Improving diameter bounds for distance-regular graphs

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Abstract

In this paper we study the sequence \((c_i)_{0 \leq i \leq d}\) for a distance-regular graph. In particular we show that if \(d \geq 2j\) and \(c_j > 1\) then \(c_{2j-1} > c_j\) holds. Using this we give improvements on diameter bounds by A. Hiraki, J.H. Koolen [An improvement of the Ivanov bound, Ann. Comb. 2 (2) (1998) 131–135], and L. Pyber [A bound for the diameter of distance-regular graphs, Combinatorica 19 (4) (1999) 549–553], respectively, by applying this inequality.

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1. Introduction

Let \(\Gamma\) be a distance-regular graph of diameter \(d \geq 2\), valency \(k \geq 2\) and intersection numbers \(c_i, a_i, b_i\ (0 \leq i \leq d)\). (For definitions, see next section.) We define

\[ h = h(\Gamma) := |\{i \mid 1 \leq i \leq d - 1 \text{ and } (c_i, a_i, b_i) = (c_1, a_1, b_1)\}|. \]
For each $1 \leq c \leq c_d$ we define
\[
\xi_c := \min \{ i \mid c_i \geq c \}, \\
\eta_c := \{|i \mid c_i = c\}|.
\]

In this paper we study for a given integer $c$, the number $\eta_c$ for a distance-regular graph. We obtain the following result:

**Theorem 1.1.** Let $\Gamma$ be a distance-regular graph of diameter $d \geq 2$. Let $c$ be an integer with $2 \leq c \leq c_d$. Then $\eta_c \leq \xi_c - 1$.

Using Theorem 1.1, we will give improvements on the diameter bounds of distance-regular graphs found by Hiraki and Koolen [5] and Pyber [8], respectively.

In [5] it was shown that the diameter of a distance-regular graph of valency $k$ is bounded by $\frac{1}{2} k^2 \eta_1$. In the next result we show we can interchange the power 2 by $\alpha$, where
\[
\alpha := \min \{ x > 0 \mid 4^x - 2^x \leq 1 \}.
\]

**Note 1.2.** We remark that $1.44 < \alpha < 1.441$.

**Theorem 1.3.** Let $\Gamma$ be a distance-regular graph of diameter $d \geq 2$ and valency $k \geq 3$. Let $C := \{c_i \mid i = 1, \ldots, d\}$. Then
\[
\xi_c \leq \frac{1}{2} \alpha^2 \eta_1 + 1 \tag{1}
\]
and
\[
\xi_c + \eta_c \leq \alpha^2 \eta_1 + 1 \tag{2}
\]
hold for all $c \in C$.

In particular if $h := h(\Gamma) \geq 2$ then
\[
\xi_c \leq \alpha^2 (h + 1) + 1 \tag{3}
\]
and
\[
\xi_c + \eta_c \leq 2 \alpha^2 (h + 1) + 1 \tag{4}
\]
hold for all $c \in C$.

**Corollary 1.4.** Let $\Gamma$ be a distance-regular graph of diameter $d \geq 2$, valency $k \geq 3$ and $h := h(\Gamma)$. Then
\[
d \leq \frac{1}{2} k^\alpha \eta_1 + 1.
\]

In particular if $h \geq 2$ then
\[
d \leq k^\alpha (h + 1) + 1
\]
holds.

In [8] Pyber showed that the diameter of distance-regular graphs is at most 5 times the 2 logarithm of the number of vertices. The following gives an improvement of this bound.
Theorem 1.5. Let \( \Gamma \) be a distance-regular graph with \( v \) vertices. Let \( d \) be the diameter of \( \Gamma \). Then
\[
d < \frac{8}{3} \log_2 v.
\]

The paper is organized as follows: in Section 2 we give definitions, in Section 3 we give the proof of Theorem 1.1, in Section 4 we give the proofs of Theorem 1.3 and Corollary 1.4, and in the last section we give the proof of Theorem 1.5.

2. Definitions

All graphs in this paper are undirected graphs without loops and multiple edges. Suppose that \( \Gamma \) is a finite connected graph with vertex set \( V\Gamma \). We define the distance between any two vertices \( x \) and \( y, d(x, y) \), to be the length of any shortest path in \( \Gamma \) between \( x \) and \( y \), and the diameter \( d \) of \( \Gamma \) to be the largest distance between any pair of vertices in \( V\Gamma \). For a vertex \( x \in V\Gamma \) and any non-negative integer \( i \) not exceeding \( d \), let \( \Gamma_i(x) \) denote the subset of vertices in \( V\Gamma \) that are at distance \( i \) from \( x \). Put \( \Gamma(x) := \Gamma_1(x) \) and \( \Gamma_{-1}(x) = \Gamma_{d+1}(x) := \emptyset \). For any two vertices \( x \) and \( y \) in \( V\Gamma \) at distance \( i \), let
\[
C_i(x, y) := \Gamma_{i-1}(x) \cap \Gamma_i(y)
\]
\[
A_i(x, y) := \Gamma_i(x) \cap \Gamma_{i+1}(y)
\]
\[
B_i(x, y) := \Gamma_{i+1}(x) \cap \Gamma_i(y).
\]

A graph \( \Gamma \) is called distance-regular if there are integers \( b_i, c_i \) \((0 \leq i \leq d)\) which satisfy
\[
c_i = |C_i(x, y)| \quad \text{and} \quad b_i = |B_i(x, y)|
\]
for any two vertices \( x \) and \( y \) in \( V\Gamma \) at distance \( i \). Clearly such a graph is regular of valency \( k := b_0 \). The numbers \( c_i, b_i, \) and \( a_i \), where
\[
a_i := k - b_i - c_i \quad (i = 0, \ldots, d)
\]
is the number of neighbors of \( y \) in \( \Gamma_i(x) \) for \( x, y \in V\Gamma \) at distance \( i \), are called the intersection numbers of \( \Gamma \). The array
\[
\begin{bmatrix}
c_0 & c_1 & c_2 & \ldots & c_{d-1} & c_d \\
0 & a_1 & a_2 & \ldots & a_{d-1} & a_d \\
b_0 & b_1 & b_2 & \ldots & b_{d-1} & b_d
\end{bmatrix}
\]
is called the intersection array of \( \Gamma \).

Now, suppose that \( \Gamma \) is a distance-regular graph of diameter \( d \geq 2 \), valency \( k \geq 2 \) and intersection numbers \( c_i, a_i, b_i, 0 \leq i \leq d \). Clearly, \( b_d = c_0 = a_0 = 0 \) and \( c_1 = 1 \). In \cite[Section 4.1]{2}, it is shown that \( \Gamma_i(x) \) contains \( k_i \) elements where
\[
k_0 := 1, \quad k_1 := k, \quad k_{i+1} := k_i b_i / c_i, \quad i = 1, \ldots, d - 1,
\]
and in \cite[Proposition 4.1.6]{2} it is shown that
\[
k = b_0 > b_1 \geq b_2 \geq \cdots \geq b_{d-1} > b_d = 0,
\]
\[
1 = c_1 \leq c_2 \leq \cdots \leq c_d \leq k \quad \text{and}
\]
\[
c_i \leq b_j \quad \text{if } i + j \leq d.
\]

For more information on distance-regular graphs, see \cite{2}.
3. Proof of Theorem 1.1

In this section we show Theorem 1.1. In order to show the theorem we use the intersection diagram with respect to an edge. Let \( \Gamma \) be a \( d \)-distance-regular graph of diameter \( d \geq 2 \) and valency \( k \geq 3 \). Let \( (u, v) \) be an edge in \( \Gamma \). For each \( 0 \leq i, j \leq d \) set \( D^i_j = D^i_j(u, v) := \cap_i \Gamma_i(u) \cap \Gamma_j(v) \). The intersection diagram with respect to \( (u, v) \) is the collection of circles indexed by the sets \( \{D^i_j\}_{0 \leq i, j \leq d} \) with lines between them. If there is no line between \( D^i_j \) and \( D^s_t \), then it means that there is no edge \((x, y)\) for any \( x \in D^i_j \) and \( y \in D^s_t \). Also if we know that \( D^i_j \) is the empty set then we suppress it.

Remark 3.1. If \( c_i = c_{i+1} \) then there are no edges between any two of \( \{D^{i+1}_j, D^i_j, D^{i+1}_j\} \).

Let \( c \) be an integer with \( 2 \leq c \leq c_d \). Let \( \eta := \eta_c \) and \( \xi := \xi_c \). Then for any edge \( (u, v) \) in \( \Gamma \) the intersection diagram with respect to \( (u, v) \) has the shape as in Fig. 1.

Lemma 3.2. If \( c_0 + c_{\xi - 1} > c_\xi \) then the following hold:

(i) There are no edges between \( D^{\xi-2}_{\xi-1} \) and \( D^{\xi-1}_{\xi-1} \).
(ii) There exists an edge \( (x_0, x_1) \) such that \( x_0 \in D^{\xi-1}_{\xi-2} \) and \( x_1 \in D^{\xi-1}_{\xi-1} \).
(iii) \( b_\eta \geq c_{\xi-1} + c_\eta \).

Proof. (i) Assume that there is an edge between \( D^{\xi-1}_{\xi-2} \) and \( D^{\xi-1}_{\xi-1} \). Then there exists a path \( (y_0, y_1, \ldots, y_\eta) \) such that \( y_\eta \in D^{\xi-1}_{\xi-2} \) and \( y_i \in D^{\xi-1}_{\xi-2+i} \) for \( 1 \leq i \leq \eta \). Clearly \( d(y_0, y_\eta) = \eta \). It follows that

\[ C_\xi(y_\eta, y_0) \supseteq \Gamma(y_\eta) \cap D^{\xi-2}_{\xi-1} \cup \Gamma(y_0) \cap D^{\xi-1}_{\xi-2+i} \supseteq C_\eta(y_\eta, y_0) \cup C_{\xi-1}(u, y_0). \]

This contradicts \( c_\eta + c_{\xi-1} > c_\xi \). This shows (i).

(ii) Let \( x_1 \in D^{\xi-1}_{\xi-1} \). As \( c_\xi > c_{\xi-1} \) and there are no edges between \( D^{\xi-1}_{\xi-1} \) and \( D^{\xi-1}_{\xi-2} \), there must be a neighbor \( x_0 \in D^{\xi-1}_{\xi-1} \) of \( x_1 \). This shows (ii).

(iii) It follows, from (ii), that there exists a path \( (x_0, x_1, \ldots, x_\eta) \) such that \( x_0 \in D^{\xi-1}_{\xi-1} \) and \( x_i \in D^{\xi-1+i}_{\xi-2+i} \) for \( 1 \leq i \leq \eta \). Now \( C_\eta(x_\eta, x_0) \subseteq \Gamma(x_0) \cap D^{\xi-1}_{\xi-1} \) and by symmetry we
find that
$$c_\xi \leq \left| \Gamma(x_0) \cap D_{\xi-1}^\xi \right| = \left| \Gamma(x_0) \cap D_{\xi}^\xi \right|.$$ We also have
$$C_{\xi-1}(u, x_0) \cup \left[ \Gamma(x_0) \cap D_{\xi-1}^\xi \right] \subseteq B_\eta(x_\eta, x_0),$$
$$C_{\xi-1}(u, x_0) \cap \left[ \Gamma(x_0) \cap D_{\xi}^\xi \right] = \emptyset,$$
which implies
$$c_{\xi-1} + c_\eta \leq b_\eta.$$ This shows (iii).

**Proof of Theorem 1.1.** Suppose \( \xi \leq \eta \). We will derive a contradiction and this shows the theorem. In order to do this we will show several claims.

Let \((u, v)\) be any edge in \( \Gamma \) and \( D_j^I = D_j^I(u, v) := I_i(u) \cap I_j(v) \). Then **Remark 3.1 and Lemma 3.2(i)** imply that the intersection diagram with respect to \((u, v)\) has the shape as in **Fig. 2**, and **Lemma 3.2(ii)** implies that there exists a path \((x_0, x_1, \ldots, x_\xi)\) of length \( \xi \) such that \( x_0 \in D_{\xi-1}^\xi \) and \( x_i \in D_{\xi+i-2}^{\xi+i-1} \) for all \( 1 \leq i \leq \xi \).

**Claim 1.**

(i) The set \( D_{\xi}^\xi \cap A_\xi(x_\xi, x_0) \) is non-empty.

(ii) Let \( z \in D_{\xi}^\xi \cap A_\xi(x_\xi, x_0) \). Then \( C_\xi(z, x_\xi) \subseteq D_{2\xi-2}^{\xi-3} \cap \Gamma_1(x_\xi) \).

(iii) Let \( z \in D_{\xi}^\xi \cap A_\xi(x_\xi, x_0) \). Then \( D_{\xi+1}^{\xi+1} \cap C_\xi(x_\xi, z) \) is non-empty.

**Proof.** (i) The set \( D_{\xi}^\xi \cap A_\xi(x_\xi, x_0) \) is non-empty, as otherwise
$$C_{\xi-1}(u, x_0) \cup B_{\xi-1}(v, x_0) \subseteq B_\xi(x_\xi, x_0),$$
$$C_{\xi-1}(u, x_0) \cap B_{\xi-1}(v, x_0) = \emptyset$$
hold and thus \( c_{\xi-1} + b_{\xi-1} \leq b_\xi \) which is a contradiction as \( \xi \geq 2 \). This shows (i).
(ii) It is clear that \( C_\xi(z, x_\xi) \subseteq D_{2\xi-3}^{2\xi-2} \cup D_{2\xi-1}^{2\xi-1} \). If there exists \( z' \in D_{2\xi-1}^{2\xi-1} \cap C_\xi(z, x_\xi) \) then
\[
C_\xi(x_0, z') \cup \{x_\xi\} \subseteq C_{2\xi-1}(v, z').
\]
This contradicts \( c_\xi = c_{2\xi-1} \) as \( \eta \geq \xi \). Thus \( D_{2\xi-1}^{2\xi-1} \cap C_\xi(z, x_\xi) = \emptyset \) and this shows (ii).

(iii) It is clear that \( C_\xi(x_\xi, z) \subseteq D_{\xi-1}^{\xi} \cup D_{\xi+1}^{\xi+1} \). If \( D_{\xi+1}^{\xi+1} \cap C_\xi(x_\xi, z) = \emptyset \) then
\[
C_\xi(x_\xi, z) \cup \{x_0\} \subseteq C_\xi(v, z)
\]
which is a contradiction. Thus we have (iii). \( \square \)

Define \( P := C_\xi(x_\xi, x_0) \) and \( Q := B_{\xi-1}(x_\xi-1, x_0) - B_\xi(x_\xi, x_0) \).

Claim 2. The set \( P \cap Q \) is empty and \( P \cup Q \subseteq D_{2\xi-3}^{2\xi-2} \).

**Proof.** It is clear that \( P \subseteq D_{\xi-1}^{\xi} \). Let \( z \in Q \). If \( z \in P \), then \( x_\xi \in D_{\xi-1}^{\xi}(x_\xi, x_\xi-1) \) and \( z \in D_{\xi-1}^{\xi}(x_\xi, x_\xi-1) \) hold. As \( \eta \geq \xi \geq 2 \), it follows by Lemma 3.2(i) that there are no edges between \( D_{\xi-1}^{\xi} \) and \( D_{\xi}^{\xi-1} \). Therefore \( P \cap Q = \emptyset \) and \( Q \subseteq A_\xi(x_\xi, x_0) \) hold. As
\[
D_{\xi-2}^{\xi-2} \cup D_{\xi-1}^{\xi-2} \cap D_{\xi-1}^{\xi-1} \cap \Gamma_1(x_0) \subseteq B_\xi(x_\xi, x_0),
\]

it follows that \( Q \subseteq D_{\xi}^{\xi} \cup D_{\xi-1}^{\xi-1} \cup D_{\xi-2}^{\xi-2} \cup D_{\xi-2}^{\xi-1} \). Suppose \( z \in [D_{\xi}^{\xi} \cup D_{\xi-1}^{\xi-1} \cup D_{\xi-2}^{\xi-2}] \). Then
\[
C_\xi(z, x_\xi) \subseteq D_{2\xi-3}^{2\xi-2} \cap \Gamma_1(x_\xi) = C_{2\xi-1}(u, x_\xi).
\]
(Eq. (6) is clear when \( z \in D_{\xi-2}^{\xi-1} \cup D_{\xi-2}^{\xi-1} \). If \( z \in D_{\xi}^{\xi} \), then it follows from Claim 1(ii).) By comparing both sides of (6) we have \( x_\xi \in C_{2\xi-1}(u, x_\xi) = C_\xi(z, x_\xi) \) which contradicts \( z \in B_{\xi-1}(x_\xi-1, x_0) \). Hence \( Q \subseteq D_{2\xi-3}^{2\xi-2} \). The claim is proved. \( \square \)

Claim 3. There exists \( u' \in B_{\xi-1}(x_1, v) - B_{\xi-1}(x_1, v) \). Define \( R := B_{\xi}(u', x_0) \). Then \( R \subseteq D_{\xi}^{\xi} \).

**Proof.** Since \( u \in B_{\xi-1}(x_1, v) - B_{\xi-1}(x_0, v) \), there exists \( u' \in B_{\xi-1}(x_0, v) - B_{\xi-1}(x_1, v) \).

Consider the intersection diagram with respect to \( (u', v) \). Then we have \( x_0 \in D_{\xi-1}^{\xi}(u', v) \) and \( x_1 \in D_{\xi-2}^{\xi-1}(u', v) \). Take any \( w_2 \in R \). Then \( w_2 \in D_{\xi+1}^{\xi+1}(u', v) \) and we can take a path \((w_2, w_3, \ldots, w_\xi)\) such that \( w_i \in D_{\xi+i-2}^{\xi+i-2}(u', v) \) for \( 2 \leq i \leq \xi \). By Claim 1(i), (iii) there exist \( z_0 \in D_{\xi}^{\xi}(u', v) \cap A_\xi(w_2, x_1) \) and \( z_1 \in D_{\xi+1}^{\xi+1}(u', v) \cap C_\xi(w_\xi, z_0) \). Let \((z_1, \ldots, z_\xi = w_\xi)\) be a shortest path connecting \( z_1 \) and \( w_\xi \). Since \( z_1 \in D_{\xi+1}^{\xi+1}(u', v) \) and \( z_\xi \in D_{2\xi-2}^{2\xi-2}(u', v) \) such that \( d(z_1, z_\xi) = \xi - 1 \) there exists an integer \( t \) with \( 2 \leq t \leq \xi - 1 \) such that \( z_t \in D_{\xi+i}^{\xi+i}(u', v) \) for all \( 0 \leq i \leq \xi - 1 \) and \( z_t \in D_{\xi+i-1}^{\xi+i-1}(u', v) \) for all \( 0 \leq i \leq \xi \) by Claim 1(ii). Next we return to the intersection diagram with respect to \( (u, v) \). Since \( x_0 \in D_{\xi-1}^{\xi-1} \) and \( w_2 \in B_{\xi-1}(x_0, x_0), \) we have \( w_2 \in D_{\xi-1}^{\xi-1} \cup D_{\xi}^{\xi} \).

Suppose \( w_2 \in D_{\xi}^{\xi-1} \). Then we have \( w_i \in D_{\xi+i-3}^{\xi+i-3} \) for all \( 2 \leq i \leq \xi \) as the former diagram gives us the distance between \( v \) and \( w_j \). Similarly we have \( z_i \in D_{\xi+i-2}^{\xi+i-2} \) for all
\[ t \leq i \leq \xi \quad \text{and} \quad z_i \in D_{\xi+i-1}^\xi \quad \text{for all} \quad 0 \leq i \leq t - 1. \] 
Hence \((x_1, z_0)\) is an edge such that \(x_1 \in D_\xi^\xi\) and \(z_0 \in D_{\xi-1}^\xi\). This contradicts Lemma 3.2(i) as \(c_\eta + c_{\xi-1} > c_\xi\) holds by the assumption \(\eta \geq \xi\). Hence we have \(w_2 \in D_\xi^\xi\) and our claim is proved. \(\square\)

Claims 2 and 3 show that the sets \(P\), \(Q\) and \(R\) are disjoint and \(P \cup Q \cup R \subseteq B_{\xi-1}(u, x_0)\).

As \(|P| = c_\xi\), \(|Q| = b_{\xi-1} - b_\xi\) and \(|R| = b_\xi\) hold, this implies \(c_\xi + (b_{\xi-1} - b_\xi) + b_\xi \leq b_{\xi-1}\) which is a contradiction. The theorem is proved. \(\square\)

4. Proofs of Theorem 1.3 and Corollary 1.4

In this section we prove Theorem 1.3 and Corollary 1.4. Recall \(C := \{c_i \mid i = 1, \ldots, d\}\).

Lemma 4.1. Let \(\Gamma\) be a distance-regular graph of diameter \(d \geq 2\) and valency \(k \geq 3\). Let \(c \in C \setminus \{1\}\). Then

(i) \(c_i + c_{\xi-i} \leq c\) for all \(1 \leq i \leq \xi - 1\).

(ii) \(\prod_{j=1}^{\xi-1} c_j \leq \left(\frac{c}{2}\right)^{\xi-1}\).

Proof. (i) This is [7, Proposition 1(ii)].

(ii) Let \(\xi := \xi_c\). Then (i) implies that

\[ 2\sqrt{c_j c_{\xi-j}} \leq c_j + c_{\xi-j} \leq c \]

holds for all \(1 \leq j \leq \xi - 1\). Hence we have

\[ \prod_{j=1}^{\xi-1} c_j = \prod_{j=1}^{\xi-1} \sqrt{c_j c_{\xi-j}} \leq \left(\frac{c}{2}\right)^{\xi-1}. \]

Lemma 4.2. Let \(\beta := \left(\frac{1}{\delta}\right)^\frac{1}{\alpha}\). For a real number \(x\) with \(\beta \leq x \leq 1\)

\[ x^\alpha + 2(1 - x)^\alpha \leq 1 \]

holds.

Proof. Define \(f : [\beta, 1] \to \mathbb{R}\) by

\[ f(x) := x^\alpha + 2(1 - x)^\alpha. \]

By definition of \(\beta\) (and \(\alpha\)), it follows easily that \(f(\beta) = 1\). Also \(f(1) = 1\). By straightforward calculation one sees that on \([\beta, 1]\), the function \(f\) has maxima at \(x = 1\) and \(x = \beta\). This shows the lemma. \(\square\)

Proof of Theorem 1.3. Let \(C = \{\gamma_1, \gamma_2, \ldots, \gamma_q\}\) such that

\[ 1 = \gamma_1 < \gamma_2 < \cdots < \gamma_q = c_d \]
holds. We will prove
\[ \xi_{\gamma_j} \leq \frac{1}{2}(\gamma_j)^{\alpha} \eta_1 + 1 \]  
and
\[ \xi_{\gamma_j} + \eta_{\gamma_j} \leq (\gamma_j)^{\alpha} \eta_1 + 1 \]
hold for all \( j = 1, 2, \ldots, q \) by induction on \( j \).

As \( 1 = \xi_1 \leq \frac{1}{2}(\eta_1 + 1) \) and \( \xi_1 + \eta_1 = \eta_1 + 1 \) hold, so (7) and (8) hold for \( j = 1 \). Now let \( s \geq 2 \) and assume that (7) and (8) hold for all \( \gamma_i \) with \( 1 \leq i < s \). Let \( c := \gamma_s, c' := \gamma_{(s-1)} \) and \( c'' = bc \) for some \( 0 < b < 1 \).

First we will prove Eq. (7) holds for \( j = s \). In order to show this we need to consider two cases, namely the case \( 0 < b \leq \beta \) and the case \( \beta < b < 1 \), respectively.

**Case 1:** \( 0 < b \leq \beta \).

**Proof.** As \( c' = bc \leq \beta c \) and \( b^\alpha = \frac{1}{2} \) hold, we find
\[ \xi_c = \xi_{c'} + \eta_{c'} \leq (bc)^{\alpha} \eta_1 + 1 \leq \frac{1}{2} c^{\alpha} \eta_1 + 1 \]
hold by our induction hypothesis. \( \square \)

**Case 2:** \( \beta < b < 1 \).

**Proof.** Let \( c'' := c(\xi_c - \xi_{c'}) \). Then Lemma 4.1(i) implies that \( c = c_{\xi_c} \geq c_{\xi_{c'}} + c_{\xi_c - \xi_{c'}} = c' + c'' \) holds and thus \( c'' \leq (1 - b)c \) holds. Therefore we find that
\[ \xi_c = \xi_{c''} + (\xi_{c''} - 1) \]
\[ \leq \frac{1}{2}(c')^{\alpha} \eta_1 + 1 + (c'')^{\alpha} \eta_1 \]
\[ \leq \frac{1}{2}(b^{\alpha} + 2(1 - b)^{\alpha}) c^{\alpha} \eta_1 + 1 \]
\[ \leq \frac{1}{2} c^{\alpha} \eta_1 + 1 \]
hold by Lemma 4.2 and our induction hypothesis. \( \square \)

Hence Eq. (7) holds for \( j = s \). The fact that Eq. (8) holds for \( j = s \) follows from Theorem 1.1. Therefore we have shown that Eqs. (7) and (8) hold for all \( 1 \leq j \leq q \).

Now assume \( h \geq 2 \). In [6, Theorem 2] it is shown that \( \eta_1 \leq 2(h + 1) \) holds. Eqs. (3) and (4) follow now immediately from Eqs. (1) and (2), respectively. \( \square \)

**Proof of Corollary 1.4.** If \( \xi_{c_d} = d \) then by Eq. (1) the following holds:
\[ d = \xi_{c_d} \leq \frac{1}{2}(c_d)^{\alpha} \eta_1 + 1 \leq \frac{1}{2} k^{\alpha} \eta_1 + 1. \]

Now we assume that \( \xi_{c_d} < d \). If \( c_d \leq \beta k \) then the result holds as
\[ d = \xi_{c_d} + \eta_{c_d} - 1 \leq (c_d)^{\alpha} \eta_1 \leq (\beta k)^{\alpha} \eta_1 = \frac{1}{2} k^{\alpha} \eta_1 \]
Proof of Theorem 1.5

(i), (ii): These follow from [2, Corollary 5.9.7] and [2, Proposition 5.5.1], assertion follows from [6, Theorem 2]. Note that by (ii) respectively. (iii) We only need to consider the case $d \leq \eta$ and holds by (2). To complete the proof we need to consider $\ell \leq \eta$ and and $\eta \leq \xi$. Hence the result follows by Theorem 1.3 and Lemma 4.2:

$$
\begin{align*}
d &\leq \xi_{cd} + (\xi_c + \eta_c - 1) \\
&\leq \frac{1}{2}(c_{cd})^2 \eta_1 + 1 + c^2 \eta_1 \\
&\leq \frac{1}{2}((\eta + 2(1 - \epsilon))^2 k^2 \eta_1 + 1 \\
&\leq \frac{1}{2}k^2 \eta_1 + 1. \quad \Box
\end{align*}
$$

5. Proof of Theorem 1.5

In the proof of Theorem 1.5 we will use the following results.

Lemma 5.1. Let $\Gamma$ be a distance-regular graph of diameter $d \geq 2$ and valency $k \geq 3$. Let $h = h(\Gamma)$. Then for any integer $1 \leq i \leq d$ the following hold:

(i) If $k_{i+1} \leq k_i$, then $d \leq 3i$.
(ii) $a_1 + 2 \leq c_i + b_{i-1}$.
(iii) If $b_{i-1} = 2$ and $c_i = 1$ then $i \leq 2h + 2$.

Proof. (i), (ii): These follow from [2, Corollary 5.9.7] and [2, Proposition 5.5.1], respectively. (iii) We only need to consider the case $h = 1$ as for the case $h \geq 2$ the assertion follows from [6, Theorem 2]. Note that by (ii) $a_1 \leq 1$ holds. Now [4, Theorems 1.1–1.2] imply that $c_\ell \neq 0$. Hence $\eta_1 \leq 3 < 2h + 2$. The lemma is shown. \Box

Lemma 5.2. Let $\Gamma$ be a distance-regular graph of diameter $d \geq 2$ and valency $k \geq 3$. Let $\ell \leq d$ be maximal such that $2c_\ell \leq b_{\ell-1}$. Then

$$
d \leq 4\ell.
$$

Proof. By definition of $\ell$, $2c_\ell \leq b_{\ell-1}$ and $2c_{\ell+1} > b_\ell$ hold. We may assume that $3\ell + 1 \leq d$. Let $c := c_{2\ell+1}$, $\xi := \xi_c$ and $\eta := \eta_c$. Then $2c_{\ell+1} > b_\ell \geq c_{2\ell+1} = c$ holds by (5). We have $c \geq 2$ as otherwise $c_{\ell+1} = 1$ and $b_\ell = 1$ hold by the definition of $\ell$, and thus $k_{\ell+1} = k_\ell$ holds which contradicts Lemma 5.1(i).

If $\eta \geq \ell + 1$, then $c_\eta + c_{\ell - 1} \geq 2c_{\ell+1} > c = c_\xi$ as $\xi - 1 \geq \eta \geq \ell + 1$ by Theorem 1.1. By Lemma 3.2(iii) we have

$$
b_\ell \geq b_\eta \geq c_{\ell-1} + c_\eta \geq 2c_{\ell+1}.
$$

But that is a contradiction with the definition of $\ell$. Hence we find $\eta \leq \ell$. As $2(\ell + 1) \leq \eta + \xi \leq 3\ell + 1$, $c_{3\ell+1} \geq c_{\eta+\xi} \geq 2c_{\ell+1} > b_\ell$ holds by Lemma 4.1(i). So $d \leq 4\ell$ holds. \Box

Proof of Theorem 1.5. Let $\ell \leq d$ be maximal such that $2c_\ell \leq b_{\ell-1}$. Let $c := c_\ell$, $\xi := \xi_\ell$, and $\eta := \eta_\ell$. In order to prove the theorem we need to consider three cases, namely $c_\ell \geq 2$, $c_\ell = 1$ and $b_{\ell-1} \geq 3$) and $c_\ell = 1$ and $b_{\ell-1} = 2$).

Case 1: $c_\ell \geq 2$. 

Proof. We have
\[ \prod_{j=1}^{\xi-1} b_{j-1} \geq (2c)^{\xi-1} \quad \text{and} \quad \prod_{j=1}^{\xi-1} c_j \leq \left( \frac{c}{2} \right)^{\xi-1} \]
from Lemma 4.1(ii). Theorem 1.1 implies that \( \ell \leq \xi + \eta - 1 \leq 2\xi - 2 \). Hence we have
\[ k_\ell = \prod_{j=1}^{\xi-1} b_{j-1} \prod_{j=1}^{\ell} b_{j-1} c_j \geq 4^{\xi-1} 2^{\ell-\xi+1} = 2^{\ell+\xi-1} \]
and thus \( \log_2 v > \log_2 k_\ell \geq \ell + \xi - 1 \geq \frac{3}{2} \ell \) holds. Therefore by Lemma 5.2
\[ d \leq 4\ell < \frac{8}{3} \log_2 v \]
holds. \( \Box \)

Case 2: \( c_\ell = 1 \) and \( b_{\ell-1} \geq 3 \).

Proof. We have \( \frac{b_{i-1}}{c_i} \geq 3 \) for all \( 1 \leq i \leq \ell \). Hence \( v > k_\ell \geq 3^\ell \) holds. By Lemma 5.2,
\[ d \leq 4\ell < 4 \log_3 v = 4(\log_3 2)(\log_2 v) < \frac{8}{3} \log_2 v \]
holds. \( \Box \)

Case 3: \( c_\ell = 1 \) and \( b_{\ell-1} = 2 \).

Proof. There are two possibilities, namely \( c_{\ell+1} = 1 \) or \( c_{\ell+1} \geq 2 \), but in each case \( k_{\ell+1} \leq k_\ell \) holds by \( b_\ell \leq b_{\ell-1} = 2 \) and \( k_{\ell+1} c_{\ell+1} = k_\ell b_\ell \). Hence Lemma 5.1(i) and (iii) imply that \( d \leq 3\ell \) and \( \ell \leq 2h + 2 \). If \( b_1 \leq 2 \) then \( k \leq 4 \) as \( a_1 \leq 1 \) by Lemma 5.1(ii).
Hence the result is proved by [1] and [3].

Now we may assume \( b_1 \geq 3 \). Then
\[ v > k_\ell = \prod_{j=1}^{\ell} b_{j-1} \geq k b_1 2^{\ell-1-h} \geq 3^{h+1} 2^{\ell-h-1} \geq 6^\ell. \]
Therefore we have \( \log_2 v > \log_2 6^\ell \geq \frac{9}{8} \ell \) and thus
\[ d \leq 3\ell < \frac{8}{3} \log_2 v \]
holds. \( \Box \)

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