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Journal of Algebra 293 (2005) 611–630
JOURNAL OF Algebra
www.elsevier.com/locate/jalgebra

# On extensions of generalized Steinberg representations

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Received 11 October 2004

Available online 17 May 2005

Communicated by Michel Broué

**Abstract**

Let $F$ be a local non-archimedean field and let $G$ be the group of $F$-valued points of a connected reductive algebraic group over $F$. In this paper we compute the Ext-groups of generalized Steinberg representations in the category of smooth $G$-representations with coefficients in a certain self-injective ring.


*Keywords:* Reductive $p$-adic groups; Elliptic representations; Ext-groups; Period domains; Generalized Steinberg representations

## 1. Introduction

The origin of the problem we treat here is the computation of the étale cohomology of $p$-adic period domains with finite coefficients. In [O] the computation yields a filtration of smooth representations of a $p$-adic Lie group on the cohomology groups, which is induced by a certain spectral sequence. A natural problem which arises in this context is to show that this filtration splits canonically. The graded pieces of the filtration are essentially

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0021-8693/$ – see front matter 
doi:10.1016/j.jalgebra.2005.03.028

generalized Steinberg representations. A natural task is therefore to study the extensions of these representations.

Let $F$ be a local non-archimedean field and let $G$ be the group of $F$-valued points of a fixed connected reductive algebraic group over $F$. The field $F$ induces a natural topology on $G$ providing it with the structure of a locally profinite group. The aim of this paper is to determine the Ext-groups of generalized Steinberg representations in the category of smooth $G$-representations with coefficients in a self-injective ring $R$. An important example of such a ring is given by a field of characteristic zero. We refer to the next chapter for the precise conditions we impose on $R$. One crucial assumption is that the pro-order of $G$ is invertible in $R$. In [V1] it is shown that this condition is sufficient for the existence of a normalized Haar measure on $G$. Using this Haar measure and the self-injectivity of $R$ ensures all the well-known properties and techniques in representation- and cohomology theory of a $p$-adic reductive group, e.g., Frobenius reciprocity, exactness of the fixed point functor for a compact open subgroup of $G$, etc., as in the classical case where $R = \mathbb{C}$. In particular we have enough injective and projective objects in the category of smooth $G$-representations.

The generalized Steinberg representations are parametrized by the subsets of a relative $F$-root basis $\Delta$ of $G$. For any subset $I \subset \Delta$, let $P_I \subset G$ be the corresponding standard-parabolic subgroup of $G$. Let $i^G_{P_I} = C^\infty(P_I \backslash G, R)$ be the smooth $G$-representation consisting of locally constant functions on $P_I \backslash G$ with values in $R$. If $J \supset I$ is another subset, then there is a natural injection $i^G_{P_J} \hookrightarrow i^G_{P_I}$. The generalized Steinberg representation with respect to $I \subset \Delta$ is the quotient

$$v^G_{P_I} = i^G_{P_I} \Big/ \sum_{\substack{I \subset J \subset \Delta \\ I \neq J}} i^G_{P_J}.$$

In the case $I = \emptyset$ we just get the ordinary Steinberg representation. In the case $R = \mathbb{C}$ it is known that the representations $v^G_{P_J}$, for $J \supset I$, are precisely the irreducible subquotients of $i^G_{P_I}$. Our main result is formulated in the following theorem.

**Theorem 1.** *Let $G$ be semi-simple. Let $I, J \subset \Delta$. Then*

$$\mathrm{Ext}^i_G\big(v^G_{P_I}, v^G_{P_J}\big) = \begin{cases} R, & i = |I \cup J| - |I \cap J|, \\ 0, & \text{otherwise}. \end{cases}$$

Note that in the case where $I = \Delta$ or $J = \Delta$, i.e., $v^G_{P_I}$ or $v^G_{P_J}$ is the trivial representation and $R$ is a field of characteristic zero, this computation was carried out by Casselman [Ca1, Ca2], respectively Borel and Wallach [BW]. If on the other extreme $I = \emptyset$ or $J = \emptyset$, the Ext-groups were computed by Schneider and Stuhler [SS].

If $G$ is not necessarily semi-simple, then we have, in addition, a contribution of the center $Z(G)$ of $G$ in the formula above. By using a Hochschild–Serre argument, we conclude from Theorem 1 the following.

**Corollary 2.** *Let $G$ be reductive with center $Z(G)$ of $F$-rank $d$. Let $I, J \subset \Delta$. Then we have*

$$Ext^i_G\big(v^G_{P_I}, v^G_{P_J}\big) = \begin{cases} R^{\binom{d}{j}}, & i = |I \cup J| - |I \cap J| + j,\ j = 0, \dots, d, \\ 0, & \textit{otherwise}. \end{cases}$$

Our proof of Theorem 1 is quite natural. One uses certain resolutions of the representations $v^G_{P_I}$ in terms of the induced representations $i^G_{P_K}$, where $K \supset I$. By a spectral sequence argument, the proof reduces to the computation of the groups $Ext^*_G(i^G_{P_I}, i^G_{P_J})$, for $I, J \subset \Delta$. This is done by Frobenius reciprocity and a description of the Jacquet modules for these kind of representations. The latter has been considered in [Ca3] in the case $R = \mathbb{C}$. It holds more generally in our situation.

A totally different proof of Theorem 1 has been given by J.-F. Dat [D]. Apart from the fact that $R$ needs not to be self-injective, his proof has the advantage of producing the extensions of generalized Steinberg representations explicitly.

## 2. Notation

Let $p$ be a prime number and let $F$ be a local non-archimedean field. We suppose that the residue field of $F$ has order $q = p^r$, $r > 0$. Let $val : F \to \mathbb{Z}$ be the discrete valuation taking a fixed uniformizer $\varpi_F \in F$ to $1 \in \mathbb{Z}$. Denote by $|\cdot|_{\mathbb{R}} : F \to \mathbb{R}$ the corresponding normalized $p$-adic norm with values in $\mathbb{R}$.

Let $\mathbf{G}$ be a connected reductive algebraic group over $F$. Fix a maximal $F$-split torus $\mathbf{S}$ and a minimal $F$-parabolic subgroup $\mathbf{P}$ in $\mathbf{G}$ containing $\mathbf{S}$. Let $\mathbf{M} = Z(\mathbf{S})$ be the centralizer of $\mathbf{S}$ in $\mathbf{G}$, which is a Levi subgroup of $\mathbf{P}$. Denote by $\mathbf{U}$ the unipotent radical of $\mathbf{P}$. Let

$$\Phi \supset \Phi^+ \supset \Delta = \{\alpha_1, \dots, \alpha_n\}$$

be the corresponding subsets of relative $F$-roots, $F$-positive roots, $F$-simple roots. To simplify matters we call them just roots instead of relative $F$-roots. For a subset $I \subset \Delta$, we let $\mathbf{P}_I \subset \mathbf{G}$ be the standard parabolic subgroup defined over $F$ such that $\Delta \setminus I$ are precisely the simple roots of the unipotent radical $\mathbf{U}_I$ of $\mathbf{P}_I$. As extreme cases we have

$$\mathbf{P}_\Delta = \mathbf{G} \quad \text{and} \quad \mathbf{P}_\emptyset = \mathbf{P}.$$

Moreover, there is for each subset $I \subset \Delta$ a unique Levi subgroup $\mathbf{M}_I$ of $\mathbf{P}_I$ which contains $\mathbf{M}$. Let

$$\Phi_I \supset \Phi_I^+ \supset I$$

be its set of roots, positive roots, simple roots with respect to $\mathbf{S} \subset \mathbf{M_I} \cap \mathbf{P}$. We denote by

$$W = N(\mathbf{S})/Z(\mathbf{S})$$

the relative Weyl group of $\mathbf{G}$. For any subset $I \subset \Delta$, let $W_I$ be the parabolic subgroup of $W$ which is generated by the reflections associated to $I$. It coincides with the Weyl group of $\mathbf{M}_I$. Thus, we have

$$W_\Delta = W \quad \text{and} \quad W_\emptyset = \{1\}.$$

Whereas we denote algebraic groups defined over $F$ by boldface letters, we use ordinary letters for their groups

$$G := \mathbf{G}(F), \qquad P_I := \mathbf{P}_I(F), \qquad M_I := \mathbf{M}_I(F), \ldots$$

of $F$-valued points. We supply these groups with the canonical topology given by $F$. These are locally profinite topological groups. For any linear algebraic group $\mathbf{H}$ defined over $F$, we denote by $X^*(\mathbf{H})_F$ its group of $F$-rational characters. Let $\mathbf{M} \subset \mathbf{G}$ be a Levi subgroup. Put

$$^0M = \bigcap_{\alpha \in X^*(\mathbf{M})_F} kern|\alpha|_{\mathbb{R}}.$$

This is a normal open subgroup generated by all compact subgroups of $M$ (cf. [BW, Chapter X 2.2]). Moreover, the quotient $M/{}^0M$ is a finitely generated free abelian group of rank equal to the $F$-rank of $Z(\mathbf{M})$. The valuation *val* on $F$ gives rise to a natural homomorphism of groups

$$\Theta_M : X^*(\mathbf{M})_F \to Hom\big(M/{}^0M, \mathbb{Z}\big) \tag{1}$$

defined by $\Theta_M(\chi) = val \circ \chi(F)$, where $\chi(F) : M \to F^\times$ is the induced homomorphism on the $F$-valued points. It is easily seen that $\Theta_M$ is injective. Furthermore, the source and the target of $\Theta_M$ are both free $\mathbb{Z}$-modules of the same rank. Therefore, we may identify $X^*(\mathbf{M})_F$ with a sublattice of $Hom(M/{}^0M, \mathbb{Z})$.

We fix a self-injective ring $R$, i.e., $R$ is an injective object in the category $Mod_R$ of $R$-modules. Let $i : \mathbb{Z} \to R$ be the canonical homomorphism. Then we have $ker(i) = d \cdot \mathbb{Z}$, for some integer $d \in \mathbb{N}$. We suppose that $R$ fulfills the following assumptions.

(1) The pro-order $|G|$ of $G$ is invertible in $R$, i.e., $|G|$ is prime to $d$ (see [V1, Chapter I, 1.5] for the definition of the pro-order). In particular, $i(q) \in R^\times$.
(2) Let

$$\rho = \det Ad_{Lie(\mathbf{U})}|\mathbf{S} \in X^*(\mathbf{S})_F$$

be the character given by the determinant of the adjoint representation of $\mathbf{P}$ on $Lie(\mathbf{U})$ restricted to $\mathbf{S}$. Write $\rho$ in the shape

$$\rho = \sum_{\alpha \in \Delta} n_\alpha \alpha,$$

with $n_\alpha \in \mathbb{N}$, $\alpha \in \Delta$. Then we impose on $R$ that $d$ is prime to

$$\prod_{r \leqslant \sup\{n_\alpha;\ \alpha \in \Delta\}} \bigl(1 - q^r\bigr).$$

(3) Let $E/F$ be a finite Galois splitting field of $\mathbf{G}$. Then we further suppose that $d$ is prime to the order of the Galois group $Gal(E/F)$, i.e., $i(|Gal(E/F)|) \in R^\times$.
(4) Finally, we suppose that the injective maps $\Theta_{M_I}$ become isomorphisms after base change to $R$ for all $I \subset \Delta$.

**Remarks.** 1. Important examples of such rings are given by fields of characteristic zero or by $R = \mathbb{Z}/n\mathbb{Z}$ with $n \in \mathbb{N}$ suitable chosen.

2. In the terminology of Vignéras, respectively Dat assumption (1) means that $R$ is *banal* (cf. [D, 3.1.5, 3.1.6], respectively [V1]) for $G$. A prime $d$ which satisfies assumption (2) is called *bon* for $G$ (cf. [D, 3.1.5]). A ring $R$ which fulfills both assumptions (1) and (2) is called *fortement banal* (cf. loc. cit. 3.1.6).

Suppose for the moment that $G$ is an arbitrary locally profinite group. We agree that all $G$-representations (sometimes we use the term $G$-module as well) in this paper are defined over $R$. Recall that a smooth $G$-representation is a representation $V$ of $G$ such that each $v \in V$ is fixed by a compact subgroup $K \subset G$. We denote the category of smooth representations by $Mod_G$. If $V$ is a smooth $G$-module, then we let $\widetilde{V}$ be its smooth dual. Any closed subgroup $H$ of $G$ gives rise to functors

$$i_H^G,\ c\text{-}i_H^G : Mod_H \to Mod_G$$

called the (unnormalized) induction, respectively induction with compact support. We recall their definitions. Let $W$ be a smooth $H$-representation. Then we have

$$i_H^G(W) := \bigl\{f : G \to W;\ f(hg) = h \cdot f(g)\ \forall h \in H,\ g \in G,\ \exists \text{ compact open subgroup } K_f \subset G \text{ s.t. } f(gk) = f(g)\ \forall g \in G,\ k \in K_f\bigr\},$$

respectively

$$c\text{-}i_H^G(W) := \bigl\{f \in i_H^G(W);\ \text{the support of } f \text{ is compact modulo } H\bigr\}.$$

It is obvious that we have

$$i_H^G = c\text{-}i_H^G,$$

if $H\backslash G$ is compact. If furthermore $W$ is admissible, i.e., $W^K$ is of finite type over $R$ for all compact open subgroups $K \subset G$, then $i_H^G(W)$ is admissible as well [V1, I, 5.6]). Finally, we denote for any $G$-module $V$ by $V^G$, respectively $V_G$, the invariants, respectively the coinvariants of $V$, with respect to $G$.

Next, we want to recall the definition of the generalized Steinberg representations. Let $\mathbf{1}$ be the trivial representation of any locally profinite group. For any subset $I \subset \Delta$, let

$$i^G_{P_I} := i^G_{P_I}(\mathbf{1}) = c\text{-}i^G_{P_I}(\mathbf{1}) = C^\infty(P_I \backslash G, R)$$

be the smooth and admissible representation of locally constant functions on $P_I \backslash G$ with values in $R$. If $\Delta \supset J \supset I$ is another subset, then there is an injection $i^G_{P_J} \hookrightarrow i^G_{P_I}$ which is induced by the natural surjection $P_I \backslash G \to P_J \backslash G$. The generalized Steinberg representation of $G$ with respect to $I \subset \Delta$ is defined to be the quotient

$$v^G_{P_I} := i^G_{P_I} \Big/ \sum_{\substack{I \subset J \subset \Delta \\ J \neq I}} i^G_{P_J}.$$

In the case $R = \mathbb{C}$ it has been shown that the generalized Steinberg representations are irreducible and not pairwise isomorphic for different $I \subset \Delta$ (cf. [Ca2, Theorem 1.1]). This result has been generalized by J.-F. Dat [D] to the case of an algebraically closed field which is *fortement banal* for $G$.

We finish this section with introducing some more notations. We fix a normalized left-invariant $R$-valued Haar measure $\mu$ on $G$ with respect to a maximal compact open subgroup of $G$. The existence of such a Haar measure is guaranteed by assumption (1) on $R$ (see [V1, I, 2.4]). Further, we denote by $|\cdot| : F \to R$ the "norm" given by the composition of

$$F \to q^{\mathbb{Z}}, \quad x \mapsto q^{-val(x)}$$

together with the natural homomorphism $\mathbb{Z}[1/q] \to R$. Finally, if $\mathbf{H}$ is any linear algebraic group over $F$, then we put

$$X(\mathbf{H}) := X^*(\mathbf{H})_F \otimes_{\mathbb{Z}} R.$$

## 3. The computation

Let $G$ be an arbitrary locally profinite group which satisfies assumption (1) on $R$. The category $Mod_G$ of smooth $G$-representations has then enough injectives and projectives [V1]. This fact provides two different choices for the computation of the Ext-groups $Ext^*_G(V, W)$ for a given pair of smooth $G$-representations $V, W$. Notice that

$$H^i(G, V) = Ext^i_G(\mathbf{1}, V)$$

is the $i$th right derived functor of

$$Mod_G \to Mod_R, \quad V \mapsto V^G,$$

whereas $H_i(G, V)$ denotes the $i$th left derived functor of the right exact functor

$$Mod_G \to Mod_R, \quad V \mapsto V_G.$$

Since $R$ is self-injective, it is easy to see that there is an isomorphism

$$H_i(G, V)^\vee = Ext^i_G(V, \mathbf{1})$$

for all smooth $G$-representations $V$ and for all $i \geqslant 0$. Here the symbol $^\vee$ indicates the $R$-dual space.

For our proof of Theorem 1, we need some statements on the cohomology of smooth representations of locally profinite groups with values in $R$. Up to Lemma 14 all the statements are well known in the classical case, i.e., where $R = \mathbb{C}$. In our situation their proofs are essentially the same. But for being on the safe side, we are going to reproduce the arguments shortly. Up to Lemma 7—apart from Lemma 4—$G$ is an arbitrary locally profinite group satisfying assumption (1) on $R$.

**Lemma 3.** *Let $K \subset G$ be an open compact subgroup. Then $i^G_K(\mathbf{1})$ is an injective object in $Mod_G$.*

**Proof.** By [V1, I, 4.10] we know that the trivial $K$-representation $\mathbf{1}$ is an injective object. Since the induction functor respects injectives (loc. cit. I, 5.9(b)), we obtain the claim. □

Let $Y$ be the Bruhat–Tits building of $\mathbf{G}$ over $F$. We denote by $C^q(Y)$, $q \in \mathbb{N}$, the space of $q$-cochains on $Y$ with values in $R$. As in the classical case we have the following fact.

**Lemma 4.** *The natural chain complex*

$$0 \to R \to C^0(Y) \to C^1(Y) \to \cdots \to C^q(Y) \to \cdots$$

*is an injective resolution of the trivial $G$-representation $\mathbf{1}$ by smooth $G$-modules.*

**Proof.** The proof coincides with the one of [BW, Chapter X, 1.11] which uses Lemma 3 and the contractibility of the Bruhat–Tits building $Y$. □

Our next lemma deals with the Hochschild–Serre spectral sequence. Let $N \subset G$ be a closed subgroup. As it has been pointed out by Casselman in [Ca2], the restriction functor from the category of smooth $G$-modules to that of $N$-modules does not preserve injective objects. For this reason, the standard arguments for proving the existence of the Hochschild–Serre spectral sequence—as in the cohomology theory of groups—breaks down. Nevertheless, the restriction functor preserves projective objects giving a homological variant of the Hochschild–Serre spectral sequence (see appendix of [Ca2]).

**Lemma 5.** *Let $N \subset G$ be a closed normal subgroup of $G$. If $V$ is a projective $G$-module, then $V_N$ is a projective $G/N$-module. Thus, we get for every pair of smooth $G$-modules $V$, $W$, such that $N$ acts trivially on $W$, a spectral sequence*

$$E_2^{p,q} = Ext^q_{G/N}\big(H_p(N,V), W\big) \Rightarrow Ext^{p+q}_G(V,W).$$

*If, furthermore, $N$, respectively $G/N$ is compact, then we have*

$$Ext^q_{G/N}(V_N, W) = Ext^q_G(V,W) \quad \forall q \in \mathbb{N},$$

*respectively*

$$Ext^0_{G/N}\big(H_p(N,V), W\big) = Ext^p_G(V,W) \quad \forall p \in \mathbb{N}.$$

**Proof.** The proof is the same as in the classical case [Ca2, A.9]. It starts with the observation that the coinvariant functor is left adjoint to the exact functor viewing a smooth $G/N$-module as a smooth $G$-module. Therefore, $V_N$ is a projective $G/N$-module if $V$ is projective. By [V1, I, 5.10] we know that the restriction functor preserves projectives. Using the standard-arguments applied to the Grothendieck spectral sequence, we obtain the first part of the claim. The reason for the second part is the exactness of the coinvariant, respectively fixed-point functor for a compact subgroup [V1, I, 4.6]. $\square$

**Lemma 6.** *Let $V$ and $W$ be smooth representations of $G$. Suppose that $W$ is admissible. Then*

$$Ext^i_G(V,W) \cong Ext^i_G\big(\widetilde{W}, \widetilde{V}\big) \quad \textit{for all } i \geqslant 0.$$

**Proof.** Let

$$0 \leftarrow V \leftarrow P^0 \leftarrow P^1 \leftarrow \cdots$$

be a projective resolution of $V$. Since $R$ is self-injective, we conclude as in [V1, I, 4.18] that the functor $W \mapsto \widetilde{W}$ from the category of smooth $G$-representations to itself is exact. By [V1, I, 4.13(2)] we see that the modules $\widetilde{P}^j$, $j \geqslant 0$, are injective objects in $Mod_G$. Hence, we obtain an injective resolution

$$0 \to \widetilde{V} \to \widetilde{P}^0 \to \widetilde{P}^1 \to \cdots$$

of $\widetilde{V}$. Moreover, we know by [V1, I, 4.13(1)] that

$$Hom_G\big(V, \widetilde{W}\big) = Hom_G\big(W, \widetilde{V}\big),$$

for any pair of smooth $G$-modules $V$, $W$. Since $W$ is admissible, we have $W = \widetilde{\widetilde{W}}$ (see [V1, 4.18(iii)]) and the claim follows. $\square$

In the special case $W = \mathbf{1}$ we obtain the following.

**Corollary 7.** *Let $V$ be a smooth representation of $G$. Then*

$$H^i\big(G, \widetilde{V}\big) \cong H_i(G, V)^\vee \quad \textit{for all } i \geqslant 0.$$

From now on, we suppose again that $G$ is the set of $F$-valued points of some reductive algebraic group defined over $F$.

**Lemma 8.** *Let $Q \subset G$ be a parabolic subgroup with Levi decomposition $Q = M \cdot N$. Let $V$, respectively $W$, be a smooth representation of $G$, respectively $M$. Extend $W$ trivially to a representation of $Q$. Then*

$$Ext^i_G\big(V, i^G_Q(W)\big) \cong Ext^i_M(V_N, W) \quad \textit{for all } i \geqslant 0.$$

**Proof.** By Frobenius reciprocity [V1, I, 5.10] we deduce that

$$Ext^*_G\big(V, i^G_Q(W)\big) = Ext^*_Q(V, W).$$

Since $N$ is the union of its compact open subgroups, we deduce from [V1, I, 4.10] the exactness of the functor

$$Mod_N \to Mod_R, \quad V \mapsto V_N.$$

Thus, the statement follows from Lemma 5. □

After having established the main techniques for computing cohomology of representations, we are able to take the first step in order to prove Theorem 1. The following proposition is also well known in the classical case. Here, assumption (4) on $R$ enters for the first time.

**Proposition 9.** *We have*

$$H^*(G, \mathbf{1}) = \Lambda^* X(\mathbf{G}),$$

*where $\Lambda^* X(\mathbf{G})$ denotes the exterior algebra of $X(\mathbf{G})$.*

**Proof.** We copy the proof of the classical case [BW, Chapter X, Proposition 2.6].

1st case. $\mathbf{G}$ is semi-simple and simply connected. Then we apply the $G$-fixed point functor to the resolution of the trivial representation in Lemma 4. The result is a constant coefficient system on a base chamber inside the Bruhat–Tits building, which is contractible. Thus, we obtain $H^*(G, \mathbf{1}) = H^0(G, \mathbf{1}) = R$.

2nd case. $\mathbf{G}$ is semi-simple. Then we consider its simply connected covering $\sigma : \mathbf{G}' \to \mathbf{G}$. The induced homomorphism $G' \to G$ has finite kernel, its image is a closed cocompact normal subgroup. We apply Lemma 5 twice, to $G \supset \sigma(G')$ and $G' \supset ker(\sigma)$.

3rd case. $\mathbf{G}$ is arbitrary reductive. Let $D\mathbf{G}$ be the derived group of $\mathbf{G}$ and put $G' = D\mathbf{G}(F)$. Then we have $G \supset {}^0G \supset DG'$, where $DG'$ denotes the derived group of $G'$.

Moreover, the quotient ${}^0G/DG'$ is compact. Therefore, we conclude by the previous case, Lemma 5 and Corollary 7 that

$$H^*\big({}^0G, \mathbf{1}\big) = H^*\big(DG', \mathbf{1}\big) = H^0\big(DG', \mathbf{1}\big) = R.$$

With the same arguments, we see that

$$H^*(G, \mathbf{1}) = H^*\big(G/{}^0G, \mathbf{1}\big).$$

Now it is known that the cohomology of a finite rank free commutative (discrete) group $L$ coincides with the cohomology of the corresponding torus:

$$H^*(L, \mathbf{1}) = \Lambda^*\big(Hom(L, \mathbb{Z})\big) \otimes_{\mathbb{Z}} R.$$

Applying this fact to $G/{}^0G$, we get

$$H^*(G, \mathbf{1}) = \Lambda^*\big(Hom\big(G/{}^0G, \mathbb{Z}\big)\big) \otimes_{\mathbb{Z}} R.$$

By assumption (4) on $R$ we have $Hom(G/{}^0G, \mathbb{Z}) \otimes_{\mathbb{Z}} R \cong X(\mathbf{G})$ from which the result follows. □

**Corollary 10.** *Let $I \subset \Delta$. Then we have*

$$H^*\big(G, i^G_{P_I}\big) = H^*(P_I, \mathbf{1}) = H^*(M_I, \mathbf{1}) = \Lambda^* X(\mathbf{M}_I).$$

**Proof.** The statement follows from Lemma 8, Proposition 9 and by our assumption (4) on $R$. □

In order to compute the cohomology of generalized Steinberg representations, we need the following proposition. For two subsets $I \subset I' \subset \Delta$ with $|I' \setminus I| = 1$, we let

$$p_{I,I'} : i^G_{P_{I'}} \to i^G_{P_I}$$

be the natural homomorphism induced by the surjection $G/P_I \to G/P_{I'}$. For arbitrary subsets $I, I' \subset \Delta$, with $|I'| - |I| = 1$ and $I' = \{\beta_1, \ldots, \beta_r\}$, we put

$$d_{I,I'} = \begin{cases} (-1)^i \, p_{I,I'}, & I' = I \cup \{\beta_i\}, \\ 0, & I \not\subset I'. \end{cases}$$

**Proposition 11.** *Let $I \subset \Delta$. The complex*

$$0 \to i^G_G \to \bigoplus_{\substack{I \subset K \subset \Delta \\ |\Delta \setminus K| = 1}} i^G_{P_K} \to \bigoplus_{\substack{I \subset K \subset \Delta \\ |\Delta \setminus K| = 2}} i^G_{P_K} \to \cdots \to \bigoplus_{\substack{I \subset K \subset \Delta \\ |K \setminus I| = 1}} i^G_{P_K} \to i^G_{P_I} \to v^G_{P_I} \to 0,$$

*with differentials induced by the $d_{K,K'}$ above is acyclic.*

**Proof.** See [SS, §6, Proposition 13] for the case of $I = \{\alpha_1, \alpha_2, \ldots, \alpha_i\}$, $i \geqslant 1$, and $G = GL_n$. The proof there is only formulated for coefficients in the ring of integers $\mathbb{Z}$. However, the proof holds for arbitrary rings, since it is of combinatorial nature.

A different approach consists of using [SS, §2, Proposition 6]. It says: let $G_1, \ldots, G_m$ be a family of subgroups in some bigger abelian group $G$. Suppose that the following identities are satisfied for all subsets $A, B \subset \{1, \ldots, m\}$:

$$\left(\sum_{i\in A} G_i\right) \cap \left(\bigcap_{j\in B} G_j\right) = \sum_{i\in A}\left(G_i \cap \left(\bigcap_{j\in B} G_j\right)\right).$$

Then the natural (oriented) complex

$$G \leftarrow \bigoplus_{i=1}^{m} G_i \leftarrow \bigoplus_{\substack{i,j=1\\ i<j}}^{m} G_i \cap G_j \leftarrow \bigoplus_{\substack{i,j,k=1\\ i<j<k}}^{m} G_i \cap G_j \cap G_k \leftarrow \cdots$$

is an acyclic resolution of $\sum_i G_i \subset G$. We apply this proposition to the $G$-modules $i^G_{P_K}$, where $I \subset K \subset \Delta$ and $|\Delta \setminus K| = 1$. The condition of the proposition is fulfilled. Indeed, we have

$$i^G_{P_I} \cap i^G_{P_J} = i^G_{P_{I\cup J}} \quad \text{and} \quad i^G_{P_I} \cap \left(i^G_{P_J} + i^G_{P_K}\right) = \left(i^G_{P_I} \cap i^G_{P_J}\right) + \left(i^G_{P_I} \cap i^G_{P_K}\right),$$

for all subsets $I, J, K \subset \Delta$. The first identity follows from the fact that $P_{I\cup J}$ is the parabolic subgroup generated by $P_I$ and $P_J$. For the second one confer [BW, 4.5, 4.6], respectively [L, 8.1, 8.1.4] (The statement there is formulated in the case where $R = \mathbb{C}$. The result holds also in our general situation. The proof relies on the exactness of the Jacquet-functor and a description of the $S$-modules $(i^G_{P_I})_U$ using the filtration in our proof of Proposition 15.) $\square$

Now, we can treat the cohomology of generalized Steinberg representations. The following theorem makes use of assumption (3) on $R$ in the case when $G$ is not split.

**Theorem 12.** *Let $G$ be semi-simple and let $I \subset \Delta$. Then we have*

$$H^i\left(G, v^G_{P_I}\right) = \begin{cases} R, & i = |\Delta \setminus I|, \\ 0, & \textit{otherwise}. \end{cases}$$

**Proof.** The proof is the same as in [BW, Chapter X, Proposition 4.7]. A not very different approach works as follows. Apply the cohomology functor $H^*(G, -)$ to the acyclic complex of Proposition 11. We obtain a complex

$$0 \to \Lambda^* X(\mathbf{G}) \to \bigoplus_{\substack{I\subset K\subset\Delta\\ |\Delta\setminus K|=1}} \Lambda^* X(\mathbf{M}_K) \to \cdots \to \bigoplus_{\substack{I\subset K\subset\Delta\\ |K\setminus I|=1}} \Lambda^* X(\mathbf{M}_K) \to \Lambda^* X(\mathbf{M}_I) \to 0.$$

Suppose that $\mathbf{G}$ is split. In this case it is well known (cf. [J, Chapter II, 1.18]) that $X^*(\mathbf{M}_K)_F$ may be identified with the submodule of $X^*(\mathbf{S})_F$ defined by

$$\left\{\chi \in X^*(\mathbf{S})_F;\ \left\langle \chi, \alpha^\vee \right\rangle = 0\ \forall \alpha \in K\right\},$$

where $\langle \cdot, \cdot \rangle : X^*(\mathbf{S})_F \times X_*(\mathbf{S})_F \to \mathbb{Z}$ is the natural pairing and $\alpha^\vee$ denotes the corresponding coroot. Using the Hochschild–Serre spectral sequence, we may assume without loss of generality that $\mathbf{G}$ is simply connected. If we denote by $\{\omega_\alpha \in X^*(\mathbf{S})_F;\ \alpha \in \Delta\}$ the fundamental weights of $\mathbf{G}$ with respect to $\mathbf{S} \subset \mathbf{P}$, then we get

$$X(\mathbf{M}_K) \cong \bigoplus_{\alpha \in \Delta \setminus K} R \cdot \omega_\alpha \subset X(\mathbf{S}).$$

Thus we see—again by using [SS, §2, Proposition 6]—that the complex above is acyclic with respect to $\Lambda^r$ for

$$r < rk\big(Z(\mathbf{M}_I)\big) = |\Delta \setminus I|.$$

In the case $rk(Z(\mathbf{M}_I)) = r$ all the entries of the complex vanish except for $\Lambda^r X(\mathbf{M}_I) = R$. Using the standard spectral sequence associated to the complex above proves the claim in the split case.

In the general case, let $E/F$ be our fixed Galois splitting field of $\mathbf{G}$. Then we deduce with the same arguments that the corresponding complex of $E$-rational characters has the desired property. Applying the $Gal(E/F)$-fixed point functor to this complex yields the claim. Note that the fixed point functor is exact by assumption (3) on $R$. □

For attacking Theorem 1 we still need two lemmas.

**Lemma 13.** *Let $V$ be a smooth representation of $G$. Suppose that there exists an element $z \in Z(G)$ in the center of $G$ and an element $c \in R$, such that $c - 1 \in R^\times$ and $z \cdot v = c \cdot v$ for all $v \in V$. Then we have*

$$H^*(G, V) = 0.$$

**Proof.** See [BW, Chapter X, Proposition 4.2] for the classical case. We repeat shortly the argument. By identifying Ext-groups with Yoneda-Ext-groups, we have to show that for all $n \in \mathbb{N}$, all $n$-extensions of $\mathbf{1}$ by $V$ are trivial. More generally, we will show that if $U$ is a $R$-module with trivial $G$-action, then there are no non-trivial extensions of $U$ by $V$. In fact, let

$$E^\bullet : 0 \to V \to E^1 \to E^2 \to \cdots \to E^n \to U \to 0$$

be an arbitrary $n$-extension. Since $z$ lies in the center of $G$, it defines an endomorphism of $E^\bullet$ and we get the identity $E^\bullet = c.E^\bullet$. Here $c.E^\bullet$ denotes the scalar multiplication of $R$ on the module $Ext^n_G(U, V)$ (confer [M, Chapter III, Theorem 2.1]). Thus, we have $0 = E^\bullet - c.E^\bullet = (1-c).E^\bullet$. Since $1 - c \in R^\times$, we conclude that $E^\bullet = 0 \in Ext^n_G(U, V)$. □

**Lemma 14.** *Let $H \subset G$ be a closed subgroup and let $W$ be a smooth representation of $H$. Then we have*

$$\widetilde{c\text{-}i_H^G(W)} \cong i_H^G\big(\widetilde{W}\delta_H\big),$$

*where $\delta_H$ is the modulus character of $H$.*

**Proof.** This follows from [V1, I, 5.11] together with the fact that $G$ is unimodular. □

**Proposition 15.** *Let $G$ be semi-simple and let $I, J \subset \Delta$. Then we have*

$$Ext_G^*\big(i_{P_I}^G, i_{P_J}^G\big) = \begin{cases} \Lambda^* X(\mathbf{M}_J), & \textit{if } J \subset I, \\ 0, & \textit{otherwise}. \end{cases}$$

**Proof.** By Lemma 8 we have for all $i \geqslant 0$ isomorphisms

$$Ext_G^i\big(i_{P_I}^G, i_{P_J}^G\big) \cong Ext_{M_J}^i\big(\big(i_{P_I}^G\big)_{U_J}, \mathbf{1}\big),$$

where $(i_{P_I}^G)_{U_J}$ is the Jacquet-module of $i_{P_I}^G$ with respect to $M_J$. In the case $R = \mathbb{C}$ there is constructed in [Ca3, 6.3]—a substitute for the Mackey formula—a decreasing $\mathbb{N}$-filtration $\mathcal{F}^\bullet$ of smooth $P_J$-submodules on $i_{P_I}^G$ defined by

$$\mathcal{F}^i = \Big\{ f \in i_{P_I}^G;\ \operatorname{supp}(f) \subset \bigcup_{\substack{w \in W_I \backslash W / W_J \\ l(w) \geqslant i}} P_I \backslash P_I w P_J \Big\}, \quad i \in \mathbb{N}.$$

Here the length $l(w)$ of a double coset $w \in W_I \backslash W / W_J$ is the length of its Kostant-representative which is the one of minimal length within its double coset. In the following, we will identify the double cosets with its Kostant-representatives. There are canonical isomorphisms

$$gr_{\mathcal{F}^\bullet}^i\big(i_{P_I}^G\big) \cong \bigoplus_{\substack{w \in W_I \backslash W / W_J \\ l(w) = i}} c\text{-}i_{P_J \cap w^{-1} P_I w}^{P_J}$$

of smooth $P_J$-modules for all $i \geqslant 0$. Furthermore, we have for every $w \in W_I \backslash W / W_J$ an isomorphism

$$\Big(c\text{-}i_{P_J \cap w^{-1} P_I w}^{P_J}\Big)_{U_J} \cong c\text{-}i_{M_J \cap w^{-1} P_I w}^{M_J}(\gamma_w)$$

of smooth $M_J$-modules, where $\gamma_w$ is the modulus character of $P_J \cap w^{-1} P_I w$ acting on $U_J / U_J \cap w^{-1} P_I w$. The first isomorphism is a corollary of Proposition 6.3.1 (loc. cit.) (see also [V1, I, 1.7(iii)]), whereas the second one is the content of Proposition 6.3.3 (loc. cit.). In the general case, i.e., for our specified ring $R$, the same formulas hold. In fact, the proof can be taken over word by word. Since $M_J \cap w^{-1} P_I w$ is a parabolic subgroup in $M_J$, we observe that

$$c\text{-}i^{M_J}_{M_J \cap w^{-1} P_I w}(\gamma_w) = i^{M_J}_{M_J \cap w^{-1} P_I w}(\gamma_w).$$

The character $\gamma_w$ is the norm of the rational character

$$\det Ad_{Lie(\mathbf{U_J})} / \det Ad_{w^{-1} Lie(\mathbf{P_I}) w \cap Lie(\mathbf{U_J})} \in X^*\big(\mathbf{P_J} \cap w^{-1} \mathbf{P_I} w\big).$$

Its restriction to $S$ is given by

$$\gamma_{w|S} = \Bigg| \prod_{\substack{\alpha \in \Phi^+ \setminus \Phi_J^+ \\ w\alpha \in \Phi^- \setminus \Phi_I^-}} \alpha \Bigg|. \tag{2}$$

Fix an element $w \in W_I \backslash W / W_J$. We are going to show that

$$Ext^*_{M_J}\Big(i^{M_J}_{M_J \cap w^{-1} P_I w}(\gamma_w), \mathbf{1}\Big) = 0,$$

unless $w = 1$ and $J \subset I$. Since the Jacquet-functor is exact, this will give by successive application of the long exact cohomology sequence with respect to the filtration $\mathcal{F}^\bullet$ the statement of our proposition. By Lemmas 6 and 14 we conclude that

$$Ext^*_{M_J}\Big(c\text{-}i^{M_J}_{M_J \cap w^{-1} P_I w}(\gamma_w), \mathbf{1}\Big) \cong Ext^*_{M_J}\Big(\mathbf{1}, i^{M_J}_{M_J \cap w^{-1} P_I w}(\tilde{\gamma}_w \delta_{M_J \cap w^{-1} P_I w})\Big),$$

where $\delta_{M_J \cap w^{-1} P_I w}$ denotes the modulus character of the parabolic subgroup $M_J \cap w^{-1} P_I w$ of $M_J$ and $\tilde{\gamma}_w$ is the smooth dual of $\gamma_w$. The Levi decomposition of the latter group is given by

$$M_J \cap w^{-1} P_I w = M_{J \cap w^{-1} I} \cdot \big(M_J \cap w^{-1} U_I w\big)$$

(see [C, Proposition 2.8.9]). So, the restriction of $\delta_{M_J \cap w^{-1} P_I w}$ to $S$ is the norm of the rational character

$$\prod_{\substack{\alpha \in \Phi_J^+ \\ w\alpha \in \Phi^+ \setminus \Phi_I^+}} \alpha,$$

i.e.,

$$\delta_{M_J \cap w^{-1} P_I w|S} = \Bigg| \prod_{\substack{\alpha \in \Phi_J^+ \\ w\alpha \in \Phi^+ \setminus \Phi_I^+}} \alpha \Bigg|. \tag{3}$$

In the case where $J \not\subset I$ or $w \neq 1$ we deduce from the following lemma the existence of an element $z$ in the center of $M_{J\cap w^{-1}I}$ such that

$$\tilde{\gamma}_w(z)\delta_{M_J\cap w^{-1}P_I w}(z) - 1 \in R^\times.$$

By Lemma 13 we conclude that

$$Ext^*_{M_J}\Big(c\text{-}i^{M_J}_{M_J\cap w^{-1}P_I w}(\gamma_w), \mathbf{1}\Big) = 0.$$

In the case $J \subset I$ we obtain therefore an isomorphism

$$Ext^*_G\big(i^G_{P_I}, i^G_{P_J}\big) \cong Ext^*_{M_J}(\mathbf{1}, \mathbf{1}) = \Lambda^* X(\mathbf{M}_J)$$

which is induced by the element $w = 1$. □

I want to stress that the following lemma uses assumption (2) on $R$.

**Lemma 16.** *Let $J \not\subset I$ or $w \neq 1$. Then there exists an element $z \in Z(M_{J\cap w^{-1}I})$ such that $\tilde{\gamma}_w(z)\delta_{M_J\cap w^{-1}P_I w}(z) - 1 \in R^\times$.*

**Proof.** 1st case. Let $w \neq 1$. Then we have $\gamma_w \neq \mathbf{1}$. In fact, $\gamma_w = 1$ would imply that

$$Lie(U_J) \subset Lie\big(w^{-1}P_I w\big)$$

or equivalently $U_J \subset w^{-1}P_I w$. But in general one has

$$P_{J\cap w^{-1}I} = \big(P_J \cap w^{-1}P_I w\big)\cdot U_J$$

(see [C, Proposition 2.8.4]). Thus, we deduce that the intersection $P_J \cap w^{-1}P_I w$ is a parabolic subgroup. This is only true if $w = 1$.

We want to recall that for any subset $K \subset \Delta$ the maximal split torus in the center $Z(M_K)$ of $M_K$ coincides with the connected component of the identity in $\bigcap_{\alpha\in K} kern(\alpha) \subset S$. Since $Z(M_J) \subset Z(M_{J\cap w^{-1}I})$, it is enough to construct an element $z \in Z(M_J)$ which has the desired property. From the representation (2) we may easily conclude the existence of an element $z \in Z(M_J)$ with $\tilde{\gamma}_w(z) \neq 1$. Our purpose is to show the existence of an element $z \in Z(M_J)$ such that $\tilde{\gamma}_w(z) - 1 \in R^\times$. We may suppose that $\mathbf{G}$ is adjoint. Let

$$\big\{\omega_\alpha \in X_*(S);\ \alpha \in \Delta\big\}$$

be the dual base (co-fundamental weights) of $\Delta$, i.e., $\langle\omega_\beta, \alpha\rangle = \delta_{\alpha,\beta}$, for all $\alpha, \beta \in \Delta$. Since $\gamma_w \neq \mathbf{1}$, it is possible to find a root $\alpha \in \Delta \setminus J$ such that $w\alpha \in \Phi^- \setminus \Phi^-_I$. Put

$$z := \omega_\alpha\big(\varpi_F^{-1}\big).$$

Then we have $z \in Z(M_J)$ and

$$\tilde{\gamma}_w(z) - 1 = q^r - 1$$

for some $1 \leqslant r \leqslant n_\alpha$. By assumption (2) on $R$, the product $\prod_{r \leqslant \sup\{n_\alpha;\ \alpha\in\Delta\}}(1-q^r)$ is invertible in $R$. Further, we see from the expression (3) that $\delta_{M_J \cap w^{-1} P_I w}(z) = 1$. This completes the proof in the first case.

2nd case. Let $w = 1$ and $J \not\subset I$. Then we have $\gamma_w = 1$. Since $J \not\subset I$, we see that the restriction of $\delta_{M_J \cap P_I}$ to $Z(M_{J\cap I})$ is not trivial. Again, we can find similarly to the first case an element $z \in Z(M_{J\cap I})$ such that $\delta_{M_J\cap P_I}(z) - 1 \in R^\times$. $\square$

**Proposition 17.** *Let $G$ be semi-simple and let $I, J \subset \Delta$. Then we have*

$$Ext^*_G\big(v^G_{P_I}, i^G_{P_J}\big) = \begin{cases} \Lambda^* X(\mathbf{M}_J)[-|\Delta \setminus I|], & \Delta = I \cup J, \\ 0, & \textit{otherwise}. \end{cases}$$

**Proof.** We apply the acyclic complex of Proposition 11 to the representation $v^G_{P_I}$. Taking an injective resolution of $i^G_{P_J}$ then gives rise in the usual way to a double complex such that its associated spectral sequence converges to $Ext^*_G(v^G_{P_I}, i^G_{P_J})$. The $E_1$ term of this spectral sequence has the shape

$$0 \to Ext^*_G\big(i^G_{P_I}, i^G_{P_J}\big) \to \bigoplus_{\substack{I\subset L\subset\Delta\\ |L\setminus I|=1}} Ext^*_G\big(i^G_{P_L}, i^G_{P_J}\big) \to \bigoplus_{\substack{I\subset L\subset\Delta\\ |L\setminus I|=2}} Ext^*_G\big(i^G_{P_L}, i^G_{P_J}\big) \to \cdots$$
$$\to \bigoplus_{\substack{I\subset L\subset\Delta\\ |\Delta\setminus L|=1}} Ext^*_G\big(i^G_{P_L}, i^G_{P_J}\big) \to Ext^*_G\big(i^G_G, i^G_{P_J}\big) \to 0.$$

By Proposition 15 we see that $K := I \cup J$ is the minimal subset of $\Delta$ containing $I$ with $Ext^*_G(i^G_{P_K}, i^G_{P_J}) \neq 0$. Hence, the $E_1$ term reduces to

$$0 \to \Lambda^* X(\mathbf{M}_J) \to \bigoplus_{\substack{K\subset L\subset\Delta\\ |L\setminus K|=1}} \Lambda^* X(\mathbf{M}_J) \to \bigoplus_{\substack{K\subset L\subset\Delta\\ |L\setminus K|=2}} \Lambda^* X(\mathbf{M}_J) \to \cdots \to \bigoplus_{\substack{K\subset L\subset\Delta\\ |\Delta\setminus L|=1}} \Lambda^* X(\mathbf{M}_J)$$
$$\to \Lambda^* X(\mathbf{M}_J) \to 0.$$

In the case of $K = \Delta$ we are obviously done. In the case $K \neq \Delta$ we see that the cohomology of the $E_1$ term vanishes, since it is a constant coefficient system on the standard simplex corresponding to the set $K$. $\square$

**Proof of Theorem 1.** This time we apply Proposition 11 to $v^G_{P_J}$. This yields by taking a projective resolution of $v^G_{P_I}$ a double complex such that its associated spectral sequence converges to $Ext^*_G(v^G_{P_I}, v^G_{P_J})$. The $E_1$ term of this spectral sequence is just

$$0 \to Ext^*_G\big(v^G_{P_I}, i^G_G\big) \to \bigoplus_{\substack{J\subset L\subset\Delta\\ |\Delta\setminus L|=1}} Ext^*_G\big(v^G_{P_I}, i^G_{P_L}\big) \to \bigoplus_{\substack{J\subset L\subset\Delta\\ |\Delta\setminus L|=2}} Ext^*_G\big(v^G_{P_I}, i^G_{P_L}\big) \to \cdots$$
$$\to \bigoplus_{\substack{J\subset L\subset\Delta\\ |L\setminus J|=1}} Ext^*_G\big(v^G_{P_I}, i^G_{P_L}\big) \to Ext^*_G\big(v^G_{P_I}, i^G_{P_J}\big) \to 0.$$

By Proposition 17 we conclude that the minimal subset $K$ of $\Delta$ containing $J$ with $Ext^*_G(v^G_{P_I}, i^G_{P_K}) \neq 0$ is

$$K = (\Delta \setminus I) \cup J = (\Delta \setminus I) \,\dot\cup\, (I \cap J).$$

Therefore, the $E_1$ term reduces to

$$0 \to \Lambda^* X(\mathbf{G})\big[-|\Delta \setminus I|\big] \to \bigoplus_{\substack{K\subset L\subset\Delta\\ |\Delta\setminus L|=1}} \Lambda^* X(\mathbf{M}_L)\big[-|\Delta \setminus I|\big] \to \cdots$$
$$\to \bigoplus_{\substack{K\subset L\subset\Delta\\ |L\setminus K|=1}} \Lambda^* X(\mathbf{M}_L)\big[-|\Delta \setminus I|\big] \to \Lambda^* X(\mathbf{M}_K)\big[-|\Delta \setminus I|\big] \to 0.$$

This complex is precisely—up to shifts—the complex for the computation of the cohomology of $v^G_{P_K}$ for a semi-simple group $G$ (cf. Theorem 12, respectively [BW, Chapter X, Proposition 4.7])! Thus, we obtain an isomorphism

$$H^*\big(G, v^G_{P_K}\big)\big[-\big(|J| - |K|\big) - |\Delta \setminus I|\big] \cong Ext^*_G\big(v^G_{P_I}, v^G_{P_J}\big).$$

It remains to compute the degree $d$, where the latter space does not vanish. The degree is by Theorem 12 equal to

$$\begin{aligned} d &= |\Delta \setminus K| + |\Delta \setminus I| + |J| - |K| \\ &= \big|\Delta \setminus \big(\Delta \setminus I \,\dot\cup\, (I \cap J)\big)\big| + |\Delta \setminus I| + |J| - \big|\Delta \setminus I \,\dot\cup\, (I \cap J)\big| \\ &= \big|I \cap \Delta \setminus (I \cap J)\big| + |J| - |I \cap J| = \big|I \setminus (I \cap J)\big| + |J| - |I \cap J| \\ &= |I| - |I \cap J| + |J| - |I \cap J| = |I \cup J| - |I \cap J|. \qquad \square \end{aligned}$$

**Remark.** An argument of J.-F. Dat shows that Theorem 1 even holds if $R$ is not self-injective. In fact, in his paper [D, Theorem 3.1.4] he first shows the statement for an algebraically closed field which is *fortement banal* for $G$. Then he uses this result to deduce the general case by elementary commutative algebra.

**Proof of Corollary 2.** Consider the projection $G \to G/Z(G)$ onto the adjoint group of $G$. The action of $Z(G)$ on $v_{P_I}^G$ and $v_{P_J}^G$ is trivial. By applying Lemma 5 to this situation, we get a spectral sequence

$$Ext^q_{G/Z(G)}\big(H_p\big(Z(G), v_{P_I}^G\big), v_{P_J}^G\big) \Rightarrow Ext_G^{p+q}\big(v_{P_I}^G, v_{P_J}^G\big).$$

By the proof of Proposition 9, we deduce that

$$H^*\big(Z(G), \mathbf{1}\big) = \Lambda^* Hom\big(Z(G)/{}^0Z(G), \mathbb{Z}\big) \otimes R \cong \Lambda^* R^d.$$

Therefore, we get

$$H_*\big(Z(G), v_{P_I}^G\big) = H^*\big(Z(G), \mathbf{1}\big)^\vee \otimes v_{P_I}^G \cong \bigoplus_{j=0}^{d} \big(v_{P_I}^G\big)^{\binom{d}{j}}.$$

Now we apply Theorem 1 together with Corollary 7. $\square$

In the remainder of this paper we give another corollary in the case of the general linear group and where $R$ is an algebraically closed field. It computes the Ext-group of elliptic representations (cf. [D] for the definition of these representations). This corollary has been pointed out to me by C. Kaiser in the case $R = \mathbb{C}$. M.-F. Vignéras has communicated to me that it also holds for algebraically closed fields $R$ of positive characteristic satisfying our assumptions.

In the following, we use the Zelevinsky classification of smooth $G$-representations in order to describe the elliptic ones [Z]. This description holds for any algebraically closed field which is banal for $R$ (cf. also [V3] for a treatment of the Zelevinsky classification in the modular case). Let $G = GL_n$ with $n = r \cdot k$ for some integers $k, r > 0$. Let $P_{r,k}$ be the upper block parabolic subgroup containing the Levi subgroup

$$\underbrace{GL_r \times \cdots \times GL_r}_{k}.$$

We fix an irreducible cuspidal representation $\sigma$ of $GL_r$. For any integer $i \geqslant 0$, we put $\sigma(i) = \sigma \otimes |\det|^i$, where $\det : GL_r \to F^\times$ is the determinant. Consider the graph $\Gamma$ consisting of the vertices $\{\sigma, \sigma(1), \ldots, \sigma(k-1)\}$ and the edges $\{\{\sigma(i), \sigma(i+1)\};\ i = 0, \ldots, k-2\}$. Thus we can illustrate $\Gamma$ in the shape

$$\sigma - \sigma(1) - \cdots - \sigma(k-1).$$

An orientation of $\Gamma$ is given by choosing a direction on each edge. We denote by $Or(\Gamma)$ the set of all orientations on $\Gamma$.

Let $\mathcal{J}$ be the set of irreducible subquotients of $\tilde{i}_{P_{r,k}}^G(\sigma \otimes \sigma(1) \otimes \cdots \otimes \sigma(k-1))$, where $\tilde{i}_{P_{r,k}}^G$ denotes the normalized induction functor. Following [Z, 2.2], there is a bijection

$$\omega : Or(\Gamma) \to \mathcal{J},$$

which we briefly describe. Let $S_k$ be the symmetric group of the set $\{0,\dots,k-1\}$. Consider the map

$$S_k \to Or(\Gamma), \quad w \mapsto \Gamma(w)$$

defined as follows. The edge $\{\sigma(i),\sigma(i+1)\}$ is oriented from $\sigma(i)$ to $\sigma(i+1)$—symbolized as $\sigma(i)\to\sigma(i+1)$—if and only if $w(i)<w(i+1)$. One easily verifies the surjectivity of this map. Let $\vec{\Gamma}$ be an orientation of $\Gamma$. Choose an element $w\in S_k$ such that $\vec{\Gamma}=\Gamma(w)$. Then $\omega(\vec{\Gamma})$ is defined to be the unique irreducible quotient of

$$\tilde{i}^G_{P_{r,k}}\big(\sigma\big(w(0)\big)\otimes\cdots\otimes\sigma\big(w(k-1)\big)\big).$$

In loc. cit. 2.7 it is shown that this representation does not depend on the chosen representative $w$.

Denote by $\Delta_k=\{\alpha_0,\dots,\alpha_{k-2}\}$ the set of simple roots of $GL_k$ with respect to the standard root system of $GL_k$. Let $\mathcal{P}(\Delta_k)$ be its power set. For a subset $I\subset\Delta_k$, we let $\Theta(I)\in Or(\Gamma)$ be the orientation of $\Gamma$ defined by $\sigma(i)\to\sigma(i+1)$ if and only if $\alpha_i\in I$, $i=0,\dots,k-2$. It is easily seen that we get in this way a bijection

$$\Theta:\mathcal{P}(\Delta_k)\to Or(\Gamma).$$

For any subset $I\subset\Delta_k$, we put finally

$$v^G_I(\sigma):=\omega\big(\Theta(I)\big). \qquad \square$$

**Example 1.** Consider the special case $r=1$ and $\sigma=|\cdot|^{(1-n)/2}$. Then we have $P_{r,k}=P$,

$$\tilde{i}^G_P\big(\sigma\otimes\cdots\otimes\sigma(n-1)\big)=i^G_P \quad \text{and} \quad v^G_I(\sigma)=v^G_{P_I}, \quad \text{for all } I\subset\Delta=\Delta_k.$$

**Corollary 18.** *Let $I,J\subset\Delta_k$. Set $i:=|I\cup J|-|I\cap J|$. Then we get*

$$Ext^*_G\big(v^G_I(\sigma),v^G_J(\sigma)\big)=R[-i]\oplus R[-i-1].$$

**Proof.** We make use of the theory of types of Bushnell and Kutzko [BK] (see also [V2,V3] for the modular case). Let $(K,\lambda)$ be the type of the block containing $v^G_\emptyset(\sigma)$. By definition $K$ is a certain compact open subgroup of $G$ and $\lambda$ is an irreducible representation of $K$, such that the functor

$$V\mapsto Hom_G\big(c\text{-}i^G_K(\lambda),V\big)$$

from the block above to the category of right $End_G(i^G_K(\lambda))$-modules is an equivalence of categories. Furthermore, there exists an unramified extension $F'/F$, such that the following holds ([BK,V2,V3, IV.6.3]). Set $G'=GL_k(F')$ and let $I'\subset G'$ be the standard Iwahori subgroup. Then there is an algebra isomorphism [BK, 7.6.19]

$$End_{G'}\big(i^G_{I'}(\mathbf{1})\big)\ \to\ End_G\big(i^G_K(\lambda)\big).$$

This isomorphism induces an equivalence between the block of unipotent $G'$-representations and the block of $G$-representations containing $v_{\emptyset}^{G}(\sigma)$. Under this identification, the representations $v_{I}^{G}(\sigma)$ and $v_{P_I}^{G}$ correspond to each other. This can be seen from the fact that the equivalence is compatible with normalized induction [BK, 7.6.21] and with twists [BK, 7.5.12]. Thus, the statement follows from Corollary 2. $\square$

## Acknowledgments

I am grateful to J.-F. Dat for his numerous remarks on this paper. He explained to me how to generalize my proof from the case $R = \mathbb{C}$ to the case of a certain self-injective ring. I thank the IHÉS and J.-F. Dat for the invitation in June 2003. I also thank A. Huber, M. Rapoport, P. Schneider, T. Wedhorn and the referee for helpful remarks. Finally, I thank C. Kaiser and M.-F. Vignéras for pointing out to me Corollary 18 as a consequence of the results above.

## References

[BK] C.J. Bushnell, P.C. Kutzko, The Admissible Dual of $GL(N)$ via Compact Open Subgroups, Ann. of Math. Stud., vol. 129, Princeton Univ. Press, Princeton, NJ, 1993.

[BW] A. Borel, N. Wallach, Continuous Cohomology, Discrete Subgroups, and Representations of Reductive Groups, second ed., Math. Surveys Monogr., vol. 67, Amer. Math. Soc., Providence, RI, 2000.

[Ca1] W. Casselman, On a $p$-adic vanishing theorem of Garland, Bull. Amer. Math. Soc. 80 (1974) 1001–1004.

[Ca2] W. Casselman, Non-unitary argument for $p$-adic representations, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 28 (1981) 907–928.

[Ca3] W. Casselman, Introduction to the theory of admissible representations and $p$-adic reductive groups, preprint (1995).

[C] R.W. Carter, Finite Groups of Lie Type, Wiley Classics Lib., Wiley, New York, 1993.

[D] J.-F. Dat, Correspondances de Langlands locale et monodromie des espaces de Drinfeld, math.NT/0407055.

[J] J.-C. Jantzen, Representations of Algebraic Groups, Pure Appl. Math., vol. 131, Academic Press, New York, 1987.

[L] G.I. Laumon, Cohomology of Drinfeld Modular Varieties, Part I: Geometry, Counting of Points and Local Harmonic Analysis, Cambridge Stud. Adv. Math., vol. 41, Cambridge Univ. Press, Cambridge, 1996.

[M] S. Mac Lane, Homology, Grundlehren Math. Wiss., vol. 114, Springer, New York, 1963.

[O] S. Orlik, The cohomology of period domains for reductive groups over local fields, preprint (2002).

[SS] P. Schneider, U. Stuhler, The cohomology of $p$-adic symmetric spaces, Invent. Math. 105 (1991) 47–122.

[V1] M.-F. Vignéras, Représentations $l$-modulaires d'un groupe réductif $p$-adique avec $l \neq p$, Progr. Math., vol. 137, Birkhäuser, Boston, 1996.

[V2] M.-F. Vignéras, Extensions between irreducible representations of a $p$-adic $\mathrm{GL}(n)$, in: Olga Taussky-Todd: in Memoriam, Pacific J. Math. (1997).

[V3] M.-F. Vignéras, Induced representations of reductive $p$-adic groups in characteristic $\ell \neq p$, Selecta Math. (N.S.) 4 (1998) 549–623.

[Z] A.V. Zelevinsky, Induced representations of reductive $p$-adic groups. II, Ann. Sci. École Norm. Sup. (4) 13 (2) (1980) 165–210.