$ by $G$, $\tilde{F} : \tilde{A} \rightarrow \tilde{A}/G$ is a Galois covering functor with the group $G$, given by the natural projection, and $F : \tilde{A} \rightarrow A$ is a certain covering functor, which commutes with automorphisms from $G$ only on objects. (Clearly, in (4.3) we identify algebras with finite locally bounded categories). Therefore, given a covering functor $F : \tilde{A} \rightarrow A$ as above, we prove a rather general result – Theorem 2.2 – comparing the dimensions of the subsets of the first kind modules $\text{mod}_A^1(\tilde{A}, t)$ and $\text{mod}_{\tilde{A}/G}^1(\tilde{A}, t)$ (of $\text{mod}_A^1(\tilde{A}, t)$ and $\text{mod}_{\tilde{A}/G}(\tilde{A}, t)$, respectively). As a conclusion of the proved inequalities, which are of the shape $\text{dim} \: \text{mod}_A^1(\tilde{A}, t) \geq \text{dim} \: \text{mod}_{\tilde{A}/G}^1(\tilde{A}, t)$, we obtain the expected equalities of dimension for the degenerations admitting the diagram (4.3), with $\tilde{A}$ being locally support finite (see Theorem 2.3). Moreover, for some special class of covering functors $F$, the so-called almost Galois covering functors of the integral type, we infer preserving of the tame and wild representation types by $F$, in some special cases (see Theorem 3.1, compare also Theorem 2.4 in a more general situation). We also discuss the geometric meaning of some technical condition appearing in the formulation of Theorem 3.1 (see Proposition 3.2).

1. Preliminaries and notation

We will use here rather the language of degenerations of locally bounded categories and varieties of modules over them, than the classical one for algebras. Therefore we briefly recall basic notions.

Throughout the paper $k$ will always denote an algebraically closed field.

1.1

Let $R$ be a $k$-category (each set $R(x, y)$ of morphisms from $x$ to $y$ in $R$, $x, y \in \text{ob} R$, is a $k$-linear space and the composition of morphisms in $R$ is $k$-bilinear). Then $R$ is called locally bounded, if all objects of $R$ have local endomorphism rings, the different objects are nonisomorphic, and the sums $\sum_{y \in R} \text{dim}_k R(x, y)$ and $\sum_{y \in R} \text{dim}_k R(y, x)$ are finite for each $x \in R$ (see [3,22]).

For any $R$ as above we denote always by $\mathcal{J}(R)$ the Jacobson radical of $R$. By an $R$-module $M$ we mean a $k$-linear contravariant functor from $R$ to the category of all $k$-vector spaces. We say that $M$ is locally finite-dimensional (resp., finite-dimensional) if all $k$-vector spaces $M(x), x \in \text{ob} R$, are finite dimensional (resp., additionally almost all of them are equal to the zero space). We denote by $\text{Mod}^\ast R$ the category of all $R$-modules and by $\text{Mod} R$ (resp., $\text{mod} R$) the full subcategories formed by all locally finite-dimensional (resp., finite dimensional) $R$-modules.

For any $M$ from $\text{Mod} R$ (resp., $\text{mod} R$), by the dimension vector (resp., dimension) of $M$ we mean the collection $\text{dim}_k M = (\text{dim}_k M(x))_{x \in \text{ob} R} \subseteq \mathbb{N}^{\text{ob} R}$ (resp., the integer $\text{dim}_k M = \sum_{x \in \text{ob} R} \text{dim}_k M(x) \in \mathbb{N}$) and by the support of $M$ the set supp$M = \{x \in \text{ob} R : M(x) \neq 0\}$. The category $\mathcal{J}(R)$ is called locally support finite, if for every $x \in \text{ob} R$, the union of all supp$M$, for indecomposable modules $M$ from $\text{mod} R$ with $M(x) \neq 0$, forms a finite set.

For locally bounded categories we have adopted the notions of the representation types: finite, tame and wild; moreover, similarly as for algebras the tame–wild dichotomy holds true (see [5,18,16]).

Let $A$ be an algebra. Then for any $\Lambda$-$R$-bimodule $B$ such that $\Lambda B$ is a finitely generated free module, we denote by $\text{rk}(B)$ the rank vector of $B$, i.e. the vector $(\text{rk} B(x))_{x \in \text{ob} R} \subseteq \mathbb{N}^{\text{ob} R}$, where $\text{rk} B(x)$ is a rank of the free $\Lambda$-module $\Lambda B(x)$.

1.2

Following the idea of Geiss [23], for any $d = (d_{i,j}) \in \mathbb{N}^{\text{ob} R \times \text{ob} R}$, with $n \in \mathbb{N}$, one considers the variety $\text{lbc}_d = \text{lbc}_d(k)$ of locally bounded $k$-categories $\mathcal{K}$ with a common (indexed) set of objects $\text{ob} \mathcal{K} = \{x_1, \ldots, x_n\}$ and a fixed dimension vector $\text{dim} \: \mathcal{K} = (d_{i,j}) = (\text{dim}_k J(\mathcal{K})(x_i, x_j))$, for all $i, j \in [n]$, where $[n] := \{1, \ldots, n\}$. The set $\text{lbc}_d$ is formed by structure constants $c = (c_{i,j}^{t,s})$, i.e. the collections of scalars $c_{i,j}^{t,s} \in k$, $i, j, t, r, s \in [n] \times [d_{i,j}] \times [d_{i,j}]$, defining (on the standard bases) the composition, which yields the structure of a locally bounded $k$-category $\mathcal{K} = R(c)$ with $\text{ob} R = \{n\}$ and $J(\mathcal{K})(i, j) = k^{d_{i,j}}$ (resp. $J(\mathcal{K})(i, j) = k^{d_{i,j}}$) for all $i, j \in [n]$. In fact, $\text{lbc}_d$ is a closed subset (in Zariski topology) of the affine space $\mathbb{A}^{N(d)}$, where $N(d) = \sum_{i \in [n]} d_{i,i}$. Clearly, for any locally bounded $k$-category $\mathcal{K}$ with $\text{ob} \mathcal{K} = \{x_1, \ldots, x_n\}$ and $\text{dim} \: \mathcal{K} = (d_{i,j})$, there exists $c \in \text{lbc}_d$ such that $R(c)$ is isomorphic to $\mathcal{K}$ (over the mapping $i \mapsto x_i, i \in [n]$). The structure constants $c$ for $R$ are uniquely determined by the choice of bases $(v_{i,j})_{i \in [n]}$ of the spaces $J(\mathcal{K})(x_i, x_j), i, j \in [n]$.

The variety $\text{lbc}_d$ admits the natural regular action of the connected affine algebraic group $H_d = \prod_{i \in [n]} \text{Gl}_{d_{i,i}}(k)$ on itself, which is given by the “base changes”. It has the property that for any $c, c' \in \text{lbc}_d$, $R(c)$ and $R(c')$ are isomorphic (over $\text{id}_{[n]}$) if and only if the orbits $H_d \cdot c$ and $H_d \cdot c'$ coincide.

Let $R_0, R_1$ be a pair of locally bounded $k$-categories such that $\text{ob} R_0 = \text{ob} R_1 = \{x_1, \ldots, x_n\}$ and $\text{dim} \: R_0 = \text{dim} \: R_1$ ($= d$). We say that $R_0$ is a degeneration of $R_1$, if for the constant structures $c^{(0)} \in \text{lbc}_d$ of the categories $R_0$ and $R_1$, respectively, the inclusion $H_d \cdot c^{(0)} \subseteq H_d \cdot c^{(1)}$ holds.
1.3

Let \( R \) be an arbitrary (not necessarily finite) locally bounded \( k \)-category. Fix basis \( \mathcal{B}_{x,y} \) of the radical subspaces \( J(y, x) \subseteq R(y, x) \), for all \( x, y \in \text{ob} \ R \), where \( J = J(R) \). Then for any dimension vector \( z \in \mathbb{N}^{\text{ob} \ R} \), the set \( \text{mod}_R(z) \), by definition, consists of all collections

\[
(M_{\alpha_{x,y}}) \in \prod_{x, y \in \text{ob} \ R, \alpha_{x,y} \in \mathcal{B}_{x,y}} M(z_{x,y}(z))(k)
\]

formed by the matrices \( M_{\alpha_{x,y}} \) of the linear structure maps \( M(\alpha_{x,y}) : M(x) \rightarrow M(y) \), with respect to the standard bases, for \( R \)-modules \( M \) such that \( \dim_k M = z \) and \( M(x) = k^{\alpha(x)} \), for every \( x \in \text{ob} \ R \). Clearly, we have the action \( \ast \) of the group

\[
G(z) = \prod_{x \in \text{ob} \ R} GL(x)(k)
\]
on \( \text{mod}_R(z) \), mapping the pair \((\gamma, M) = ((\gamma_x, (M_{\alpha_{x,y}}))) \in G(z) \times \text{mod}_R(z) \) into the collection \( \gamma \ast M = (M'_{\alpha_{x,y}}) \in \text{mod}_R(z) \), given by the formula \( M'_{\alpha_{x,y}} = \gamma_x M_{\alpha_{x,y}} \gamma^{-1}_y \), where \( x, y \in \text{ob} \ R \) and \( \alpha_{x,y} \in \mathcal{B}_{x,y} \).

Recall that if \( |z| := \sum_{x \in \text{ob} \ R} z(x) < \infty \), then \( \text{mod}_R(z) \) carries a structure of the affine variety. More precisely, \( \text{mod}_R(z) \) is closed in the Zariski topology subset of the (finite dimensional) affine space \( R^N(z) = \prod_{x, y \in \text{ob} \ R, \alpha_{x,y} \in \mathcal{B}_{x,y}} M(z_{x,y}(z))(k) \), where \( N(z) = \sum_{x, y \in \text{ob} \ R} z(x) \) is the dimension of \( x, y \) \( \dim_k (R_R, R) \), given by the equations of the form

\[
M_{\alpha_{x,y}} M_{\alpha_{x,y}} = \sum_{\alpha_{x,z}} c_{\alpha_{x,y}, \alpha_{x,z}} M_{\alpha_{x,z}}
\]

for \( \alpha_{x,y} \in \mathcal{B}_{x,y}, \alpha_{y,z} \in \mathcal{B}_{y,z} \) and \( \alpha_{x,z} \in \mathcal{B}_{x,z} \), where the constant structures \( c_{\alpha_{x,y}, \alpha_{x,z}} \) are defined by the equalities

\[
\alpha_{x,y} \cdot \alpha_{y,z} = \sum_{\alpha_{x,z}} c_{\alpha_{x,y}, \alpha_{x,z}} \alpha_{x,z}
\]

Moreover, \( \ast \) is then a regular action of the connected affine algebraic group \( G(z) \) on the variety \( \text{mod}_R(z) \).

From now on we assume that all the considered dimension vectors \( z \) are finite, i.e. that they belong to the set \( \mathbb{N}^{\text{ob} \ R}_0 := \{ z' \in \mathbb{N}^{\text{ob} \ R} : |z'| < \infty \} \).

For any \( z \in \mathbb{N}^{\text{ob} \ R}_0 \) and positive integer \( t \), we set

\[
\text{mod}_R(z, t) := \{ M \in \text{mod}_R(z) : \dim_k \text{End}_R M \geq t \}.
\]

We collect below the well known properties of the introduced sets (proved in [23, 24, 28]), which will play an important role in further considerations.

**Proposition.** (a) Let \( R \) be a locally bounded \( k \)-category. For any \( z \in \mathbb{N}^{\text{ob} \ R}_0 \) and \( t \in \mathbb{N} \), \( \text{mod}_R(z, t) \) is a closed \( G(z) \)-invariant subset of the variety \( \text{mod}_R(z) \). Moreover, in case \( R \) is finite, \( R \) is tame if and only if

\[
\dim \text{mod}_R(z, t) \leq |z| + (|z|_2 - t)
\]

for all pairs \((z, t)\) such that \( 1 \leq t \leq |z|_2 \), where \( |z|_2 = \sum_{x \in \text{ob} \ R} z(x)^2 \).

(b) Let \( R_0, R_1 \) be a pair of finite locally bounded \( k \)-categories. If \( R_0 \) is a degeneration of \( R_1 \) then

\[
\dim \text{mod}_{R_1}(z, t) \leq \dim \text{mod}_{R_0}(z, t)
\]

for all pairs \((z, t)\).

2. Dimension of module varieties and coverings

We start by recalling basic concepts of coverings, next we prove our main result — Theorem 2.2, and finally we formulate several facts being its consequences.

2.1

Let \( R \) and \( R' \) be a pair of locally bounded \( k \)-categories. Recall [3, 22] that a functor \( F : R \rightarrow R' \) is called a covering functor, if \( F \) is dense and for any pair of objects \( x \in \text{ob} \ R, a \in \text{ob} \ R' \), \( F \) induces two \( k \)-isomorphisms:

\[
\bigoplus_{y \in F^{-1}(a)} R(x, y) \cong R'(F(x), a) \quad \text{and} \quad \bigoplus_{y \in F^{-1}(a)} R(y, x) \cong R'(a, F(x)).
\]
Given a covering functor $F : R \to R'$ one can study interrelations between the module categories $\text{MOD} R$ and $\text{MOD} R'$ by using the pair of functors

$$\text{MOD} R \xrightarrow{F_*} \text{MOD} R'$$

where $F_* : \text{MOD} R' \to \text{MOD} R$ is the “pull-up” functor associated with functor $F$, assigning to each $X$ in $\text{MOD} R'$ the $R$-module $X \circ F$, and the “push-down” functor $F_* : \text{MOD} R \to \text{MOD} R'$ is the left adjoint to $F_* \ (\text{see} \ [26])$. The $R$-module $F_y(N)(a)$, for $N$ in $\text{MOD} R$, is defined as follows: $F_y(N)(a) = \bigoplus_{\alpha \in F^{-1}(a)} N(\alpha)$, for $a \in \text{ob} R$, and $F_x(N)(a) = [N(\alpha)] : \bigoplus_{\alpha \in F^{-1}(a)} N(\alpha)$ \rightarrow $\bigoplus_{\alpha \in F^{-1}(a)} N(\alpha)$, for $a \in \text{Ob} R$, where $\sum_{\alpha \in F^{-1}(b)} F_y(\alpha) = \alpha$, for $x \in F^{-1}(a)$.

Clearly, we have $F_*(\text{mod} R') \subseteq \text{mod} R$ and $F_* \ (\text{mod} R) \subseteq \text{mod} R'$.

Let $G \subseteq \text{Aut}_{\text{cat}}(R)$ be a group of $k$-linear automorphisms of a locally bounded $k$-category $R$. Then $G$ acts also on the category $\text{MOD} R$ by translations $s(-)$, which assign to each $M$ in $\text{MOD} R$ the $R$-module $sM = M \circ g^{-1}$. Given $M$ in $\text{MOD} R$, we set $G_M = \{g \in G : sM \simeq M\}$. We say that $G$ acts freely on $\text{Ind} R$ if $G_M = \{id_R\}$, for every indecomposable $M$ from $\text{mod} R$.

Assume now that $G$ acts freely on objects of $R$ (i.e. that $G_x = \{id_R\}$, for every $x \in \text{ob} R$). Then the covering functor $F : R \to R'$ is called a $G$-covering, if the set $F^{-1}(a)$ is $G$-invariant and the action of $G$ on $F^{-1}(a)$ is transitive, for every $a \in \text{ob} R'$. Recall that for $G$ as above there exists one distinguished $G$-covering functor. Namely, we can always form the quotient (orbit category) $R/G$, which is again locally bounded (we set $\text{ob} (R/G) = (\text{ob} R)/G$, the morphism spaces are defined in terms of $G$-orbits of morphisms in $R$, see $[3,22]$ for the precise definition). Then the natural projection yields a $G$-covering functor $F : R \to R/G$ such that $F_\gamma = F$ for all $\gamma \in G$, called a Galois covering.

Galois covering functors have nice properties and are well understood (see $[3,22]$). There exist many results concerning the nice behaviour of Galois coverings with respect to preserving the representation types in specific situations (see $[22,14,13,15]$, also $[8,9]$).

2.2

Let $F : R \to R'$ be an arbitrary $k$-linear functor between locally bounded $k$-categories. Then by $f = f(F) : N_0^{ob R} \to N_0^{ob R'}$, we denote the associated map dependent only on $F \circ \text{ob} : \text{ob} R \to \text{ob} R'$ and given by the formula

$$f(z)(a) = \sum_{x \in F^{-1}(a)} z(x),$$

where $z \in N_0^{ob R}$ and $a \in \text{ob} R'$. Assume now that $F$ is a covering functor. Then for any $a, b \in \text{ob} R'$, the set $\{F(\tilde{a}) : \tilde{a} \in \bigcap \{y \in F^{-1}(b) : B_{x,y}\}\}$ forms a base of the space $J(R')(b, a)$, hence for any fixed $x \in F^{-1}(a)$ there exists a uniquely determined (invertible) matrix $(c_{a,a}^x)$ such that $\sum_{a \in \bigcap \{y \in F^{-1}(b) : B_{x,y}\}} c_{a,a}^x F(\tilde{a}) = \alpha$, for every $\alpha \in B_{x,b}$. Now for any vector $z \in N_0^{ob R}$, we define the regular map

$$f_z = f_z(F) : \text{mod} R(z) \to \text{mod} R(f(z))$$

by setting $f_z((M_a)) = (M_a)$, where $M_a \in M_{f(z)(b) \times f(z)(a)}(k)$ are given by the block matrices

$$\begin{bmatrix}
\sum_{a \in B_{x,y}} c_{a,a}^x M_a
\end{bmatrix}_{y \in F^{-1}(b), x \in F^{-1}(a)}$$

for $a, b \in \text{ob} R'$ and $\alpha \in B_{a,b}$. It is easily seen that in this situation we have $F_* (\tilde{M}) = M$, where $\tilde{M}$ is an $R$-module determined by $(M_a) \in \text{mod}_R(z)$ and $M$ is an $R'$-module determined by $(M_a) \in \text{mod}_{R'}(z)$.

For a covering functor $F : R \to R'$ and $z \in N_0^{ob R'}$, we define the sets

$$\text{mod}^1_{R'}(z) := \bigcup_{\tilde{z} \in F^{-1}(z)} G(z) \ast (f_{\tilde{z}}(\text{mod}_R(\tilde{z}))),$$

and

$$\text{mod}^1_{R'}(z, t) := \text{mod}^1_{R'}(z) \cap \text{mod}^1_{R'}(z, t),$$

$t \in N$, where $f : N_0^{ob R} \to N_0^{ob R'}, f_z : \text{mod}_R(z) \to \text{mod}_{R'}(z)$ are the maps induced by $F$, introduced above. The defined sets consist of all modules of the first kind with respect to $F$ (in the sense of $[16]$) contained in $\text{mod}^1_{R'}(z)$ (resp., in $\text{mod}^1_{R'}(z, t)$), and they form dense subcategories in the full subcategory of $\text{mod} R'$ consisting of all first kind modules with the dimension vector $z$ (resp., of all first kind modules with the dimension vector $z$, whose endomorphism algebras have at least the dimension $t$).

To any dimension vector $z \in N_0^{ob R'}$ we associate two subsets $D$, $D' \subseteq N_0^{ob R}$. Fix $x_1, \ldots, x_r \in \text{ob} R$ such that $F(x_i) = a_i$, where $\text{supp} z = \{a_1, \ldots, a_r\} \subseteq \text{ob} R'$. By $D = D(z; x_1, \ldots, x_r)$ we denote the set of all $\tilde{z} \in F^{-1}(z)$ such that each
Let $\mathcal{D} = \mathcal{D}'(z; x_1, \ldots, x_r)$ the set of all $\tilde{z} \in \mathbb{N}_0^{obR}$ such that $\text{supp} \tilde{z}$ is connected, $\text{supp} \tilde{z} \cap (x_1, \ldots, x_r) \neq \emptyset$ and $f(\tilde{z})(a) \leq z(a)$ for all $a \in \text{ob} R'$ (then $|\tilde{z}| \leq |z|$ and $\text{supp} \tilde{z} \subseteq F^{-1}(\text{supp} z)$).

Note that the sets $\mathcal{D}$ and $\mathcal{D}'$ are finite, since for every $i$, the number of full connected subcategories $C \subseteq R$, such that $x_i \in \text{ob} C$ and $|\text{ob} C| \leq |z|$, is finite.

**Theorem.** Let $F^0 : R \to R^0$, $F^1 : R \to R_1$ be a pair of $G$-covering functors such that $F^0$ is a Galois one, where $G \subseteq \text{Aut}_{k-	ext{cat}}(R)$ is a group of $k$-automorphisms of a locally bounded $k$-category $R$ acting freely on $\text{ob} R$. Given a dimension vector $z \in \mathbb{N}_0^{obR}$.

(a) if

\[
(*) : F^1_1F^1_1N \cong \bigoplus_{g \in G} gN \ \text{for all } N \ \text{in } \mathcal{D} \ \text{such that } \dim N \in \mathcal{D} = \mathcal{D}(z; x_1, \ldots, x_r),
\]

then $\dim \mod_{R_0}^1(z, t) \leq \dim \mod_{R_1}^1(z, t)$ for every $t \in \mathbb{N}$, in particular,

\[
\dim \mod_{R_0}^1(z) \leq \dim \mod_{R_1}^1(z);
\]

(b) moreover, if additionally

\[
(**) : F^1_1(N') \cong F^1_1(\beta(N')) \ \text{for all } g \in G \ \text{and } N' \ \text{in } \mathcal{D} \ \text{such that } \dim N' \in \mathcal{D}' = \mathcal{D}'(z; x_1, \ldots, x_r),
\]

then $\dim \mod_{R_0}^1(z, t) = \dim \mod_{R_1}^1(z, t)$ for every $t \in \mathbb{N}$, in particular,

\[
\dim \mod_{R_0}^1(z) = \dim \mod_{R_1}^1(z).
\]

**Proof.** We prove the assertions (a) and (b) simultaneously.

Fix $z \in \mathbb{N}_0^{obR}$ such that $(*)$ holds for $z$, and set

\[
\mod_{R_0}^1(z) := \bigcup_{\tilde{z} \in \mathcal{D}} G(z) \ast (f_2^0(\text{mod}G(\tilde{z})))
\]

for $s = 0, 1$, where $f_2^s = f_2(F^s)$. Clearly, we have $\mod_{R_0}^1(z) \subseteq \mod_{R_0}^1(z)$. Moreover, $\mod_{R_0}^1(z) = \mod_{R_1}^1(z)$ for $s = 0$, and also for $s = 1$, provided $(**)$ holds. This follows from the observation that for a decomposition $N = \bigoplus_{l=1}^m N^0_l$ of an $R$-module $N$, with $\dim N \in f^{-1}(z)$, into a direct sum of indecomposable modules, there exist $g_1, \ldots, g_m \in G$ such that $(\text{supp} g_i N^0_l) \cap (x_1, \ldots, x_r) \neq \emptyset$ for every $l$. Consequently, we have $\dim g_i N^0_l \in \mathcal{D}$ and $\dim N' \in \mathcal{D}$, where $N' = \bigoplus_{l=1}^m g_i N^0_l$, and therefore

\[
F^1_1N \cong \bigoplus_{l=1}^m F^1_1N^0_l \cong \bigoplus_{l=1}^m F^1_1(g_i N^0_l) \cong F^1_1N'
\]

(for $s = 0$ the isomorphism is obvious, for $s = 1$ we apply $(**)$).

For any $\tilde{z} \in \mathcal{D}$, we set

\[
\mod_{R_0}(\tilde{z}) := \{ N \in \mod_{R_0}(\tilde{z}) : \sum_{g \in G} [N, gN] \geq t \}
\]

where $[N, gN] := \dim_k \text{Hom}_{R_0}(N, gN)$. Note that $\text{Hom}_{R_0}(N, gN) = 0$ for almost all $g \in G$, since $G$ acts freely on $\text{ob} R$ so the set \{ $g \in G : gS \cap S \neq \emptyset$ \} is finite, where $S = \text{supp} N$. Moreover, \$ \sum_{g \in G} [N, gN] \geq t \$ if and only if $\dim \text{End}_{R_0}(F^1_1N) \geq t$, for $s = 0$ (resp. $s = 1$), since $\text{End}_{R_0}(F^1_1N) \cong \text{Hom}_{R_0}(N, F^1_1F^1_1N)$ and $F^1_1N \cong \bigoplus_{g \in G} gN$ (see $(*)$, for $s = 1$). Hence, it follows that

\[
(f_2^0)^{-1}(\mod_{R_0}(z, t)) = \mod_{R_0}(\tilde{z}) = (f_2^0)^{-1}(\mod_{R_1}(z, t)).
\]

In conclusion, the set $\mod_{R_0}(\tilde{z})$ is closed and $G(\tilde{z})$-invariant, due to the existence of the commutative diagrams

\[
\begin{array}{ccc}
\ast : G(\tilde{z}) \times \mod_{R_0}(\tilde{z}) & \longrightarrow & \mod_{R_0}(\tilde{z}) \\
\epsilon \times f_2 & \downarrow & \ \downarrow f_2 \\
\ast : G(z) \times \mod_{R_0}(z) & \longrightarrow & \mod_{R_0}(z)
\end{array}
\]

where $\epsilon : G(\tilde{z}) \to G(z)$ is a diagonal embedding given by the mapping $(\gamma_{2(\alpha)}) \mapsto (\gamma_{2(\alpha)})$, with $\gamma_{2(\alpha)} = \text{diag}(\gamma_{2(\alpha)})$.\{ $x \in F^{-1}(\alpha)$ \}$.

We have also the equalities

\[
\bigcup_{\tilde{z} \in \mathcal{D}} G(z) \ast f_2^0(\mod_{R_0}(\tilde{z})) = \mod_{R_0}^1(z, t)
\]


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and
\[ \bigcup_{i \in D} G(z) \ast f^i_s(\text{mod}_g(\tilde{z})) = \text{mod}^i_{R_1}(z, t) \]
where \( \text{mod}^i_{R_1}(z, t) := \text{mod}^{i-1}_{R_1}(z) \cap \text{mod}_R(z, t) \).

For a fixed \( t \in \mathbb{N} \), let \( \text{mod}_g(\tilde{z}) = \bigcup_{i \in I^t} \text{mod}_g(\tilde{z})_{t,i} \) be a decomposition of \( \text{mod}_g(\tilde{z}) \) into a union of irreducible components, where \( I^t = I(t) \). We set
\[ X^0_{t,i} := (X^0_{t,i}(t)) := G(z) \ast f^i_s(\text{mod}_g(\tilde{z})_{t,i}) \]
where \( i \in I^t, \tilde{z} \in D \). Then each of the sets \( X^0_{t,i} \) is irreducible as an image of the regular map
\[ \psi^s_{t,i} := (\psi^s_{t,i}(t)) := \ast \circ (\text{id} \times f^i_s) : G(z) \times \text{mod}_g(\tilde{z})_{t,i} \to \text{mod}_R(z, t). \]
In this way we obtain decompositions
\[ \text{mod}^i_{R_1}(z) = \bigcup_{i \in I^t \times I^t} X^i_{t,i} \]
into unions of irreducible sets, for \( s = 0, 1 \). Therefore, it suffices to show that \( \text{dim} X^0_{t,i} \geq \text{dim} X^1_{t,i} \) (resp., \( \text{dim} X^1_{t,i} = \text{dim} X^0_{t,i} \)) in the case (***) holds, for every pair \((\tilde{z}, i)\).

Fix a pair \((\tilde{z}, i)\). The regular map
\[ \psi_i := \psi^1_{t,i} : G(z) \times \text{mod}_g(\tilde{z})_{t,i} \to X^0_{t,i} \]
is a dense map between irreducible varieties, so by “fibre dimension theorem” [25,31] there exists a nonempty open subset\( U = U(t) \subset G(z) \times \text{mod}_g(\tilde{z})_{t,i} \) such that
\[ \text{dim} \psi_i^{-1}_1(\gamma, N) = \text{dim} G(z) + \text{dim} \text{mod}_g(\tilde{z})_{t,i} - \text{dim} X^0_{t,i} \]
for any \((\gamma, N) \in U\). Thus, it suffices to show that \( \text{dim} \psi^{-1}_1(\gamma, N) \leq \text{dim} \psi^{-1}_0(\gamma, N) \) (resp., \( \text{dim} \psi^{-1}_1(\gamma, N) = \text{dim} \psi^{-1}_0(\gamma, N) \), if (**) holds), for all \( \gamma \in G(z) \) and \( N \in \text{mod}_g(\tilde{z})_{t,i} \).

Fix \((\gamma, N) \in G(z) \times \text{mod}_g(\tilde{z})_{t,i} \) and set
\[ Z^0 := \psi_i^{-1}_0(\gamma, N) = \{ (\gamma', N') : \gamma' f^1_j(N') = \gamma f^1_j(N) \}. \]

Let \( N_1 = N_2, \ldots, N_h \) be all up to isomorphism \( R \)-modules of the form \( N' = \bigoplus_{l=1}^m \oplus N(l) \), which belong to \( \text{mod}_g(\tilde{z})_{t,i} \), where \( N = \bigoplus_{l=1}^m N(l) \) is a fixed decomposition of \( N \) into a direct sum of indecomposable modules. Note that since \( \text{dim} N' = \tilde{z} \), the number of such modules is really finite. Then clearly \( F^0_{l,k} N_j \cong F^0_{l,n} N \), for all \( j \); moreover,
\[ \{ N' \in \text{mod}_g(\tilde{z})_{t,i} : F^0_{l,k} N' \cong F^0_{l,n} N \} = \bigcup_{j=1}^h G(\tilde{z}) \cdot N_j \]
since \( F^0_{l,k} N' \cong F^0_{l,n} N \) implies \( F^0_{l,k} N' \cong F^0_{l,n} N \), and \( N' \) is a direct summand of the \( R \)-module \( \bigoplus_{g \in G} N' \cong \bigoplus_{g \in G} N = \bigoplus_{g \in G} \bigoplus_{l=1}^m \oplus N(l) \); hence, \( N' \cong N_j \), for some \( j \). Analogously, by (**), we have the inclusion
\[ \{ N' \in \text{mod}_g(\tilde{z})_{t,i} : F^1_{l,k} N' \cong F^1_{l,n} N \} \subseteq \bigcup_{j=1}^h G(\tilde{z}) \cdot N_j \]
which is an equality, if (**) holds. (Note that \( F^1_{l,k} N_j \cong F^1_{l,n} N \) for all \( j \), since \( \text{dim} g(N(l)) \in D' \), for some \( g(l) \in G \), and hence \( F^1_{l,k}(g(N(l))) = F^1_{l,n}(N(l)) \), for all \( l \).) In consequence, we obtain the factorizations
\[ Z^0 \xrightarrow{\pi_s = \pi_{2i}} \text{mod}_R(\tilde{z})_{t,i} \]
for \( s = 0, 1 \), where \( \pi_2 : G(z) \times \text{mod}_g(\tilde{z})_{t,i} \to \text{mod}_R(\tilde{z})_{t,i} \) is a projection on the second component. For any \( j = 1, \ldots, h \), we set
\[ Z^0_j = (\pi^s)^{-1}(\theta(N_j)) = \{ (\gamma', N') \in Z^0 : N' \in \theta(N_j) \}. \]
Note that $Z^0_j \neq \emptyset$ for any $j$, since $F^0_j N_j \cong F^0_j N$ (resp., $Z^1_j \neq \emptyset$ for any $j$ and $s = 0, 1$, if (**) holds, since $F^1_j N_j \cong F^1_j N$). Moreover, we have the decompositions

$$Z^1 = \bigcup_{i=1}^h Z^1_i$$

into disjoint sums of locally closed sets, for $s = 0, 1$. Therefore to complete the proof, we show that $\dim Z^1_j \leq \dim Z^0_j$ (resp., $\dim Z^1_j = \dim Z^0_j$, if (***) holds). In fact, it suffices only to prove that $\dim Z^1_j = \dim Z^0_j$, if $Z^1_j \neq \emptyset$. In the proof we apply the properties of the associated fibre bundles (see [4, Theorem 5.15] for a definition).

Fix $j$ such that $Z^1_j \neq \emptyset$. Consider the $G(\tilde z)$-actions

$$\star^i : G(\tilde z) \times Z^j_i \to Z^j_i$$

given by the mappings $(\delta, (\gamma', N')) \mapsto (\gamma' \epsilon(\delta^{-1}), \delta N')$, for $s = 0, 1$. (Note that $\star^i$ is well defined, since $\gamma' \epsilon(\delta^{-1})f^j_z(\delta N') = \gamma' f^j_z(\delta^{-1} \delta N') = \gamma' f^j_z(N')$.) Then the maps

$$\pi^j_i = \pi^j_i : Z^j_i \to \sigma(N_j)$$

are $G(\tilde z)$-equivariant.

Set $H_j := G(\tilde z)_{N_j}$ and $F^j_s := (\pi^j_s)^{-1}(N_j)$, for $s = 0, 1$. Recall that the orbit map $G(\tilde z) \to \sigma(N_j)$ induces $G(\tilde z)$-equivariant regular isomorphism between homogeneous space $G(\tilde z)/H_j$ and the orbit $\sigma(N_j)$, mapping $eH_j$ on $N_j$ ([4, p. 16]). In consequence, by [4, Lemma 5.17] applied to the maps $\pi^j_i$, we obtain the isomorphisms

$$Z^1_j \cong G(\tilde z) \times^{H_j} F^j_s$$

of varieties.

Denote by $\phi^s : G(\tilde z) \times F^j_s \to Z^j_s$ the corresponding geometric quotient maps under the standard action of $H_j$ on $G(\tilde z) \times F^j_s$, for $s = 0, 1$. Observe that

$$F^j_s \cong \{\gamma' \in G(\tilde z) : \gamma' f^j_s(N_j) = \gamma f^j_s(N)\} \cong G(z)_{\gamma f^j_s(N)} \cong G(z)_{\gamma f^j_s(N)}$$

since $\gamma' f^j_s(N') = \gamma f^j_s(N)$, for $\gamma' \in G(z)$, if and only if $\gamma' (\gamma'^{-1})_0 \in G(z)_{\gamma^j_s(N)}$, where $\gamma'^{-1} \in G(z)$ such that $\gamma'^{-1}_0 \gamma f^j_s(N_j) = \gamma f^j_s(N)$ is fixed. Thus, the set $F^j_s$ is irreducible, since $G(z)_{\gamma f^j_s(N)} \cong \text{ AUT}_R(f^j_s(N))$ and $\text{ AUT}_R(f^j_s(N))$ is an open set in the linear space $\text{ End}_R(f^j_s(N))$. The group $G(\tilde z)$ is connected, so the sets $Z^j_0$ and $Z^j_1$ are also irreducible. Hence, from the fibre dimension formulas for the flat maps $\phi^s$, we obtain the equalities

$$\dim Z^j_s = \dim G(\tilde z) + \dim F^j_s - \dim H_j$$

for $s = 0, 1$. (The fibres of $\phi^s$ are simply the orbits of the standard action of $H_j$ on $G(\tilde z) \times F^j_s$, so they all are isomorphic to $H_j$.) On the other hand, by the previous considerations we have

$$\dim F^j_s = \dim G(z)_{\gamma f^j_s(N)} = \dim \text{ End}_R(f^j_s(N)) = \sum_g [N, gN]$$

(for $s = 1$ the last equality follows by (**)). Consequently, $\dim F^j_s$ does not depend on $s$ and $\dim Z^0_j = \dim Z^1_j$.

In this way, the proof of our claim and of the whole theorem is complete. □

**Remark.** (a) In fact we have shown that if for a fixed $z \in \mathbb{N}^{|G|} \cdot A$, $F^1_z F^1_z N \cong \bigoplus_{g' \in G'} \mathcal{S} N$ and $F^1_z (\mathcal{S} N) \cong F^1_z N$ for all $g \in G$ hold for any indecomposable $N$ in mod $R$ with $\dim N \in D^j(z; x_1, \ldots, x_n)$ then the same hold for any $N$ in mod $R$ with $\dim N \in f^{-1}(z)$; in particular, for $N$ with $\dim N \in D(z; x_1, \ldots, x_n)$.

(b) The assertion (for a fixed $z$) remain valid, if $F^0$ is not necessarily a Galois covering. In fact, it suffices only to require that the analogs of the conditions (**) and (***) hold simultaneously for $F^0$.

(c) Let $F : R \to R'$ be a $G$-covering functor. If $F$ is a Galois covering then for any $z \in \mathbb{N}^{|G|}$ the set $mod^1_R(z)$ is constructible. However this is not the case for an arbitrary $G$-covering. (One easily constructs an example of a non-Galois $\mathbb{Z}$-covering functor $F$, where $R'$ is the Kronecker category and $R'$ its universal cover, with the property that already $mod^1_R(z)$ is not a constructible set for $z = (1, 1)$.)
The following two facts follows easily from Theorem 2.2.

**Corollary.** Let $F^0$ and $F^1$ be as above. Assume that the functor $F^0$ is dense. Then for any dimension vector $z \in \mathbb{N}^{ob R_0}$ as in the assumption of 2.2(a) the inequality
\[
\dim \text{mod}_{R_0}(z, t) \leq \dim \text{mod}_{R_1}(z, t)
\]
holds for every $t \geq 1$.

**Proof.** An immediate consequence of the equality $\text{mod} R_0 = \text{mod}^1 R_0$. $\Box$

**Theorem.** Let $R_0$ and $R_1$ be finite locally bounded $k$-categories such that $R_0$ is a degeneration of $R_1$. Assume that these categories admit $G$-covering functors, $F^0 : R \to R_0$ and $F^1 : R \to R_1$, respectively, by the same locally support finite locally bounded category $R$, where $G \subseteq \text{Aut}_{k\text{-cat}}(R)$ is a group of $k$-linear automorphisms of $R$ acting freely on $\text{ind} R$; moreover, $F^0$ is a Galois covering. Then for any dimension vector $z \in \mathbb{N}^{ob R_0}$ as in the assumption of 2.2(a), the equality
\[
\dim \text{mod}_{R_0}(z, t) = \dim \text{mod}_{R_1}(z, t),
\]
holds for every $t \geq 1$, in particular,
\[
\dim \text{mod}_{R_0}(z) = \dim \text{mod}_{R_1}(z).
\]

**Proof.** Follows by Proposition 1.3(b) and the corollary. (Recall that since $R$ is locally support finite, the functor $F^0$ is dense under our assumptions, see [13–15].) $\Box$

**Example.** Let $(Q, I)$, for $t = 0, 1$, be bounded quivers given by
\[
\begin{align*}
Q : & \xrightarrow{\beta} \circ \\
& \xrightarrow{\gamma} \circ
\end{align*}
\]
and
\[
I_t = (\alpha^2 - \beta \gamma, \alpha^2 \beta, \gamma \alpha^2, t \gamma \alpha \beta - \gamma \beta)
\]
(see [30]). Denote by $R_0$ and $R_1$, respectively, the corresponding locally bounded categories. It is well known that $R_0$ is a degeneration of $R_1$ (see [3, Corollary 5.2]). Moreover, let $(\tilde{Q}, \tilde{I})$ be the universal cover of $(Q, I_0)$ in the sense of [27], given by
\[
\begin{align*}
\tilde{Q} : & \xrightarrow{\tilde{\beta}_1} \circ \\
& \xrightarrow{\tilde{\gamma}_1} \circ \\
& \xrightarrow{\tilde{\beta}_2} \circ \\
& \xrightarrow{\tilde{\gamma}_2} \circ \\
& \cdots
\end{align*}
\]
and
\[
\tilde{I} = \{(\tilde{\alpha}_n \tilde{\alpha}_{n+1} - \tilde{\beta}_n \tilde{\gamma}_{n+1}, \tilde{\alpha}_n \tilde{\alpha}_{n+1} \tilde{\beta}_{n+1} \tilde{\gamma}_{n-1} \tilde{\alpha}_n \tilde{\alpha}_{n+1}, \tilde{\gamma}_n \tilde{\beta}_{n+1} : n \in \mathbb{Z})\}
\]
and $R$ be the locally bounded $k$-category defined by $(\tilde{Q}, \tilde{I})$. The fundamental group $G := \Pi_1(Q, I_0)$ is an infinite cyclic group generated by $[\alpha]$ and under the identification $G = \mathbb{Z}$ given by $[\alpha^m] \mapsto m$, it acts on $R$ by a shift of the index by $m$, for $m \in G$. Clearly, $R$ is a locally support finite, since $R_0$ is representation finite, and as a torsionfree group $G$ acts freely on $\text{ind} R$.

Denote by $F^0 : R \to R_0$ that canonical Galois $G$-covering functor and by $F^1 : R \to R_1$ the functor given by setting
\[
\begin{align*}
F^1(\tilde{\alpha}_n + \tilde{I}) &= \alpha + I_1, \\
F^1(\tilde{\beta}_n + \tilde{I}) &= \beta + b_n \alpha \beta + I_1, \\
F^1(\tilde{\gamma}_n + \tilde{I}) &= \gamma + c_n \gamma \alpha + I_1,
\end{align*}
\]
for $n \in \mathbb{Z}$, where
\[
\begin{align*}
b_n &= \begin{cases} 1 & \text{when } n = 2i, \\
1 & \text{when } n = 2i - 1 \end{cases} \\
c_n &= \begin{cases} -i & \text{when } n = 2i, \\
i & \text{when } n = 2i + 1 \end{cases}
\end{align*}
\]
It is not hard to show that $F^1$ is well defined and that it is a $G$-covering functor. Moreover, one proves that for any indecomposable $N$ in mod $R$ we have $\operatorname{Ext}^1_R(N, {}^N N) = 0$, for all $m < 0$. The last condition follows from the fact that $\operatorname{Hom}_R(\psi^m N, {}^N N) = 0$ for all $m < 0$, where $\tau_R$ denotes the Auslander–Reiten translate for $R$ (we apply the Auslander–Reiten formula [1, Theorem 2.13], the fact that $R$ is standard and the information on the shape of the Auslander–Reiten quiver of $R$). Finally, we observe that $F^1$ is an almost Galois covering functor of integral type (see 3.1 for the precise definition). Then, by [12, Theorem 3.1(b)], we infer that $F^*_\tau F_\tau(N) \cong \bigoplus_{g \in G} {}^g N$, for any $N$ in mod $R$.

In consequence, by the theorem we obtain the equalities of the respective module variety dimensions, for any $z \in \mathbb{N}^{\text{ob } R}$.

2.4

To complete the first approach to the problem of preserving by $G$-coverings the infinite representation types, we briefly discuss the wild case. We start with the following definition.

Let $R$ be a wild locally bounded $k$-category. We say that $z \in \mathbb{N}^{\text{ob } R}$ is a wild dimension vector, if there exists a $k(x, y)$-$R$-bimodule $B = B(z)$ as in the definition of wildness (i.e. such that $k(x, y)B$ is finitely generated free and the functor $\Phi = -k(x, y) \otimes B : \operatorname{mod} k(x, y) \to \operatorname{mod} R$ induces an injection between the corresponding sets of isoclasses of indecomposable modules), which satisfies $\tau_k(B) = z$.

**Theorem.** Let $F : R \to R'$ be a $G$-covering functor, where $G \subseteq \operatorname{Aut}_k(-R)$ is a torsion-free group of automorphisms of a wild locally bounded category $R$, acting freely on $ob R$ with $|ob R/G| < \infty$. If there exists a wild vector $z \in \mathbb{N}^{\text{ob } R}$ such that

$$F^*_\tau F_\tau N \cong \bigoplus_{g \in G} {}^g N$$

for all $N$ in mod $R$ with $\dim N = nz$, where $n = n(z)$ is some integer such that $n \geq |z|$, then $R'$ is wild.

In the proof we apply the following fact proved by Geiss and de la Peña (see [23, Proposition 2.3], [24, Proposition 2], [28, Proposition 1.4]).

**Lemma.** Let $\Phi = -k(x, y) \otimes B : \operatorname{mod} k(x, y) \to \operatorname{mod} R$ be a functor given by the bimodule $k(x, y)B_R$ being finitely generated free as $k(x, y)$-module, with $\tau_k(B) = z$. Assume that there exists $n \geq |z|$ such that for any pair $X, Y$ of indecomposable right $n$-dimensional $k(x, y)$-modules, $\Phi(X) \cong \Phi(Y)$ if and only if $X \cong Y$. Then $\dim \operatorname{mod}_k(nz, t) \geq |nz| + |nz|_2 - t + 1$, for some $1 \leq t \leq |nz|_2$.

**Proof of Theorem 2.4.** Let $(z, n(z)) \in \mathbb{N}^{\text{ob } R} \times \mathbb{N}$ be a pair consisting of a wild dimension vector together with a natural number, satisfying the assumptions of the theorem, and $B = B(z)$ be a bimodule realizing the definition of a wild vector for $z$. Consider the composite functor

$$\Phi' : \operatorname{mod} k(x, y) \xrightarrow{\Phi} \operatorname{mod} R \xrightarrow{F_\tau} \operatorname{mod} R'$$

where $\Phi = -k(x, y) \otimes B$. Then $\Phi' = -k(x, y) \otimes B'$, where $B' = F_\tau(B)$. Clearly, $B'$ is a finitely generated free $k(x, y)$-module, $\tau_k(B') = f(\tau_k(B)) = f(z)$ and $|f(z)| = |z|$, where $f = f(F)$. We show first that $\Phi'$ satisfies the assumption of the lemma, for $n = n(z)$.

Fix a pair $X, Y$ of $n$-dimensional indecomposable modules in $\operatorname{mod} k(x, y)$ and assume that $\Phi'(X) \cong \Phi'(Y)$. Then $F_\tau \Phi(X) \cong F_\tau \Phi(Y)$, and by the assumptions, we have $\bigoplus_{g \in G} {}^g X \cong \bigoplus_{g \in G} {}^g Y$, since $\dim X = nz = \dim Y$, where $\bar{X} = \Phi(X)$ and $\bar{Y} = \Phi(Y)$. The $R$-modules $\bar{X}, \bar{Y}$ are indecomposable, so by the uniqueness of decomposition into a direct sum of indecomposables in $\operatorname{mod} R$, we have $\bar{X} \cong {}^g \bar{Y}$, for some $g \in G$ (see [15, Lemma 2.1]). Thus, $gS := S$, where $S = \supp \bar{X} = \supp \bar{Y}$; hence, $g = \id_G$, since $G$ is torsion-free, and therefore $X \cong Y$.

Now, applying the lemma, we infer that $\dim \operatorname{mod}_k(z', t) \geq |z'| + |z'|_2 - t + 1$, for some $1 \leq t \leq |z'|_2$, where $z' = nf(z)$. Consequently, by Proposition 1.3(a), $R'$ is wild. □

3. Almost Galois coverings of the integral type

In this section we discuss the “estimation problem” for dimension of the module varieties over algebras admitting some special covering, similar to the Galois one.

3.1

Let $(H, \preceq)$ be an ordered set and $\psi = \{ \psi_{h', h} : \bigoplus_{h \in H} W_h \to \bigoplus_{h' \in H} W_{h'} \}$ a $k$-linear endomorphism of the space $W = \bigoplus_{h \in H} W_h$ such that $W_h = 0$ for almost all $h \in H$. We say that $\psi$ is lower $\preceq$-unitriangular (shortly, $\preceq$-unitriangular), if $\psi_{h', h} = 0$ for all $h, h' \in H$ such that $h' \preceq h$ or $h, h'$ are incomparable and $\psi_{h, h} = \id_{W_h}$ for all $h \in H$. Note that then $\psi$ is clearly an isomorphism, and $\psi^{-1}$ is also $\preceq$-unitriangular. Moreover, the composition $\psi' \psi$ is again $\preceq$-unitriangular, if so is $\psi'$. 

Let $H$ be a group. Recall [12] that $H$ is \emph{$\mathbb{Z}$-totally ordered} by the relation $\preceq$, if there exists a surjective group homomorphism $p : H \rightarrow \mathbb{Z}$ such that $h_1 < h_2$ if and only if $p(h_1) < p(h_2)$ (resp. $h_1 \leq h_2$ if and only if $p(h_1) < p(h_2)$ or $h_1 = h_2$), for any $h_1, h_2 \in H$, where $\preceq$ denotes the standard ordering relation in $\mathbb{Z}$. (Clearly these conditions can be regarded as the definition of $\preceq$.) Note that always $(H, \preceq)$ is an ordered group, and that a free (resp. an abelian free) group is a $\mathbb{Z}$-totally ordered group in a canonical way (free generators are mapped to 1).

Let $G \subseteq \text{Aut}(R)$ be a group of automorphisms acting freely on $\text{ob} R$ and $(\text{ob} R)_0$ be a fixed set of representatives of $R$-orbits in $\text{ob} R$. Then we identify all morphisms in each $G$-orbit by setting $R(\bar{x}, \bar{y})_{g_2^{-1}g_1} := R(g_1\bar{x}, g_2\bar{y})$, for all $g_1, g_2 \in G$ and $\bar{x}, \bar{y} \in (\text{ob} R)_0$. Following [12] a $G$-covering functor $F : R \rightarrow R'$ is called an \emph{almost Galois covering of the integral type} (with the group $G$), if $G$ admits a $\mathbb{Z}$-total order $\preceq$ such that all automorphisms

$$\varphi_{g_1}(\bar{x}, \bar{y}) = \bigoplus_{g \in G} R(\bar{x}, \bar{y})_g \rightarrow \bigoplus_{g' \in G} R(\bar{x}, \bar{y})_{g'}$$

for $g_1 \in G$ and $\bar{x}, \bar{y} \in (\text{ob} R)_0$, are $\preceq$-unitary under the identifications above, where

$$\varphi_{g_1}(\bar{x}, \bar{y}) = \varphi_{g_1}(\bar{x}, \bar{y})^{(g', \bar{s})} : \bigoplus_{g \in G} R(\bar{x}, \bar{y})_g \cong \bigoplus_{g_2 \in G} R(g_1\bar{x}, g_2\bar{y}) \xrightarrow{\varphi_{g_1}} R'(\bar{x}, \bar{y}) \xrightarrow{(g')^{-1}} \bigoplus_{g_2' \in G} R(g_1\bar{x}, g_2'\bar{y}) \cong \bigoplus_{g' \in G} R(\bar{x}, \bar{y})_{g'}$$

with $\varphi_{g_1}^{(g', \bar{s})}$, $\varphi_{g_1}$ being the isomorphism induced by $F(g = g_2^{-1}g_1, g' = (g'_2)^{-1})$. One shows that the definition does not depend on the choice of the set $(\text{ob} R)_0$.

For important examples of the (nonstandard) algebras which admit almost Galois covering of the integral type we refer to [2, 30] [see [12, 23 and 4.1]].

**Theorem.** Let $R$ be a locally support finite locally bounded $k$-category, $G \subseteq \text{Aut}_{k-\text{cat}}(R)$ be a group of $k$-linear automorphisms of $R$ admitting a $\mathbb{Z}$-total order $\preceq$, acting freely on $\text{ind} R$, and having a finite number of orbits in $\text{ob} R$; moreover, $F : R \rightarrow R'$ be an \emph{almost Galois $G$-covering of the integral type} (with respect to $\preceq$). Then for any $z \in \mathbb{N}^{\text{ob} R}$ the equality

$$\dim \text{mod}_{R}(z, t) = \dim \text{mod}_{R/G}(z, t),$$

holds for every $t \geq 1$, in particular,

$$\dim \text{mod}_{R}(z) = \dim \text{mod}_{R/G}(z),$$

provided $\text{Ext}^1_{R}(N, \delta N) = 0$ for all $N$ such that $\dim N \in f^{-1}(z)$ and $g \in G^{\preceq e}$.

**Proof.** It is known that, since $F$ is an almost Galois $G$-covering functor of the integral type, the finite category $R/G$ is a degeneration of $R'$ in the sense of 1.2 [see [12, 24, 2.5]]. Moreover, by [12, Theorem 3.1(b)], we have $F_{*}F_{*}(N) \cong \bigoplus_{g \in G} \mathbb{F} N$, for any $N$ in mod $R$ such that $\text{Ext}^1_{R}(N, \delta N) = 0$, for all $g \in G^{\preceq e}$.

Denote by $\bar{F} : R \rightarrow R/G$ the standard Galois covering functor given by the canonical projection. Now applying Theorem 2.3 for the pair of covering $G$-functors $F^0 = \bar{F}$ and $F^1 = F$, we immediately obtain the assertion. \qed

### 3.2

Finally, we discuss a geometric meaning of the “Ext-vanishing condition” from Theorem 3.1.

**Proposition.** Let $G \subseteq \text{Aut}_{k-\text{cat}}(R)$ be a group of $k$-automorphisms of a locally bounded category $R$ acting freely on $\text{ob} R$, $G' \subseteq G$ a subset of $G$, and $z \in \mathbb{N}^{\text{ob} R}$ a fixed dimension vector. Then

$$V := \{N \in \text{mod}_{R}(z) : \text{Ext}^1_{R}(\delta N, N) = 0 \forall g \in G'\}$$

forms an open set in the variety $\text{mod}_{R}(z)$.

In the proof we use the following version of [7, Lemma 4.3] obtained by replacing an algebra $A$ by a locally bounded $k$-category $R$ (resp., dimensions by finite dimension vectors, i.e. elements of $\mathbb{N}^{\text{ob} R}$), and by applying the interpretation of the first extension group for $R$-modules in terms of derivations in the appropriate sense, analogous to the classical one (see [10, 3.2]).

**Lemma.** Let $R$ be a locally bounded $k$-category. Then for any finite dimension vectors $z, z' \in \mathbb{N}^{\text{ob} R}$ the function

$$E_{z, z'} : \text{mod}_{R}(z) \times \text{mod}_{R}(z') \rightarrow \mathbb{N}$$

defined by the formula $E_{z, z'}(M, N) = \dim_{k} \text{Ext}^1_{R}(M, N)$ is upper semi-continuous, i.e. the sets

$$Y_{t} := \{(M, N) \in \text{mod}_{R}(z) \times \text{mod}_{R}(z') : \dim_{k} \text{Ext}^1_{R}(M, N) \leq t\}$$

$t \in \mathbb{N}$, form open subsets in $\text{mod}_{R}(z) \times \text{mod}_{R}(z')$. 


Proof of Proposition 3.2. We start by observing that for any \( g \in G \)

\[
V_g := \{ N \in \text{mod}_g(z) : \text{Ext}_g^1(\mathcal{E} N, N) = 0 \}
\]

is an open set in the module variety \( \text{mod}_g(z) \), since \( V_g = \Delta_g^{-1}(Y_0) \), where

\[
\Delta_g : \text{mod}_g(z) \to \text{mod}_g(\mathcal{E} z) \times \text{mod}_g(z)
\]

is a regular embedding given by \( N \mapsto (\mathcal{E} N, N) \).

To show the main assertion, denote by \( \text{supp} z \) a (finite) set consisting of all \( x \in \text{ob} R \) such that \( R(y, x) \neq 0 \) or \( R(x, y) \neq 0 \), for some \( y \in \text{supp} z \). Since \( G \) acts freely on \( \text{ob} R \), the set \( G_0 := \{ g \in G : \text{supp} z \cap g(\text{supp} z) \neq \emptyset \} \) is finite. Moreover, if \( g \in G \) does not belong to \( G_0 \), then \( \text{Ext}_g^1(\mathcal{E} N, N) = 0 \) for every \( N \in \text{mod}_g(z) \), and \( V_g = \text{mod}_g(z) \). Consequently,

\[
V = \bigcup_{g \in G'} V_g = \bigcup_{g \in G' \cap G_0} V_g
\]

and \( V \) is open. \( \square \)

References